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# THEORY OF STRUCTURES

BY

ARTHUR MORLEY, M.Sc., M.I.MECH.E.

PROFESSOR OF MECHANICAL ENGINEERING IN UNIVERSITY COLLEGE NOTTINGHAM

*WITH 320 DIAGRAMS AND NUMEROUS EXAMPLES*

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# PREFACE

THE object of the following pages is mainly to set forth the theory of the simpler structures so far as it relates to strength, stiffness, and stability. The subject is largely based upon statics and the elastic properties of material, and has much in common with that called Strength of Materials. Consequently I have taken a considerable amount of matter in seven chapters out of the first nine, without great modification from my earlier book, "Strength of Materials," to which the present volume forms a companion.

Worked-out examples form an important feature of the text, and are generally essential to obtaining a sound knowledge of the subject. I have not hesitated to use examples which may be called academic, because they are simplified to illustrate particular points without unnecessary arithmetic complication; this is particularly the case with statically indeterminate structures and secondary stresses on which little more than the principle is given as an introduction to the larger treatises. Students are apt to forget how many stress computations in structural design are necessarily of a conventional nature, and the attempt has been made to point out when this is specially the case. In some instances more exact estimates have been made to indicate the nature and degree of possible error involved by conventional assumptions.

Fairly free use has been made of influence lines, which form such clear and instructive means of understanding the stresses arising from moving loads.

The practical design of structures involves so much outside of what may reasonably be called theory that it can only be thoroughly learned in the drawing office, but a few examples have been included to illustrate the application of the theory to practice.

Reinforced concrete structures are becoming so important as to demand a complete volume for their treatment, and no attempt has

been made to deal with this subject except incidentally as an example of a beam of composite cross section.

I take this opportunity of thanking numerous friends who have generously assisted me in reading proofs, preparation of designs or diagrams, and checking examples; particularly Messrs. S. W. Budd, R. T. McCallum, B.Sc., and W. N. Thomas, B.Sc. I also thank Sir Wm. Arrol & Co., Ltd., Messrs. Dorman Long & Co., Ltd., and Messrs. R. A. Skelton & Co., for the use of tables, diagrams, and technical information; and Mr. H. S. Prichard for much information regarding American practice relating to the treatment of live loads.

I should be grateful for intimation of any errors which readers may observe in the book.

ARTHUR MORLEY.

UNIVERSITY COLLEGE,
NOTTINGHAM.
*April*, 1912.

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# THEORY OF STRUCTURES

## CHAPTER I

### *STRESS AND STRAIN*

**1. Introductory.**—The subject generally known as the *Theory of Structures* or *Mechanics of Structures* includes the study of the forces carried by structures and by the individual members of structures. It is largely an application of the subject of statics, but frequently the complexity of a structure or the uncertainty of the conditions of loading prevent anything like an exact mathematical analysis of the stresses, and assumptions have to be made which it is necessary to test by experiment and practical experience. It is important to realise the limits of much of our theory and the extent to which stress computations are frequently quite conventional rather than representing an actual physical state; *e.g.* the maximum intensity of stress in a flat bar axially pulled is not known within wide limits if the bar is perforated by a single hole.

The mechanics of structures is fundamental to structural design, but successful design involves commercial questions, such as cost and durability, which are not treated as theory, and which cannot well be taken into account except as the result of practical experience.

The "Theory of Structures" is closely related to the subject of the "Strength of Materials," and any boundary between the two is necessarily an arbitrary one. "Strength of Materials" has been treated in a separate volume, but to make this book serviceable to the reader who is concerned with structures only and not with machines, sufficient of the theory of stresses and strains in single pieces has been included to make it complete in itself.

**2. Stress.**—The equal and opposite action and reaction which take place between two bodies, or two parts of the same body, transmitting forces constitute a stress. If we imagine a body which transmits a force to be divided into two parts by an ideal surface, and interaction takes place across this surface, the material there is said to be stressed or in a state of stress. The constituent forces, and therefore the stress itself, are distributed over the separating surface either uniformly or in some other manner. The *intensity of the stress* at a surface, generally referred to with less exactness as merely the stress, is estimated by the

force transmitted per unit of area in the case of uniform distribution; if the distribution is not uniform, the stress intensity at a point in the surface must be looked upon as the limit of the ratio of units of force to units of area when each is decreased indefinitely. The intensity of stress is also sometimes called the unit stress.

**3. Simple Stresses.**—There are two specially simple states of stress which may exist within a body. More complex stresses may be split into component parts.

(1) *Tensile stress* between two parts of a body exists when each draws the other towards itself. The simplest example of material subject to tensile stress is that of a tie-bar sustaining a pull. If the pull on the tie-bar is say P lbs., and we consider any imaginary plane of section X perpendicular to the axis of the bar, of area $a$ square inches, dividing the bar into two parts A and B (Fig. 1), the material at the section X is under a tensile stress. The portion B, say, exerts a pull on the portion A which just balances P, and is therefore equal and opposite to it. The average force exerted per square inch of section is

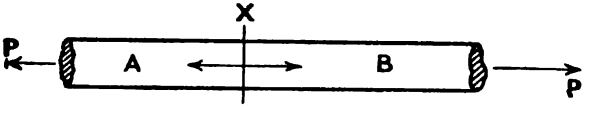


FIG. 1.

$$p = \frac{P}{a}$$

and this value $p$ is the mean intensity of tensile stress at this section.

(2) *Compressive stress* between two parts of a body exists when each pushes the other from it.

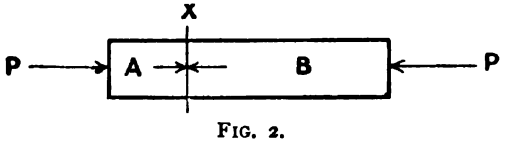


FIG. 2.

If a bar (Fig. 2) sustains an axial thrust of P tons at each end, at a transverse section X of area $a$ square inches, dividing the bar into two parts A and B, the material is under compressive stress. The portion A, say, exerts a push on the portion B equal and opposite to that on the far end of B. The average force per square inch of section is

$$p = \frac{P}{a}$$

and this value $p$ is the mean intensity of compressive stress at the section X.

*Shear stress* exists between two parts of a body in contact when the two parts exert equal and opposite forces on each other laterally in a direction tangential to their surface of contact. As an example, there is a shear stress at the section XY of a pin or rivet (Fig. 3) when the two plates which it holds together sustain a pull P in the plane of the section XY. If the area of section XY is $a$ square inches, and the pull is P

tons, the total shear at the section XY is P tons, and the average force per square inch is

$$q = \frac{P}{a}$$

This value $q$ is the mean intensity of shear stress at the section XY.

**4. Strain.**—Strain is the alteration of shape or dimensions resulting from stress.

(1) Tensile strain is the stretch, and often results from a pull which causes a condition of tensile stress to be set up. It is in the direction of the tensile stress, and is measured by the fractional elongation. Thus, if a length $l$ units is increased to $l + \delta l$, the strain is

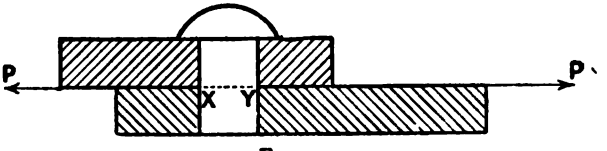


FIG. 3.

$$\frac{\delta l}{l}$$

The strain is obviously equal numerically to the stretch per unit of length.

(2) Compressive strain is the contraction which is often due to compressive stress, and is measured by the ratio of the contraction to the original length. If a length $l$ contracts to $l - \delta l$, the compressive strain is

$$\frac{\delta l}{l}$$

Tensile stress causes a contraction perpendicular to its own direction, and compressive stress causes an elongation perpendicular to its own direction.

(3) Distortional or shear strain is the angular displacement produced by shear stress. If a piece of material be subjected to a pure shear stress in a certain plane, the change in inclination (estimated in radians) between the plane and a line originally perpendicular to it, is the numerical measure of the resulting shear strain (see Art. 10).

**5. Elastic Limits.**—The limits of stress for a given material within which the resulting strain completely disappears after the removal of the stress are called the elastic limits. If a stress beyond an elastic limit is applied, part of the resulting strain remains after the removal of the stress; such a residual strain is called a permanent set. The determination of an elastic limit will evidently depend upon the detection of the smallest possible permanent set, and gives a lower stress when instruments of great precision are employed than with cruder methods. In some materials the time allowed for strain to develop or to disappear will affect the result obtained.

Elastic strain is that produced by stress within the limits of elasticity;

but the same term is often applied to the portion of strain which disappears with the removal of stress even when the elastic limits have been exceeded.

*Hooke's Law* states that within the elastic limits the strain produced is proportional to the stress producing it. The law refers to all kinds of stress.

This law is not exactly true for all materials, but is approximately so for many.

**6. Modulus of Elasticity.**—Assuming the truth of Hooke's Law, we may write

$$\text{intensity of stress} \propto \text{strain}$$

or

$$\text{stress intensity} = \text{strain} \times \text{constant}$$

The constant in this equation is called the modulus or coefficient of elasticity, and will vary with the kind of stress and strain contemplated, there being for each kind of stress a different kind of modulus. Since the strain is measured as a mere number, and has no dimensions of length, time, or force, the constant is a quantity of the same kind as a stress intensity, being measured in units of force per unit of area, such as pounds or tons per square inch. We might define the modulus of elasticity as the intensity of stress which would cause unit strain, if the material continued to follow the same law outside the elastic limits as within them, or as the intensity of stress per unit of strain.

**7. Components of Oblique Stresses.**—When the stress across any given surface in a material is neither normal nor tangential to that surface, we may conveniently resolve it into rectangular components, normal to the surface and tangential to it. The normal stresses are tensile or compressive according to their directions, and the tangential components are shear stresses.

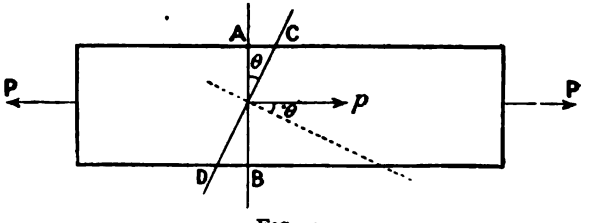



FIG. 4.

A simple example will illustrate the method of resolution of stress. If a parallel bar of cross-section $a$ square inches be subjected to a pull of P tons, the intensity of tensile stress $p$ is $\frac{P}{a}$ in the direction of the length of the bar, or, in other words, normal to a surface, AB (Fig. 4), perpendicular to the line of pull.

Let $p_n$ and $p_t$ be the component stress intensities, normal and tangential respectively, to a surface, CD, which makes an angle $\theta$ with the surface AB. Resolving the whole force P normal to CD, the component is

$$P_n = P \cos \theta$$

and the area of the surface CD is $a \sec \theta$, hence

$$p_n = \frac{P \cos \theta}{a \sec \theta} = \frac{P}{a} \cos^2 \theta = p \cos^2 \theta$$

and resolving along CD, the tangential component of the whole force is

$$P_t = P \sin \theta$$

$$p_t = \frac{P \sin \theta}{a \sec \theta} = \frac{P}{a} \sin \theta \cos \theta = p \sin \theta \cos \theta, \text{ or } \frac{p}{2} \sin 2\theta$$

Evidently $p_t$ reaches a maximum value $\frac{1}{2}p$ when $\theta = 45°$, so that all surfaces, curved or plane, inclined 45° to AB (and therefore also to the axis of pull) are subjected to maximum shear stress. In testing materials in tension or compression, it often happens that fracture takes place by shearing at surfaces inclined at angles other than 90° to the axis of pull.

EXAMPLE.—The material of a tie-bar has a uniform tensile stress of 5 tons per square inch. What is the intensity of shear stress on a plane the normal of which is inclined 40° to the axis of the bar? What is the intensity of normal stress on this plane, and what is the resultant intensity of stress?

Considering a portion of the bar, the section of which is 1 square inch normal to the axis, the pull is 5 tons. The area on which this load is spread on a plane inclined 40° to the perpendicular cross-section is

$$(1 \times \sec 40°) \text{ square inch}$$

and the amount of force resolved parallel to this oblique surface is

$$(5 \times \sin 40°) \text{ tons}$$

hence the intensity of shearing stress is

$$5 \sin 40° \div \sec 40° = 5 \sin 40° \cos 40° = 5 \times 0{\cdot}6428 \times 0{\cdot}7660$$
$$= 2{\cdot}462 \text{ tons per square inch}$$

The force normal to this oblique surface is $5 \cos 40°$, hence the intensity of normal stress is

$$5 \cos 40° \div \sec 40° = 5 \cos^2 40° = 5 \times 0{\cdot}766 \times 0{\cdot}766$$
$$= 2{\cdot}933 \text{ tons per square inch}$$

The resultant stress is in the direction of the axis of the bar, and its intensity is

$$5 \div \sec 40° = 5 \cos 40° = 3{\cdot}83 \text{ tons per square inch}$$

**8. Complementary Shear Stresses. State of Simple Shear.**—A shear stress in a given direction cannot exist without a balancing shear stress of equal intensity in a direction at right angles to it.

If we consider a small rectangular block, ABCD, of material (Fig. 5) under shear stress of intensity $q$, we cannot have equilibrium with merely equal and opposite tangential forces on the parallel pair of faces

AB and CD: these forces constitute a couple, and alone exert a turning moment. Statical considerations of equilibrium show that in this case no additional system of forces can balance the couple and produce the equilibrium unless they result in a couple contrary to the previous one;

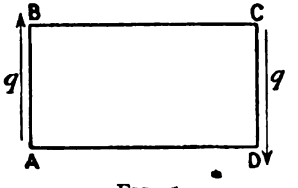



FIG. 5.

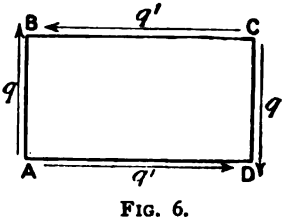



FIG. 6.

hence there must be tangential components along AD and CB, such as to balance the moments of the forces on AC and CD whether there are in addition normal forces or not. If there is a tangential stress exerting force along AD and CB (Fig. 6), and its intensity be $q'$, and the thickness of the block ABCD perpendicular to the figure be $l$, the forces on AB, BC, CD, and DA are

$$AB.l.q,\ BC.l.q',\ CD.l.q,\ \text{and}\ DA.l.q'$$

respectively, and equating the moments of the two couples produced

$$AB.l.q \times BC = BC.l.q' \times AB$$

hence

$$q = q'$$

That is, the intensities of shearing stresses across two planes at right angles are equal; this will remain true whatever normal stresses may act, or, in other words, whether $q$ and $q'$ are component or resultant stresses on the perpendicular planes.

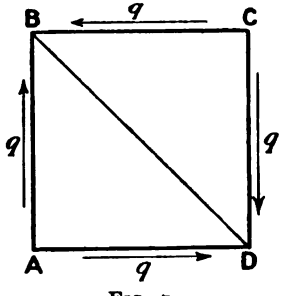



FIG. 7.

*Simple Shear.*—The state of stress shown in Fig. 6, where there are only the shear stresses of equal intensity $q$, is called simple shear. To find the stress existing in other special directions, take a small block ABCD (Fig. 7), the sides of the square face ABCD being each $s$ and the length of the block perpendicular to the figure being $l$. Considering the equilibrium of the piece BCD, resolve the forces $q$ perpendicularly to the diagonal BD, and we must have a force

$$2.q.s.l \cos 45^\circ, \text{ or } 2\,\frac{qsl}{\sqrt{2}}$$

acting on the face BD.

The area of BD is $BD \times l = \sqrt{2}.s.l$

Therefore, if $p_n$ is the intensity of normal stress on the face BD,

$$p_n \times \sqrt{2} . s . l = \frac{2}{\sqrt{2}} . q . s . l$$

hence $$p_n = q$$

and $p_n$ is evidently compressive.

Similarly the intensity of *tensile* stress on a plane AC is evidently equal numerically to $q$.

Further by resolving along BD or AC the intensity of the tangential stress on such planes is evidently zero. Hence a state of simple shear produces pure tensile and compressive stresses across planes inclined 45° to those of pure shear, and the intensities of these direct stresses are each equal to the intensities of the pure shear stress.

**9. Three Important Elastic Constants.**—Three moduli of elasticity (Art. 6) corresponding to three simple states of stress are important.

**Young's Modulus**, also called the Stretch or Direct Modulus, is the Modulus of Elasticity for pure tension with no other stress acting; it has in most materials practically the same value for compression; it is always denoted by the letter E. This direct modulus of elasticity is equal to the tensile (or compressive) stress per unit of linear strain (Art. 6). If a tensile stress $p$ tons per square inch cause a tensile strain $e$ (Art. 4), intensity of tensile stress = tensile strain × E

or $$p = e \times E$$

hence $$E = \frac{p}{e} = \frac{\text{tensile stress intensity}}{\text{tensile strain}}$$

and is expressed in the same units (tons per square inch here) as the stress $p$.

The value of E for steel or wrought iron is about 13,000 tons per square inch.

EXAMPLE 1.—Find the elongation in a steel tie-bar 10 feet long and 1·5 inches diameter, due to a pull of 12 tons.

$$\text{Area of section} = 1{\cdot}5 \times 1{\cdot}5 \times 0{\cdot}7854 = 1{\cdot}767 \text{ square inch}$$

$$\text{Stress intensity} = \frac{12}{1{\cdot}767} = 6{\cdot}79 \text{ tons per square inch}$$

$$\text{Strain} = \frac{6{\cdot}79}{13{,}000}$$

$$\text{Elongation} = \frac{6{\cdot}79}{13{,}000} \times 10 \times 12 = 0{\cdot}0627 \text{ inch}$$

EXAMPLE 2.—A copper and a steel wire, both exactly the same length, the former 0·1 and the latter 0·2 square inches in cross-sectional area, are joined together at their ends and are then stretched by a force, W. Find the tension taken by each wire, taking E as 6000 for copper and 13,000 for steel in tons per square inch.

The essential fact is that the stretch of the two wires must be the

same. Let P be the pull in the steel; then W − P is the pull borne by the copper. Then, if $l$ = length of both wires

$$\text{Stretch of the steel} = l \times \frac{\text{unit stress}}{E} = l \times \frac{P}{0{\cdot}2 \times 13{,}000}$$

$$\text{Stretch of the copper} = l \times \frac{W - P}{0{\cdot}1 \times 6000}$$

Equating the two stretches

$$\frac{P}{26} = \frac{W - P}{6}$$

hence $$P = \tfrac{13}{16}W \text{ and } W - P = \tfrac{3}{16}W$$

**10. Modulus of Rigidity**, Modulus of Transverse Elasticity, or Shearing Modulus, is the modulus expressing the relation between the intensity of shear stress and the amount of shear strain. It is denoted by the letter N, also sometimes by C or G. If the shearing strain (Art. 4) is $\phi$ (radians) due to a shear stress of intensity $q$ tons per square inch, then

$$\text{shear stress} = \text{shear strain} \times N$$

or

$$q = \phi \times N$$

$$N \text{ (tons per square in.)} = \frac{q}{\phi} = \frac{\text{shear stress}}{\text{shear strain}}$$

The value of N for steel is about $\frac{2}{5}$ of the value of E.

*Strains in Simple Shear.*—A square face, ABCD (Fig. 8), of a piece of material under simple shear stress, as in Art. 8, will suffer a strain

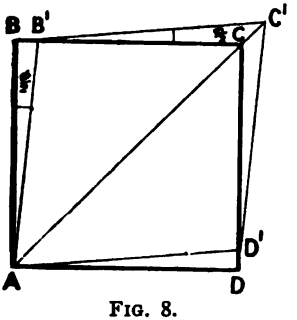



FIG. 8.

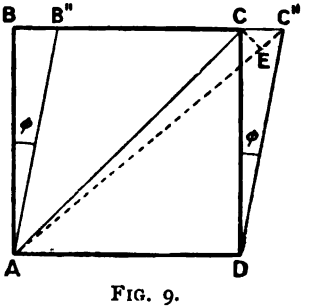



FIG. 9.

such as is indicated, by taking the new shape AB'C'D'. For expressing the strain it is slightly more convenient to consider the side AD, say, fixed, and the new shape accordingly, as in Fig. 9, AB"C"D. The strains being extremely small quantities, the straight line BB" practically coincides with an arc struck with centre A, and a line CE drawn perpendicular to AC" is substantially the same as an arc centred at A. The shear strain (Art. 4) $\phi$ radians is (Fig 9)

$$\frac{BB''}{AB} \text{ or } \frac{CC''}{CD}, \text{ and is equal to } \frac{q}{N} \text{ as above.}$$

The elongation of the diagonal AC is equal to EC″, and the linear strain is

$$\frac{EC''}{AC} = \frac{CC'' \times \frac{1}{\sqrt{2}}}{CD \times \sqrt{2}} = \tfrac{1}{2} \cdot \frac{CC''}{CD} = \tfrac{1}{2}\phi \text{ or } \tfrac{1}{2} \cdot \frac{q}{N}$$

That is, the strain in this direction is numerically half the amount of the shear strain. Similarly, the strain along the direction BD is $\frac{1}{2}\phi$, but dimensions in this direction are shortened. These are the strains corresponding to the direct stresses of intensities equal to $q$ produced across diagonal planes, as in Art. 8, by the shear stresses. Note that the strain along AC is *not* simply $\frac{p_n}{E}$, because in addition to the tensile stress $p_n$ there is a compressive stress of equal intensity at right angles to it.

**11. Bulk Modulus** is that corresponding to the volumetric strain resulting from three mutually perpendicular and equal direct stresses, such as the slight reduction in bulk a body suffers, for example, when immersed in a liquid under pressure: this modulus is generally denoted by the letter K.

If the intensities of the equal normal stresses are each $p$,

$$\frac{p}{K} = \text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}}$$

The volumetric strain is three times the accompanying linear strain, for if we consider a cube of side $a$ strained so that each side becomes

$$a \pm \delta a,$$

where $\delta a$ is very small, the linear strain is $\frac{\delta a}{a}$

The volumetric change is $(a \pm \delta a)^3 - a^3$, or $\pm 3a^2\delta a$ to the first order of small quantities. The strain then is

$$\frac{3a^2\delta a}{a^3} = 3 \cdot \frac{\delta a}{a}$$

which is three times the linear strain $\frac{\delta a}{a}$, or, in other words, the linear strain is one-third of the volumetric strain.

**12. Poisson's Ratio.**—Direct stress produces a strain in its own direction and an opposite kind of strain in every direction perpendicular to its own. Thus a tie-bar under tensile stress extends longitudinally and contracts laterally. Within the elastic limits the ratio

$$\frac{\text{lateral strain}}{\text{longitudinal strain}}$$

generally denoted by $\frac{1}{m}$, is a constant for a given material. The value of $m$ is usually from 3 to 4, the ratio $\frac{1}{m}$ being about $\frac{1}{4}$ for many metals. This ratio, which was formerly suggested as being for all materials $\frac{1}{4}$, is known as *Poisson's Ratio.*

**13. Relations between the Elastic Constants.**—Some relations between the above quantities E, N, K, and $m$ may be simply deduced. The strain of the diagonal of a square block of material in simple shear of intensity $q$ or $p$ was (Art. 10) found to be $\frac{1}{2}\frac{q}{N}$, which by Art. 8 may be replaced by $\frac{1}{2}\frac{p}{N}$, where $p$ is the intensity of the equal and opposite direct stresses across diagonal planes.

The resulting direct stress $p$ (Art. 8) in the direction of a diagonal would, if acting alone, cause a strain $\frac{p}{E}$ in the direction of that diagonal, and the opposite kind of direct stress in the direction of the diagonal perpendicular to the first would, acting alone, cause a similar kind of strain to the above one, amounting to $\frac{1}{m} \cdot \frac{p}{E}$ in the direction of the first-mentioned diagonal.

Hence, the total strain of the diagonal is

$$\frac{1}{2}\frac{p}{N} = \frac{p}{E}\left(1 + \frac{1}{m}\right)$$

from which

$$\frac{1}{2N} = \frac{1}{E}\left(1 + \frac{1}{m}\right)$$

or

$$E = 2N\left(1 + \frac{1}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Note that if $m = 4$, $\frac{N}{E} = \frac{5}{2}$.

Again, consider a cube of material under a direct normal stress $p$, say compressive, in each of the three perpendicular directions parallel to its edges (Fig. 10). Each edge is shortened by the action of the forces parallel to that edge, and the amount of such strain is

$$\frac{p}{E}$$

Again each edge is lengthened by the action of the two pairs of forces perpendicular to that edge and the amount of such strain is

$$2 \times \frac{1}{m} \cdot \frac{p}{E}$$

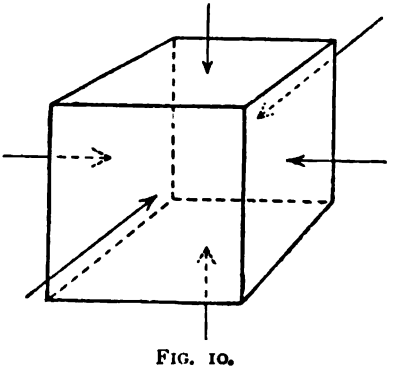


FIG. 10.

The total linear strain of each edge is then

$$\frac{p}{E}\left(1 - \frac{2}{m}\right)$$

and the volumetric strain is therefore

$$3 \cdot \frac{p}{E}\left(1 - \frac{2}{m}\right) \text{ (Art. 11)}$$

which is also by definition

$$\frac{p}{K}$$

where K is the bulk modulus.

Therefore $\frac{p}{K} = 3\frac{p}{E}\left(1 - \frac{2}{m}\right)$ or $\frac{1}{K} = \frac{3}{E}\left(1 - \frac{2}{m}\right)$

$$E = 3K\left(1 - \frac{2}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence from (1) and (2)

$$E = 2N\left(1 + \frac{1}{m}\right) = 3K\left(1 - \frac{2}{m}\right)$$

Eliminating E, this gives

$$\frac{1}{m} = \frac{3K - 2N}{6K + 2N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

also, eliminating $m$,

$$E = \frac{9KN}{N + 3K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

**14. Compound Stresses.**—When a body is under the action of several forces which cause wholly normal or wholly tangential stresses across different planes in known directions, we may find the state of stress across other planes by adding algebraically the various tangential components and the components normal to such planes, and combining the sums according to the rules of statics.

*Principal Planes.*—Planes through a point within a material such that the resultant stress across them is wholly a normal stress are called *Principal Planes*, and the normal stresses across them are called the *Principal Stresses* at that point: the direction of the principal stresses are called the axes of stress.

However complex the state of stress at a point within a body, there always exist three mutually perpendicular principal planes, and stresses at that point may be resolved wholly into the three corresponding normal stresses: further, the stress intensity across one of these principal planes is, at the point, greater than in any other direction, and another of the principal stresses is less than the stress in any other direction.

In many practical cases there is a plane perpendicular to which there is practically no stress, or in other words, one of the principal stresses is zero or negligibly small; in these cases resolution and compounding of stresses becomes a two-dimensional problem as in coplanar statics. We now proceed to investigate a few simple cases.

**15. Two Perpendicular Normal Stresses.**—If there be known normal stresses across two mutually perpendicular planes and no stress across the plane perpendicular to both of them, it is required to find the stress across any oblique interface perpendicular to that plane across

which there is no stress. Let $p_x$ and $p_y$ be the given stress intensities normal to the mutually perpendicular planes, say in directions OX and OY. If $p_x$ and $p_y$ vary along the directions OX and OY, we might consider the equilibrium of an indefinitely small element of material. If not, however, we may take a piece such as EGFH (Fig. 11), of unit thickness perpendicular to the figure. Our problem is to find the magnitude and direction of the resultant stress on a plane face EF, inclined $\theta$ to all planes which are perpendicular to the axis OX, or the normal ON of which is inclined $\theta$ to OX, $\left(\frac{\pi}{2} - \theta\right)$ to OY and in the plane of the figure, perpendicular to which the stress is *nil*. The stresses $p_x$ and $p_y$ are here shown alike, but for unlike stresses the problem is not seriously altered.

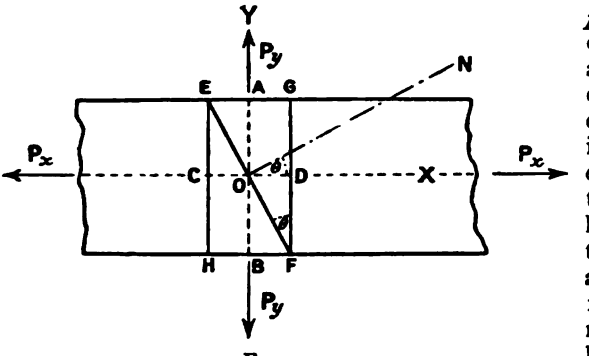



FIG. 11.

The whole normal force on the face FG is $P_x = p_x \times FG$, the area being FG × unity.

The wholly normal force on EG is $P_y = p_y \times EG$.

Let $p_n$ and $p_t$ be the normal and tangential stress intensities respectively on the face EF reckoned positive in the directions ON and OF. Then considering the equilibrium of the wedge EGF, resolving forces in the direction ON,

$$p_n \times EF = P_x \cos\theta + P_y \cos\left(\frac{\pi}{2} - \theta\right)$$
$$= p_x . FG . \cos\theta + p_y . EG . \sin\theta$$

dividing by EF

$$p_n = p_x \cos^2\theta + p_y \sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Resolving in direction OF

$$p_t \times EF = P_x \sin\theta - P_y \cos\theta$$
$$= p_x . FG . \sin\theta - p_y . EG . \cos\theta$$

dividing by EF

$$p_t = (p_x - p_y) \sin\theta \cos\theta = \frac{p_x - p_y}{2} \sin 2\theta \quad . \quad . \quad (2)$$

If $\theta = 45°$, the shear stress intensity

$$p_t = \frac{p_x - p_y}{2}$$

and is a maximum.

Across the same plane the direct (tensile) stress intensity is

$$p_n = p_x \cos^2 45° + p_y \sin^2 45° = \frac{p_x + p_y}{2}$$

Combining (1) and (2), if $p$ is the intensity of the resultant stress, since the two forces $P_x$ and $P_y$ are equal to the rectangular components of the force $p \times EF$,

$$\begin{aligned} p \cdot EF &= \sqrt{P_x^2 + P_y^2} \\ &= \sqrt{(p_x \cdot FG)^2 + (p_y \cdot EG)^2} \\ &= EF \sqrt{p_x^2 \cos^2\theta + p_y^2 \sin^2\theta} \\ p &= \sqrt{p_x^2 \cos^2\theta + p_y^2 \sin^2\theta} = \sqrt{p_n^2 + p_t^2} \quad . \quad . \quad (3) \end{aligned}$$

and since the component forces in directions OX and OY on unit area of the plane EF are $p_x \cos\theta$ and $p_y \sin\theta$, $p$ evidently makes an angle $\alpha$ with OX such that

$$\tan\alpha = \frac{p_y \sin\theta}{p_x \cos\theta} = \frac{p_y}{p_x} \cdot \tan\theta \quad . \quad . \quad . \quad . \quad . \quad (4)$$

And $p$ makes an angle $\beta$ with the plane EF, across which it acts, such that

$$\tan\beta = \frac{p_n}{p_t} \text{ or } \frac{p_x \cos^2\theta + p_y \sin^2\theta}{(p_x - p_y)\cos\theta \sin\theta} = \cot\phi \quad . \quad . \quad (5)$$

where $\phi$ is the angle which the resultant stress makes with the normal to the plane EF.

EXAMPLE.—Find the plane across which the resultant stress is most inclined to the normal.

Let $\phi$ be the maximum inclination to the normal. Then

$$\tan\phi = \frac{p_t}{p} = \frac{(p_x - p_y)\cos\theta \sin\theta}{p_x \cos^2\theta + p_y \sin^2\theta} \quad . \quad . \quad . \quad . \quad (6)$$

When $\phi$ is a maximum, $\tan\phi$ is a maximum, and

$$\frac{d(\tan\phi)}{d\theta} = 0$$

Therefore, differentiating and dividing out common factors,

$$(p_x \cos^2\theta + p_y \sin^2\theta) \cos 2\theta + (p_x - p_y) \sin\theta \cos\theta \times \sin 2\theta = 0$$

$$p_n \cos 2\theta + p_t \sin 2\theta = 0$$

$$\tan 2\theta = -\frac{p_n}{p_t} = -\cot\phi = \tan\left(\frac{\pi}{2} + \phi\right)$$

$$2\theta = \frac{\pi}{2} + \phi$$

$$\theta = \frac{\pi}{4} + \frac{\phi}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting this value of $\theta$ in equation (6) we get

$$\tan\phi = \frac{(p_x - p_y)\cos\phi}{p_x(1 - \sin\phi) + p_y(1 + \sin\phi)}$$

hence $$\frac{p_y}{p_x} = \frac{1 - \sin\phi}{1 + \sin\phi} \quad \ldots\ldots\ldots (8)$$

or $$\sin\phi = \frac{p_x - p_y}{p_x + p_y} \quad \ldots\ldots\ldots (9)$$

Equation (9) gives the maximum inclination to the normal, and equation (7) gives the inclination of the normal to the axis of the direct stress $p_x$.

*Unlike Stresses.*—If the two given stresses $p_x$ and $p_y$ are unlike, say $p_x$ tensile and $p_y$ compressive, we have the slight modifications

$$p_n = p_x \cos^2\theta - p_y \sin^2\theta \text{ (tensile)}$$
$$p_t = (p_x + p_y) \sin\theta \cos\theta = \tfrac{1}{2}(p_x + p_y) \sin 2\theta$$

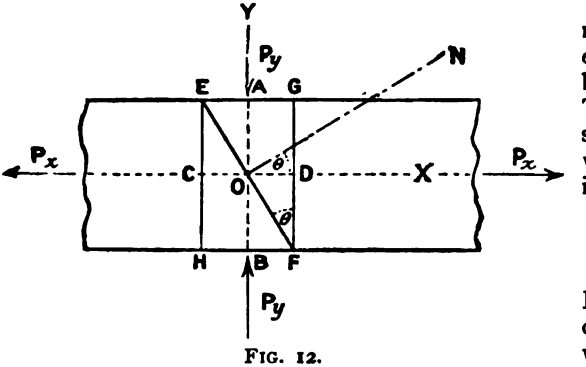



FIG. 12.

These results might be obtained just as before, but using Fig. 12. The maximum shear is again when $\theta = 45°$, and its value is

$$\frac{p_x + p_y}{2}$$

In the special case of unlike stresses, where $p_x$ and $p_y$ are numerically equal, the values for $\theta = 45°$ are

$$p_t = \frac{p_x + p_y}{2} = p_x = p_y$$
$$p_n = 0$$

These correspond exactly with the case of pure shear in Art. 8.

**16. Ellipse of Stress.**—In the last article we supposed two principal stresses $p_x$ and $p_y$ given, and the third to be zero, *i.e.* no stress perpendicular to Figs. 11 and 12. In this case, using the same notation and like stresses, the direction and magnitude of the resultant stress on any plane can easily be found graphically by the following means.

Describe, with O as centre (Fig. 13), two circles, CQD and ARB, their radii being proportional to $p_x$ and $p_y$ respectively. Draw OQ normal to the interface EF (Art. 15) to meet the larger circle in Q and the smaller in R. Draw QN perpendicular to OX and RP perpendicular to OY to meet QN in P. Then OP represents the resultant stress $p$ both in magnitude of intensity and in direction. The locus of P for various values of $\theta$, *i.e.* for different oblique interfaces, is evidently an ellipse, for the co-ordinate ON along OX is

$$OQ \cos\theta \text{ or } p_x \cos\theta$$

and PN, the co-ordinate along OY, is

$$\text{OR} \sin\theta \text{ or } p_y \sin\theta$$

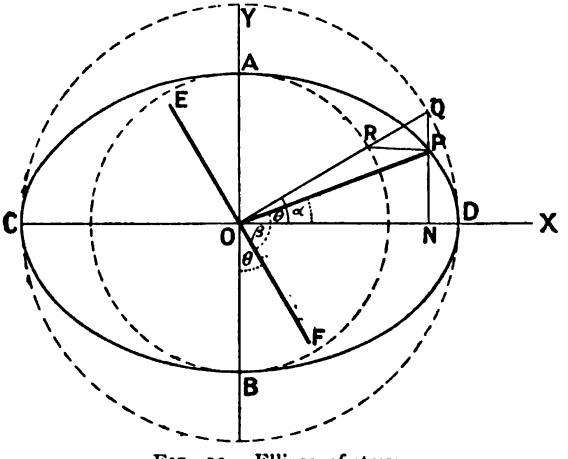



FIG. 13 —Ellipse of stress.

The axes of the ellipse are the axes of stress (Art. 14).

Also that $$\tan\alpha = \frac{p_y \sin\theta}{p_x \cos\theta} = \frac{p_y}{p_x}\tan\theta$$

is obvious from the figure.

In the second case where, say, $p_y$ is negative and $p_x$ is positive, OP

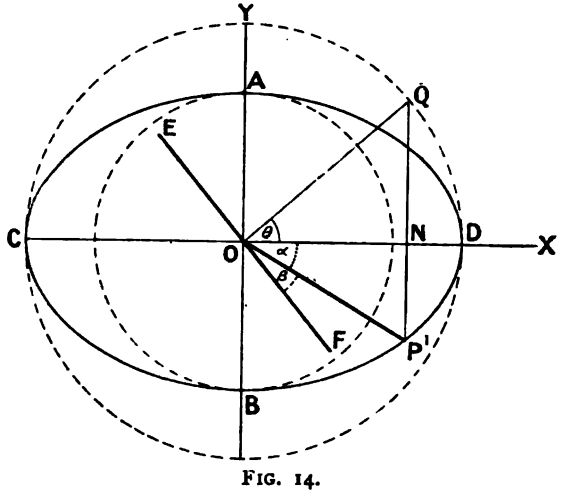



FIG. 14.

(Fig. 14) will represent the stress in magnitude and direction: here $\tan\alpha$ is negative and $\beta$ is obviously less than $\beta$ in Fig. 13.

EXAMPLE.—A piece of material is subjected to tensile stresses of 6 tons per square inch, and 3 tons per square inch, at right angles to each other. Find fully the stresses on a plane, the normal of which makes an angle of 30° with the 6-ton stress.

The intensity of normal stress on such a plane is

$$p_n = 6\cos^2 30° + 3\sin^2 30°$$
$$= 6 \times \tfrac{3}{4} + 3 \times \tfrac{1}{4} = 4\tfrac{1}{2} + \tfrac{3}{4} = 5\tfrac{1}{4} \text{ tons per square inch}$$

And the intensity of tangential stress is

$$p_t = 6\sin 30° \cos 30° - 3\sin 30° \cos 30°$$
$$= 3 \times \tfrac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4} = 1{\cdot}299 \text{ tons per square inch}$$

The resultant stress then has an intensity,

$$p = \sqrt{\left(\frac{21}{4}\right)^2 + \left(\frac{3\sqrt{3}}{4}\right)^2} = \tfrac{1}{4}\sqrt{441+27} = \frac{10{\cdot}82}{2} = 5{\cdot}41 \text{ tons per sq. in.}$$

and makes an angle $\alpha$ with the direction of the 6-ton stress, such that

$$\tan\alpha = \frac{3\sin 30°}{6\cos 30°} = \tfrac{1}{2}\tan 30° = 0{\cdot}288$$

which is the tangent of 16° 4′.

This is the angle which the resultant stress makes with the 6-ton stress. It makes, with the normal to the plane across which it acts, an angle

$$30° - 16°\,4' = 13°\,56'$$

To check this, the cotangent of the angle the resultant stress makes with the normal, or the tangent of that it makes with the plane, is

$$\frac{p_n}{p_t} = \frac{5{\cdot}25}{1{\cdot}299} = 4{\cdot}035$$

which is tangent of 76° 4′, and therefore the cotangent of 13° 56′.

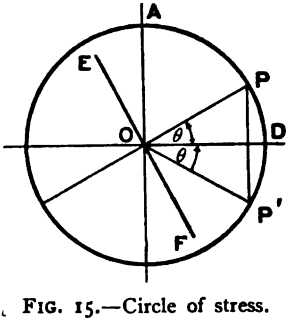


FIG. 15.—Circle of stress.

**17. Circle of Stress.**—In the particular cases when the principal stress intensities $p_x$ and $p_y$ are of equal magnitude the ellipse of stress evidently becomes a circle (see Fig. 15). And if the principal stresses are of the same sign the resultant stress on any and every oblique plane EF perpendicular to the figure, is normal to that plane. If the stress $p_y$ is of opposite sign to $p_x$ the resultant stress is in the direction OP′.

Cases of unequal principal stress may be treated by the circle of stress by writing

$$p_x = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \quad \ldots\ldots (1)$$

$$p_y = \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2} \quad \ldots\ldots (2)$$

Every unit area of the face EF is then subject to equal like normal stresses $\frac{p_x + p_y}{2}$ and to equal and opposite normal stresses $\frac{p_x - p_y}{2}$. The resultant of the two like stresses $\frac{p_x + p_y}{2}$ (Fig. 16) is a

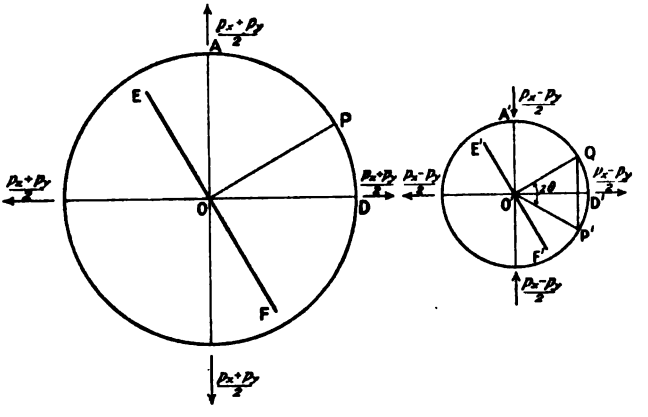



FIG. 16.—Circles of stress.

normal stress of magnitude $\frac{p_x + p_y}{2}$ shown at OP. The resultant of the two unlike stresses $\frac{p_x - p_y}{2}$, is of magnitude $\frac{p_x - p_y}{2}$ inclined $2\theta$ to the normal to EF, as shown at O'P'. These two stresses represented by OP and O'P' may then be geometrically added as shown in Fig. 17, where the vector $ab$ of length $\frac{p_x + p_y}{2}$ represents OP (Fig. 16), and $bc$ of length $\frac{p_x - p_y}{2}$ represents O'P'. The resultant is $ac$, and all the results of Art. 15 may easily be deduced from the trigonometrical solution of the triangle $abc$, *e.g.*

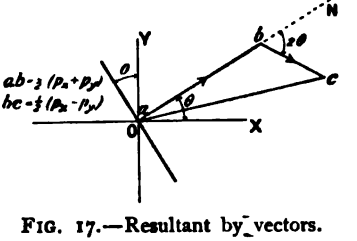



FIG. 17.—Resultant by vectors.

$$p^2 \text{ or } ac^2 = ab^2 + bc^2 - 2ab \,.\, bc \cos 2\theta$$

$$p^2 = \tfrac{1}{4}(p_x + p_y)^2 + \tfrac{1}{4}(p_x - p_y)^2 + \tfrac{1}{2}(p_x + p_y)(p_x - p_y)\cos 2\theta$$

$$p^2 = \tfrac{1}{2}p_x^2(1 + \cos 2\theta) + \tfrac{1}{2}p_y^2(1 - \cos 2\theta)$$

$$p^2 = p_x^2 \cos^2\theta + p_y \sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in agreement with (3) Art. 15.

The solution of the example given in Art. 15 follows particularly easily by this method, for in Fig. 17 the angle $\hat{abc}$ (or $\phi$)

is to be a maximum. If the fixed length $ab$ be set off in any direction (Fig. 18) to represent $\frac{p_x + p_y}{2}$ and a circle of radius representing $\frac{p_x - p_y}{2}$ be described about $b$ as centre, the side $ac$ which meets the circle is evidently most inclined to $ab$ when $ac$ is tangent to the circle, *i.e.*, when $ac$ and $bc$ are at right angles. Then from the right-angled triangle $abc$ it follows clearly that

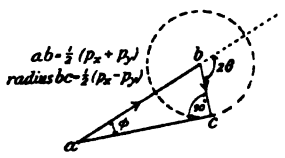



FIG. 18.—Most oblique resultant.

$$2\theta = \phi + \frac{\pi}{2} \text{ as in (7) Art 15} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Also that $$\sin \phi = \frac{bc}{ab} = \frac{p_x - p_y}{p_x + p_y} \text{ as in (9) Art 15} \quad . \quad . \quad . \quad (5)$$

and hence $$\frac{p_y}{p_x} = \frac{1 - \sin \phi}{1 + \sin \phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

a result used in the theory of earth pressure.

**18. Principal Stresses.**—When bodies are subjected to known stresses in certain directions, and these are not all wholly normal stresses, the stresses on various planes may be found by the methods of the two previous articles, provided we first find the principal planes and principal stresses (see Art. 14). It is also often important in itself, in such cases, to find the principal stresses, as one of these is, as previously stated, the greatest stress to which the material is subjected. We proceed to find principal stresses and planes in a few simple, two-dimensional cases where the stress perpendicular to the figure is *nil*.

As a very simple example, we have found in Art. 8 that the two shear stresses of equal intensity, on two mutually perpendicular planes, with no stress on planes perpendicular to the other two, give principal stresses of intensity equal to that of the shear stresses, on planes inclined $\frac{\pi}{4}$ to the two perpendicular planes to which the pure shear stresses are tangential.

As a second example, let there be, on mutually perpendicular planes, normal stresses, one of intensity $p_1$ and the other of intensity $p_2$, in addition to the two equal shear stresses of intensity $q$, as in Fig. 19, which represents a rectangular block of the material unit thickness perpendicular to the plane of the figure, across all planes parallel to which there is no stress; we may imagine the block so small that the variation of stress intensity over any plane section is negligible. The stresses $p_1$, $p_2$, and $q$ may be looked upon as independent known stresses arising from several different kinds of external straining actions, or as rectangular components, normal and tangential (Art. 7), into which oblique stresses, on the faces perpendicular to the figure, have been resolved.

It is required to know the direction of the principal planes and the

intensity of the (normal) principal stresses upon them. Fig. 19 represents the given normal stresses as tensions: the work is practically the same in the case of compressive stresses, or if one stress be compressive and the other tensile.

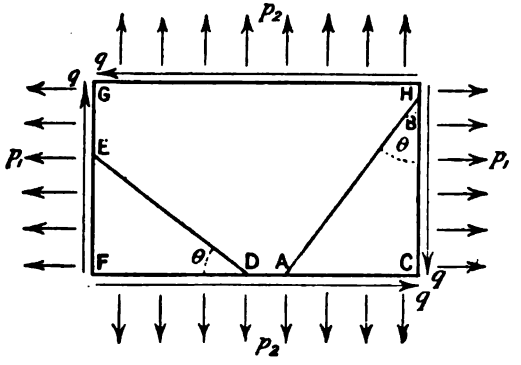



FIG. 19.

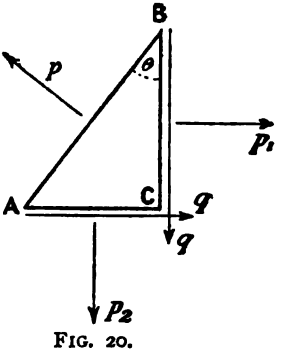



FIG. 20.

Let $\theta$ be the inclination of one principal plane to the face BC. Then an interface, AB, is a principal plane, and the stress $p$ upon it is wholly normal to AB. Consider the equilibrium of a wedge, ABC (Figs. 19 and 20), cut off by such a plane.

Resolving forces parallel to AC

$$p \,.\, \mathrm{AB} \times \cos\theta = p_1 \,.\, \mathrm{BC} + q \,.\, \mathrm{AC}$$
$$= p_1 \,.\, \mathrm{AB} \cos\theta + q \,.\, \mathrm{AB} \sin\theta$$

hence
$$(p - p_1)\cos\theta = q \sin\theta$$
$$p - p_1 = q \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Resolving parallel to BC

$$p \,.\, \mathrm{AB} \times \sin\theta = p_2 \,.\, \mathrm{AC} + q \,.\, \mathrm{BC}$$
$$= p_2 \,.\, \mathrm{AB} \sin\theta + q\,\mathrm{AB} \cos\theta$$
$$(p - p_2)\sin\theta = q \cos\theta$$
$$p - p_2 = q \cot\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting equation (1) from equation (2)

$$p_1 - p_2 = q(\cot\theta - \tan\theta) = \frac{2q}{\tan 2\theta}$$
$$\tan 2\theta = \frac{2q}{p_1 - p_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From which two values of $\theta$ differing by a right angle may be found, *i.e.* the inclinations to BC of two principal planes which are mutually perpendicular.

Further, multiplying (1) by (2)

$$(p - p_1)(p - p_2) = q^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$p^2 - p(p_1 + p_2) - (q^2 - p_1 p_2) = 0$$
$$p = \tfrac{1}{2}(p_1 + p_2) \pm \sqrt{\tfrac{1}{4}(p_1 + p_2)^2 + (q^2 - p_1 p_2)} \quad (5)$$

or,
$$\tfrac{1}{2}(p_1 + p_2) \pm \sqrt{\tfrac{1}{4}(p_1 - p_2)^2 + q^2}$$

These two values of $p$ are the values of the (normal) stress intensities on the two principal planes. The larger value (where the upper sign is taken) will be the stress intensity on such a plane as AB (Figs. 19 and 20), and will be of the same sign as $p_1$ and $p_2$; the smaller value, say $p'$, will be that on such a plane as ED (Figs. 19 and 21) perpendicular to AB, and will be of opposite sign to $p_1$ and $p_2$ if $q^2$ is greater than $p_1 p_2$.

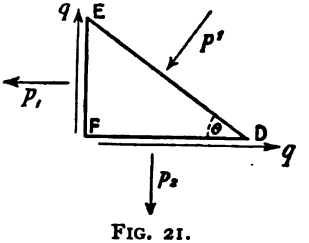


FIG. 21.

The planes on which there are maximum shear stresses are inclined 45° to the principal planes found, and the maximum intensity of shear stress is (Art. 15)

$$\frac{p - p'}{2} = \sqrt{\tfrac{1}{4}(p_1 + p_2)^2 + q^2 - p_1 p_2} = \sqrt{\tfrac{1}{4}(p_1 - p_2)^2 + q^2}$$

The modifications necessary in (3) and (4), if $p_1$ or $p_2$ is of negative sign, are obvious. If, say, $p^2$ is zero, the results from substituting this value in (3) and (4) are simple. This special case is of sufficient importance to be worth setting out briefly by itself in the next article instead of deducing it from the more general case.

**19. Principal Planes and Stresses when complementary shear stresses are accompanied by a normal stress on the plane of one shear stress.**—Fig. 22 shows the forces on a rectanglar block, GHCF, of unit thickness perpendicular to the figure, and of indefinitely small dimensions parallel to the figure, unless the stresses are uniform. Let $\theta$ be the inclination of a principal plane AB to the plane BC, which has normal stress of intensity $p_1$ and a shear stress of intensity $q$ acting on it, and let $p$ be the intensity of the wholly normal stress on AB.

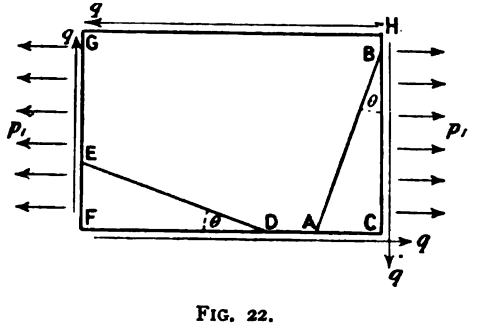


FIG. 22.

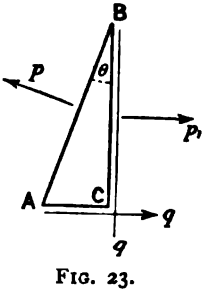


FIG. 23.

The face FC has only the shear stress of intensity $q$ acting tangentially to it.

Consider the equilibrium of the wedge ABC; resolving the forces parallel to AC (Figs. 22 and 23)

$$p \cdot AB \times \cos\theta = p_1 \cdot BC + q \cdot AC$$
$$= p_1 AB \cos\theta + q \cdot AB \sin\theta$$
$$(p - p_1)\cos\theta = q \sin\theta$$
$$(p - p_1) = q \tan\theta \quad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving parallel to BC

$$p \cdot AB \cdot \sin\theta = q \cdot BC = q \cdot AB \cos\theta$$

$$\tan\theta = \frac{q}{p} \quad \ldots\ldots\ldots\ldots \quad (2)$$

Substituting for $\tan\theta$ in (1)

$$(p - p_1) = \frac{q^2}{p}$$
$$p^2 - p_1 p - q^2 = 0$$
$$p = \tfrac{1}{2}p_1 \pm \sqrt{\tfrac{1}{4}p_1^2 + q^2} \quad \ldots\ldots\ldots \quad (3)$$

and the values of $\theta$ may be found by substituting these values of $p$ in (2). The two values differ by a right angle, the principal planes being at right angles. AB (Fig. 23) shows a principal plane of greatest stress corresponding to

$$p = \tfrac{1}{2}p_1 + \sqrt{\tfrac{1}{4}p_1^2 + q^2}$$

and ED (Fig. 24) shows the other principal plane on which the normal stress is

$$p' = \tfrac{1}{2}p_1 - \sqrt{\tfrac{1}{4}p_1^2 + q^2}$$

of opposite sign to $p_1$.

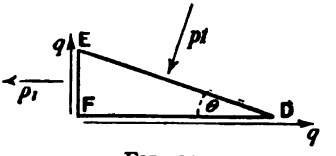


FIG. 24.

The planes of greatest shear stress are (Art. 15) those inclined 45° to the principal planes, and the intensity of shear stress upon them is

$$\frac{p - p'}{2} = \sqrt{\tfrac{1}{4}p_1^2 + q^2}. \quad \ldots\ldots\ldots \quad (4)$$

EXAMPLE.—At a point in material under stress the intensity of the resultant stress on a certain plane is 4 tons per square inch (tensile) inclined 30° to the normal of that plane. The stress on a plane at right angles to this has a normal tensile component of intensity $2\frac{1}{2}$ tons per square inch. Find fully (1) the resultant stress on the second plane, (2) the principal planes and stresses.

(1) On the first plane the tangential stress is

$$q = 4 \sin 30° = 2 \text{ tons per square inch}$$

Hence on the second plane the tangential stress is 2 tons per square inch (Art. 8). And the resultant stress is

$$p = \sqrt{2{\cdot}5^2 + 2^2} = \tfrac{1}{2}\sqrt{41} = 3{\cdot}2 \text{ tons per square inch}$$

(2) The intensity of stress normal to the first plane is

$$4 \cos 30° = 3{\cdot}464 \text{ tons per square inch}$$

Hence the principal stresses are (Art. 18 (5))

$$p = \frac{3{\cdot}464 + 2{\cdot}5}{2} \pm \sqrt{\tfrac{1}{4}(3{\cdot}462 - 2{\cdot}5)^2 + 2^2}$$
$$= 2{\cdot}982 \pm \sqrt{0{\cdot}23 + 4}$$
$$= 2{\cdot}982 \pm 2{\cdot}06$$
$$= 5{\cdot}042 \text{ tons per square inch tension and } 0{\cdot}922 \text{ ton per square inch tension.}$$

If $\theta$ be the angle made by a principal plane with the first-mentioned plane, by Art. 18 (3),

$$\tan 2\theta = \frac{2 \times 2}{3{\cdot}464 - 2{\cdot}5} = \frac{4}{0{\cdot}964} = 4{\cdot}149$$
$$2\theta = 76^\circ\ 27'$$
$$\theta = 38^\circ\ 13{\cdot}5'$$

The principal planes and stresses are then one plane inclined $38^\circ\ 13{\cdot}5'$ to the first given plane, and having a tensile stress $5{\cdot}042$ tons per square inch across it, and a second at right angles to the other or inclined $51^\circ\ 46{\cdot}5'$ to the first given plane, and having a tensile stress $0{\cdot}922$ ton per square inch across it. The planes are shown in Fig. 25.

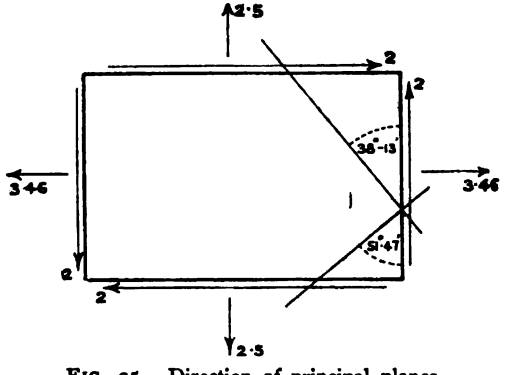



FIG. 25.—Direction of principal planes.

**20. Principal Strains.**—In a bar of material within limits of perfect elasticity a (say tensile) stress intensity $p_1$ alone will produce a strain $e_1$, in its own direction such that

$$e_1 = \frac{p_1}{E}$$

where E is Young's modulus of elasticity or the stretch modulus, provided there is freedom of lateral contraction. The contraction in all directions at right angles to the axis of the stress $p_1$ will be represented by a strain

$$\frac{p_1}{mE}$$

where $\frac{1}{m}$ is Poisson's ratio.

In an isotropic material, *i.e.* one having the same elastic properties in all directions, the effect of a stress $p_2$ acting alone at right angles to the direction of $p_1$ would be to produce a strain in its own direction, $e_2$, such that

$$e_2 = \frac{p_2}{E}$$

and at right angles to this, including the direction of the strain $\frac{p_1}{E}$, a contraction strain

$$\frac{p_2}{mE}$$

Similarly a stress $p_3$, the direction of which is perpendicular to both the previously mentioned stresses, will produce in addition to its longitudinal strain a contraction strain

$$\frac{p_3}{mE}$$

in all directions perpendicular to its direction, including the direction of the stress $p_1$.

If we have at a point in isotropic material three principal stresses of intensities $p_1$, $p_2$, and $p_3$, each will independently produce the same strains which it would cause if acting alone. Taking all the stresses of the same sign the total strain produced in the direction of the stress $p_1$ will then be

$$e_1 = \frac{p_1}{E} - \frac{p_2 + p_3}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the direction of $p_2$ the strain

$$e_2 = \frac{p_2}{E} - \frac{p_1 + p_3}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and in the direction of $p_3$ the strain

$$e_3 = \frac{p_3}{E} - \frac{p_1 + p_2}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If any one of the above stresses is of opposite kind, *i.e.* compressive in this case, the strains will be found by changing the sign of that stress in each of the above equations.

EXAMPLE.—The intensities of the three principal stresses in a boiler-plate are at a certain point 4 tons per square inch tensile in one direction, 3 tons per square inch tensile in a second, and zero in a third. Find what stress acting alone would produce the same strain in the direction of the 4-ton stress, given the ratio of Young's modulus to the modulus of rigidity is $\frac{5}{2}$.

By Art. 13 (1)

$$\frac{1}{m} = \frac{E}{2N} - 1$$
$$= \tfrac{5}{4} - 1 = \tfrac{1}{4}$$

Hence, in the direction of the 4-ton stress,

$$\text{Strain} = \frac{4}{E} - \tfrac{1}{4}\frac{3}{E} = \tfrac{13}{4} \times \frac{1}{E}$$

If $p$ is the stress intensity to produce this strain when acting alone

$$\frac{p}{E} = \frac{13}{4} \cdot \frac{1}{E}$$

or,
$$p = \frac{13}{4} = 3\frac{1}{4} \text{ tons per square inch}$$

## EXAMPLES I.

1. A round tie-bar of mild steel, 18 feet long and 1¼ inch diameter, lengthens $\frac{1}{16}$ inch under a pull of 7 tons. Find the intensity of tensile stress in the bar, the value of the stretch modulus, and the greatest intensity of shear stress on any oblique section.

2. A rod of steel is subjected to a tension of 3 tons per square inch of cross-section. The shear stress across a plane oblique to the axis is 1 ton per square inch. What is the inclination of the normal of this plane to the axis? What is the intensity of the normal stress across the plane, and what is the intensity of the resultant stress across it? Of the two possible solutions, take the plane with normal least inclined to the axis of the rod.

3. On a plane oblique to the axis of the bar in question 1, the intensity of shear stress is 1·5 ton per square inch. What is the intensity of normal stress across this plane? Also what is the intensity of resultant stress across it? Take the plane most inclined to the axis.

4. A hollow cylindrical cast-iron column is 10 inches external and 8 inches internal diameter and 10 feet long. How much will it shorten under a load of 60 tons? Take E as 8000 tons per square inch.

5. The stretch modulus of elasticity for a specimen of steel is found to be 28,500,000 lbs. per square inch, and the transverse modulus is 11,000,000 lbs. per square inch. What is the modulus of elasticity of bulk for this material, and how many times greater is the longitudinal strain caused by a pull than the accompanying lateral strain?

6. The tensile (principal) stresses at a point within a boiler-plate across the three principal planes are 0, 2, and 4 tons per square inch. Find the component normal and tangential stress intensities, and the intensity and direction of the resultant stress, at this point, across a plane perpendicular to the first principal plane, and inclined 30° to the plane having a 4-ton principal stress.

7. With the same data as question 6, find the inclination of the normal, to the axis of the 4-ton stress, of a plane on which the resultant stress is inclined 15° to the normal. What is the intensity of this resultant stress?

8. At a point in strained material the principal stresses are 0, 5 tons per square inch tensile, and 3 tons per square inch compressive. Find the resultant stress in intensity and direction on a plane inclined 60° to the axis of the 5-ton stress, and perpendicular to the plane which has no stress. What is the maximum intensity of sheer stress in the material?

9. If a material is so strained that at a certain point the intensities of normal stress across two planes at right angles are 5 tons and 3 tons per square inch, both tensile, and if the shear stress across these planes is 4 tons per square inch, find the maximum direct stress and the plane to which it is normal.

10. Solve question 9 if the stress of 3 tons per square inch is compressive.

11. At a point in a cross-section of a girder there is a tensile stress of 4 tons per square inch normal to the cross-section; there is also a shear

stress of 2 tons per square inch on that section. Find the principal planes and stresses.

12. In a shaft there is at a certain point a shear stress of 3 tons per square inch in the plane of a cross-section, and a tensile stress of 2 tons per square inch normal to this plane. Find the greatest intensities of direct stress and of shear stress.

13. In a boiler-plate the tensile stress in the direction of the axis of the shell is $2\frac{1}{2}$ tons per square inch, and perpendicular to a plane through the axis the tensile stress is 5 tons per square inch. Find what intensity of tensile stress acting alone would produce the same maximum tensile strain if Poisson's ratio is $\frac{1}{4}$.

14. A cylindrical piece of metal undergoes compression in the direction of its axis. A well-fitted metal casing, extending almost the whole length, reduces the lateral expansion by half the amount it would otherwise be. Find in terms of "$m$" the ratio of the axial strain to that in a cylinder quite free to expand in diameter. $\left(\text{Poisson's ratio} = \frac{1}{m}.\right)$

15. Three long parallel wires, equal in length and in the same vertical plane, jointly support a load of 3000 lbs. The middle wire is steel, and the two outer ones are brass, and each is $\frac{1}{4}$ square inch in section. After the wires have been so adjusted as to each carry $\frac{1}{3}$ of the load a further load of 7000 lbs. is added. Find the stress in each wire, and the fraction of the whole load carried by the steel wire. E for steel $30 \times 10^6$ lbs. per square inch, and for brass $12 \times 10^6$ lbs. per square inch.

# CHAPTER II

## *WORKING STRESSES*

**21. Elasticity.**—A material is said to be perfectly elastic if the whole of the strain produced by a stress disappears when the stress is removed. Within certain limits (Art. 5) many materials exhibit practically perfect elasticity.

*Plasticity.*—A material may be said to be *perfectly* plastic when no strain disappears when it is relieved from stress.

In a plastic state, a solid shows the phenomenon of "flow" under unequal stresses in different directions, much in the same way as a liquid. This property of "flowing" is utilized in the "squirting" of lead pipe, the drawing of wire, the stamping of coins, forging, etc.

*Ductility* is that property of a material which allows of its being drawn out by tension to a smaller section, as for example when a wire is made by drawing out metal through a hole. During ductile extension, a material generally shows a certain degree of elasticity, together with a considerable amount of plasticity. *Brittleness* is lack of ductility.

When a material can be beaten or rolled into plates, it is said to be malleable; malleability is a very similar property to ductility.

**22. Tensile Strain of Ductile Metals.**—If a ductile metal be subjected to a gradually increasing tension, it is found that the resulting strains, both longitudinal and lateral, increase at first proportionally to the stress. When the elastic limit is reached, the tensile strain begins to increase more quickly, and continues to grow at an increasing rate as the load is augmented. At a stress a little greater than the elastic limit some metals, notably soft irons and steels, show a marked breakdown, the elongation becoming many times greater than previously with little or no increase of stress. The stress at which this sudden stretch occurs is called the "yield point" of the material.

Fig. 26 is a "stress-strain" curve for a round steel bar 10 inches long and 1 inch diameter, of which the ordinates represent the stress intensities and the abscissæ the corresponding strains. The limit of elasticity occurs about A, the line OA being straight. The point B marks the "yield point," AB being slightly curved. After the yield-point stress is reached, the ductile extensions take place, the strains increasing at an accelerating rate with greater stresses as indicated by the portion of the curve between C and D. Strains produced at loads above the yield point do not develop in the same way as those below

the elastic limit. The greater part of the strain occurs very quickly, but this is followed without any further loading by a small additional extension which increases with time but at a diminishing rate. The phenomenon of the slow growth of a strain under a steady tensile stress has been called "creeping" by Prof. Ewing. On account of the part which takes time to develop, the total amount of strain produced by a given load and the shape of the stress-strain curve will be slightly modified by the rate of loading. At D, just before the greatest load is reached, the material is almost perfectly plastic, the tensile strain increasing greatly for very slight increase of load. It should be noted that in this diagram both stress intensity and strain are reckoned on the original dimensions of the material.

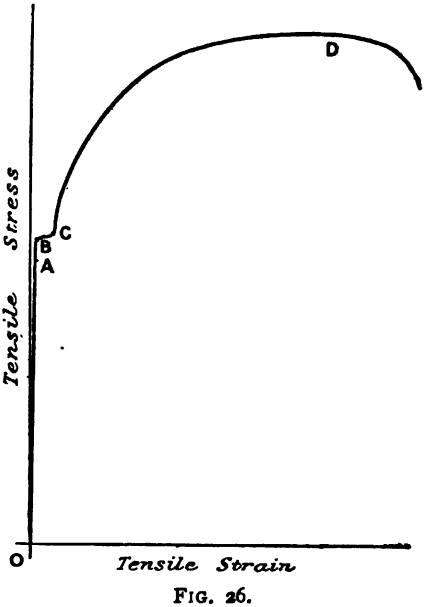


FIG. 26.

During the ductile elongation, the area of cross-section decreases in practically the same proportion that the length increases, or in other words, the volume of the material remains practically unchanged. The reduction in area of section is generally fairly uniform along the bar.

After the maximum load is reached, a sudden local stretching takes place, extending over a short length of the bar and forming a "waist." The local reduction in area is such that the load necessary to break the bar at the waist is considerably less than the maximum load on the bar before the local extension takes place. Nevertheless the breaking load divided by the reduced area of section shows that the "actual stress intensity" is greater than at any previous load. If the load be divided by the original area of cross-section, the result is the "nominal intensity of stress," which is less, in such a ductile material as soft steel, at the breaking load than at the maximum load sustained at the point D on Fig. 26.

**23. Elastic Limit and Yield Point.**—The elastic limit (Art. 5) in tension is the greatest stress after which no permanent elongation remains when all stress is removed. In nearly all metals, and particularly in soft and ductile ones, instruments of great precision will reveal slight permanent extensions resulting from very low stresses, and particularly in material which has never before been subjected to such tensile stress. In many metals, however, notably wrought iron and steel, if we neglect permanent extensions less than, say,

$\frac{1}{100000}$ of the length of a test-bar (*i.e.* strains less than 0·00001), stresses up to a considerable proportion of the maximum cause purely elastic and proportional elongation. The proportionality of the strain to the stress in Fig. 26 is indicated by OA being a straight line. For such metals as wrought iron and steel, the proportionality holds good up to the elastic limit—that is, the end of the straight line at A indicates the elastic limit, or in other words, Hooke's Law (Art. 5) is substantially true.

*Commercial Elastic Limit.*—In commercial tests of metals exhibiting a yield point, the stress at which this marked breakdown occurs is often called the elastic limit; it is generally a little above the true elastic limit.

There are, then, three noticeable limits of stress.

(1) The elastic limit, as defined in Art. 5.

(2) The limit of proportionality of stress to strain.

(3) The stress at yield point—the commercial elastic limit.

In wrought iron and steel the first two are practically the same, and the third is somewhat higher.

**24. Ultimate and Elastic Strength and Factor of Safety.**—The maximum load necessary to rupture a specimen in simple tension or shear, divided by the *original area* of section at the place of fracture, gives the nominal maximum stress necessary for fracture, and is called the ultimate strength of the material under that particular kind of stress. It is usually reckoned in pounds or tons per square inch. The ultimate strength in tension is also called the *Tenacity*. The greatest calculated stress to which a part of a machine or structure is ever subjected is called the working stress, and the ratio

$$\frac{\text{ultimate strength}}{\text{working stress}}$$

is called the Factor of Safety.

It is, of course, usual to ensure that the working stress shall be below the elastic limit of the material; but this is not sufficient, and designers, when allowing a given working stress, generally specify or assume, amongst other properties, an ultimate strength for the material, greater than the working stress in the ratio of a reasonable factor of safety. The factor of safety varies very greatly according to the nature of the stresses, whether constant, variable or alternating, simple or compound, it is frequently made to cover an allowance for straining actions, such as shocks, no reliable estimate of which can in some instances be made, diminution of section by corrosion, and other contingencies.

*Elastic Strength.*—If it is desired to keep working stresses by a certain margin within the limits of elasticity, it becomes important to know for other than simple direct stresses whether the breaking down occurs for a given value of the greatest principal stress (Art. 14) or for a given value of the greatest principal strain, which is influenced by the lateral strains produced by the other principal stresses (see Art. 20).

There are three theories as to when elastic failure takes place, viz.—

(1) For a certain value of the maximum principal stress.

(2) For a certain value of the maximum principal strain.

(3) For a certain value of the maximum shearing stress, this being proportional to the greatest difference between principal stress (see (2), Art. 15, and (4), Art. 19).

If the second theory is correct, the elastic strength of a piece of material in which the maximum principal stress is tensile, for example, will be lessened by lateral compression and increased by lateral tension. Accounts of some very interesting experiments bearing on this question have been published by Mr. J. J. Guest and others;[1] these experiments tend to confirm the third theory for ductile materials, and the first one for brittle materials, intermediate materials following some intermediate law. Mr. Scoble[2] has suggested that the condition

$$p - ap' = b$$

where $p$ and $p'$ are the greatest and least principal stresses and $a$ and $b$ are constants may be the law of yielding for all materials, the constants being different in different materials. Thus $a$ might be near zero in a brittle material and approximate to unity in ductile materials.

The different conclusions from the three theories may be well illustrated by the common case of one direct stress, $p_1$, with shear stress, $q$, on the same plane as in Art. 18.

The first theory gives a maximum principal stress

$$p = \tfrac{1}{2}p_1 + \sqrt{(\tfrac{1}{4}p_1^2 + q^2)} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The second theory gives a maximum principal strain (see Art. 19)

$$e_1 = \frac{p}{\mathrm{E}} - \frac{p'}{m\mathrm{E}}$$

$$= \frac{1}{\mathrm{E}}\left\{\tfrac{1}{2}p_1 + \sqrt{(\tfrac{1}{4}p_1^2 + q^2)} - \frac{1}{m}(\tfrac{1}{2}p_1 - \sqrt{(\tfrac{1}{4}p_1^2 + q^2)})\right\}$$

or,

$$\mathrm{E}e_1 = \tfrac{1}{2}p_1\left(1 - \frac{1}{m}\right) + \sqrt{(\tfrac{1}{4}p_1^2 + q^2)}\left(1 + \frac{1}{m}\right)$$

where $\frac{1}{m}$ is Poisson's ratio (Art. 12).

[1] "Strength of Ductile Materials under Combined Stress," *Phil. Mag.*, July, 1900, and *Proc. Physical Society*, May, 1900. Also see "Effects of Combined Stresses on Elastic Properties of Steel," by Prof. E. L. Hancock, *Phil. Mag.*, 1906, p. 276, vol. xi., and p. 418, vol. xii. Also "Strength of Ductile Materials under Combined Stress," by Mr. W. A. Scoble, *Phil. Mag.*, 1906, vol. xii. p. 533, and later, *Phil. Mag.*, January, 1910, p. 116, or *Proc. Physical Society*, vol. xxii. p. 130, and "Strength of Brittle Materials under Combined Stress," *Proc. Physical Society*, vol. xx. p. 453, and a second paper, *Phil. Mag.*, June, 1910. Also papers by Mr. W. Mason and by Mr. C. A. M. Smith in *Proc. Inst. Mech. Eng.* for December, 1909, and various articles and letters in *Engineering* during 1909 and 1910. Also "Strength of Thick Hollow Cylinders under Internal Pressure," by Messrs. Cook and Robertson in *Engineering*, Dec. 15, 1911.

[2] *Phil. Mag.*, January, 1910.

And if $m = 4$, and equivalent simple stress

$$E\epsilon_1 = \tfrac{3}{8}p_1 + \tfrac{5}{4}\sqrt{\tfrac{1}{4}p_1^2 + q^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The third theory gives a maximum *shear* stress (see Art. 19 (4)) of intensity

$$\frac{p - p'}{2} = \sqrt{\tfrac{1}{4}p_1^2 + q^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

**25. Importance of Ductility.**—In a machine or structure it is usual to provide such a section as shall prevent the stresses within the material from reaching the elastic limit. But the elastic limit can, in manufacture, by modification of composition or treatment be made high, and generally such treatment will reduce the ductility and cause greater brittleness or liability to fracture from vibration or shock. Ductile materials, on the other hand, are not brittle, and a lower elastic limit is usually found with greater ductility. Local ductile yielding in a complex structure will relieve a high local stress, due to imperfect workmanship or other causes, thereby preventing a member accidentally stressed beyond its elastic limit from reaching a much higher stress such as might be produced in a less plastic material. Thus in many applications the property of ductility is of equal importance to that of strength.

It is the practice of some engineers to specify that the steel used in a structure shall have an ultimate tensile strength between certain limits; the reason for fixing an upper limit is the possibility that greater tensile strength may be accompanied by a decrease in ductility or in power to resist damage by shock.

The usual criteria of the ductility of a metal are the percentages of elongation and contraction of sectional area in a test piece fractured by tension. Probably the percentage elongation is the better one; smaller elongation is sometimes accompanied by greater contraction of area.

**26. Percentage Elongation.**—It was noticed in Art. 22 that in fracturing a piece of mild steel by tension there was produced previous to the maximum load a fairly uniform elongation, and subsequently an

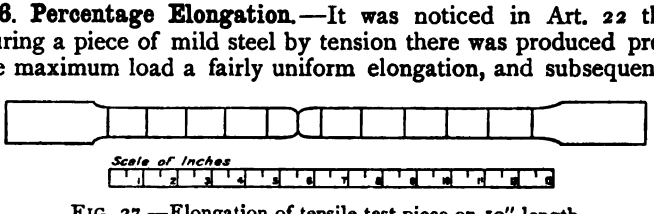



FIG. 27.—Elongation of tensile test piece on 10" length.

increased local elongation about the section of fracture (see Fig. 27). In such a case the extensions on each of 10 inches, marked out on a bar 1 inch diameter before straining, were as follow:—

| Inch . . . . | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Extension (inches) | 0·20 | 0·21 | 0·22 | 0·25 | 0·30 | 0·52 | 0·52 | 0·28 | 0·27 | 0·23 |

Fracture occurs near the division, 6 inches from one end of the marked length. Reckoning the percentage extension on the 2 inches nearest to the fracture, which include a large proportion of the local extension, the elongation is 1·04 inch, or 52 per cent. On any greater length the local extension will not affect so large a part of the length, and the percentage extension will accordingly be less. Thus, always including the fracture as centrally as possible, the elongations are

| Length (in inches) . . . | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| Elongation per cent. . . . | 52 | 40·5 | 35·7 | 33·6 | 31·0 |

If any length $l$ increases to a length $l'$, then the elongation expressed as a percentage of the *original* length is

$$\frac{l' - l}{l} \times 100$$

From the above figures it is evident that in stating a percentage elongation it is necessary to state the length on which it has been measured. Extensions are often measured on a length of 8 inches. This does not give truly comparative results for bars of different sectional areas. For example, if on a round bar 1 inch diameter the local contraction of section and extension of length is mainly on, say, 2 inches, *i.e.* on a quarter of the whole length, in a bar of $\frac{1}{2}$ inch in diameter the local effect will be mainly on about 1 inch, *i.e.* on one-eighth of the whole length. The local contraction on the thicker bar will consequently add more to the total percentage elongation on the 8 inches, since the 2-inch length undergoing much local strain is a greater proportion of the whole. The general extension which occurs before the maximum load is reached is practically independent of the area of section of the bar, and would form a suitable criterion of ductility were it not too troublesome to measure it just before any waist is formed. It cannot be measured satisfactorily after fracture, as the contraction of fracture influences the ultimate extension for some distance from the fracture, the metal "flowing" in towards the waist. It is, however, sometimes calculated by subtracting the local extension on 2 inches at fracture from the whole extension, and expressing the difference as a general extension on a length 2 inches shorter than the whole gauge length.

Professor Unwin[1] has pointed out that another possible method of comparing the ductilities of two bars of unequal areas of cross-section is to make the length over which elongation is measured proportional to the diameter (or the square root of the area in the case of other than round bars); in other words, to use pieces which are geometrically similar. This plan is in use in Germany, where the relation between

[1] *Proc. Inst. C.E.*, vol. clv. p. 170.

the gauge length $l$, over which extension is measured, and the area of cross-section $a$, is

$$l = 11{\cdot}3\sqrt{a}$$

This corresponds with a length of 8 inches (or centimetres) for a bar of half-a-square-inch (or centimetre) area.

The British practice is to use a gauge length of 8 or 10 inches irrespective of the area of section, and test pieces in which the ratio $\frac{\text{length}}{\text{square root of area of section}}$ is constant have not been commercially adopted on account of increased expense involved in preparing specimens. Professor Unwin finds that with fixed length and fixed area of section the shape of the cross-section in rectangles, having sides of different proportions, does not seriously affect the percentage elongation.

Within considerable limits the variation in percentage extension, due to various dimensions, may be very clearly stated algebraically thus—

If $e$ = total extension and $l$ = gauge length, $e$ is made up of a general extension proportional to $l$, say $b \times l$, and a local extension $a$ nearly independent of $l$. That is

$$e = a + bl$$

and percentage elongation, $100 \cdot \frac{e}{l} = 100\left(\frac{a}{l} + b\right)$, a quantity which (for a given sectional area) decreases and approaches $100b$ as $l$ is increased.

Further, the local extension $a$ is practically proportional to the square root of the area of cross-section A, say

$$a = c\sqrt{\text{A}}$$

hence $$\text{percentage elongation} = 100\left(\frac{c\sqrt{\text{A}}}{l} + b\right)$$

a quantity which increases with increase of A and decreases with increase of $l$.

The Engineering Standards Committee have not, on account of the increased cost which would be involved in machining test pieces, considered it desirable to depart from the standard length of 8 inches for measurement of elongation for strips of plate; but on account of the greater elongation produced on this fixed length by using larger cross-sectional areas, a maximum allowable limit of width has been fixed for every thickness of plate, thus limiting the area without making it absolutely fixed for the fixed gauge length.

**27. Percentage Contraction of Section.**—If a test piece is of uniform section throughout its length, and during extension uniform contraction of area goes on throughout the length, as in perfectly plastic material, the percentage contraction of area reckoned on the original area is the same as the percentage elongation reckoned on the *final* length at the time of measurement. This statement will only hold good provided that the volume of the gauged length of material remains

constant, which is always very nearly true, as shown by density tests. For if $l$ and $l'$ are the initial and final lengths, and A and A′ the initial and final areas of cross-section respectively, since the volume is practically constant

$$l . A = l' . A', \text{ or } \frac{l}{l'} = \frac{A'}{A}$$

and subtracting unity from each

$$\frac{l - l'}{l'} = \frac{A' - A}{A}, \text{ or } \frac{l' - l}{l'} = \frac{A - A'}{A}$$

The left-hand side represents the elongation reckoned on the final length, and the right-hand side represents the proportional reduction of the original area. In materials which finally draw out to a waist or neck, the proportional contraction *at fracture* will be greater than this amount, which may be looked upon as a minimum of contraction possible, except in the rare case of a specimen breaking owing to local hardness or brittleness at a place where the section is substantially larger than the remaining portions, which have become reduced by drawing out.

**28. Tenacity and Other Properties of Various Metals.**—The behaviour of a typical ductile metal has been described fully in Art. 23. Stress-strain curves for two varieties of steel and a very good quality of wrought iron are shown in Fig. 28; all of these refer to round pieces of metal 1 inch diameter, and extensions are measured on a length of 8 inches. The straight line representing the elastic stage of extension has been plotted on a scale 250 times larger than that for the later stages of strain.

*Cast iron* is a brittle material, *i.e.* it breaks with very little elongation or lateral contraction, and at a rather low stress. The stress-strain curve for a sample of good cast iron is shown on the large scale of Fig. 28, the ultimate strength or tenacity being just over 10 tons per square inch, and the strain being then just above $\frac{1}{400}$. Little if any part of the curve for cast iron is straight, the increase of extension per ton increase of stress being greater at higher stresses. It is to be noticed that the value of the direct or stretch modulus of elasticity (E), which is proportional to the gradient of the curve, will differ according as it is measured on, say, the first ton per square inch of stress or over the whole range; in the former case it would be about 6000 tons per square inch, and in the latter about 4000 tons per square inch. The higher value is the more correct, as measurement should be made within the elastic limit. The elastic limit is very low for cast iron, it may be almost zero, for slight permanent sets may be detected under very low stresses.

The ultimate strength of cast iron in tension is usually from 7 to 10 tons per square inch; in compression it is often about 50 tons per square inch. Great differences are found in test pieces from different parts of a casting, and the properties are much modified by the rate of

cooling. Thus a cast bar would generally give a different result tested in the rough with the skin on from that obtained from a similar bar with the outer material machined off; the former would show greater ultimate strength.

Owing to the liability to porosity, initial stress in cooling, etc., the working strength allowable in cast iron does not usually exceed about

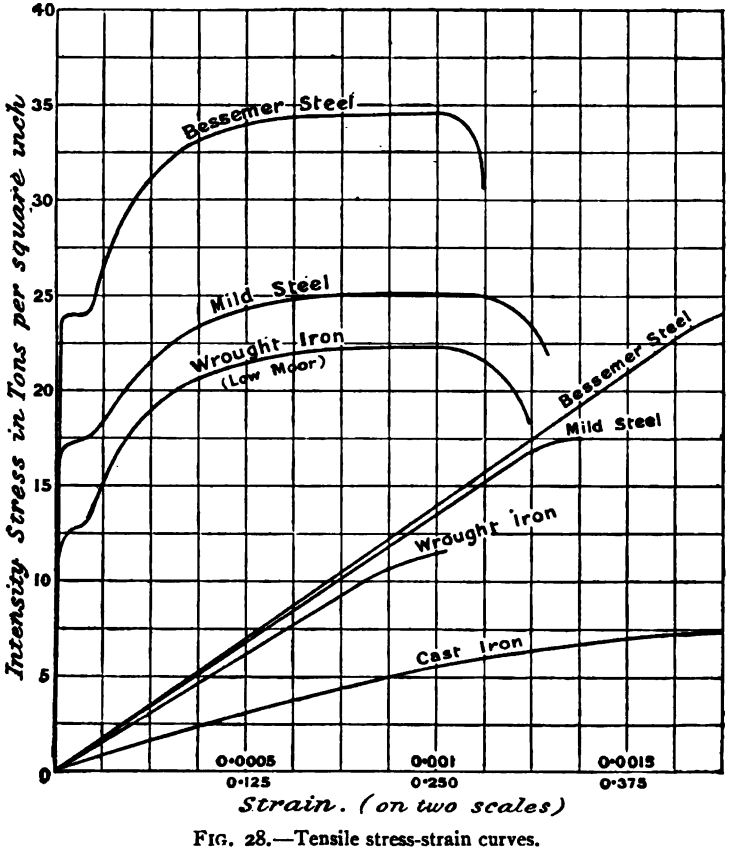



FIG. 28.—Tensile stress-strain curves.

1 ton per square inch in tension and 8 tons per square inch in compression.

*Wrought Iron.*—Wrought iron is a typical ductile metal, and contains over 99 per cent. of pure iron, and only about one-tenth per cent. of carbon. It comes from the puddling furnace in a spongy or pasty state (not liquid), and subsequent hammering and rolling do not expel all traces of slag, which may be traced in layers in the finished product. The structure appears from a fractured specimen to be fibrous or

laminated: this results from the rolling and working up of the crude product, but the metal itself, when examined under the microscope, is found to consist of crystalline grains. Both the tenacity and ductility are greater in the direction of the fibres than across them. The mechanical properties differ considerably in different qualities; those of a high quality are represented in Fig. 28; lower qualities have a lower ultimate strength and smaller elongation (see table at end of chapter).

The composition of wrought iron varies in different qualities. It is desirable to keep phosphorus below $\frac{1}{4}$ per cent. and sulphur below 0·05 per cent. Phosphorus makes the metal brittle when it is cold, and sulphur causes brittleness at a red heat.

*Steel.*—Steel was the term formerly applied to various qualities of iron which hardened by being cooled quickly from a red heat. Such material contained over $\frac{1}{2}$ per cent. of carbon chemically combined with the iron. The tenacity and ductility of these steels is not of so much interest as that of the softer varieties. The high carbon steels are not ductile, but have a high tensile strength.

Now, much more ductile materials, having a lower tensile strength, are produced by the Bessemer, Siemens, and other processes, and are classed as mild steels. The mild steels have for many purposes replaced wrought iron, being stronger, more uniform, and more ductile; unlike wrought iron they can be cast, and when required for bars, etc., they are first cast in ingots and then rolled; the ingot being obtained from the liquid state no fibre is produced in the subsequent rolling or forging, and the metal is more homogeneous than wrought iron, and often has as little carbon present, but it is not so reliable for welding, and when a weld is necessary good wrought iron is used. These steels contain less than $\frac{1}{2}$ per cent. of carbon, the quantity varying according to the purpose for which the steel is required. Thus steel rails may have from 0·3 to 0·4 per cent., structural steel about 0·25 per cent., and rivet steel about 0·1 per cent. of carbon.

Other constituents even in small quantities also greatly modify the properties of steels, and apart from chemical composition the mechanical and thermal treatment which the metal receives will greatly modify the strength and ductility. Comparatively recently, steels containing small quantities of nickel,[1] chromium, vanadium, or manganese have been produced, having very high tensile strengths combined with a considerable degree of ductility.

The qualities desirable in steel for structural ship-building and machine purposes are indicated by the Standard Specifications drawn up by the British Standards Committee and published for them.[2] The chief requirements with respect to tensile tests and composition (when specified) are shown in the following table. All the strengths and

[1] See paper by Mr. Hadfield on "Alloys of Iron and Nickel," in *Proc. Inst. C.E.*, vol. cxxviii.; also paper on "Chrome-Vanadium Steel," *Proc. Inst. Mech. Eng.*, Dec., 1904; and a paper in the *Proc. Inst. C.E.*, vol. xciii., on "Manganese Steel."

[2] By Crosby Lockwood & Sons.

elongations are to be measured on test pieces of standard dimensions (see complete specifications), and other mechanical tests are specified.

| Material and use. | Composition. | | Tenacity in tons per square inch. | | Minimum elongation on 8 inches (per cent.). | Remarks. |
|---|---|---|---|---|---|---|
| | Maximum sulphur per cent. | Maximum phosphorus per cent. | Minimum. | Maximum. | | |
| Structural steel for bridges and general building construction, plates, angles, etc. | 0·06 | { 0·06* / 0·07† } | 28 | 32 | 20 | * Open hearth process. |
| Rivet bars for above . | — | — | 26 | 30 | 25 | † Bessemer. |
| Ship plates . . . . | — | — | 28 | 32 | 20‡ | |
| Angles, bulb angles, channel sections, etc., for ship-building . | — | — | 28 | 33 | 20 | ‡ 16 per cent. for plates below ⅝ in. |
| Rivet bars for ships . | — | — | 25 | 30 | 25 | |
| Railway axles . . . | 0·035 | 0·035 | 35 to 40 | — | { 25 / 20 } | |

The strength and ductility of steel forgings and castings is dependent upon many circumstances, and varies considerably in different parts of large pieces of material. Some idea of the values is given in the table at the end of the chapter.

**29. Compression.**—Metals have generally practically the same limit of elasticity and modulus of elasticity (E) in direct compression as in tension, and the tension test being much easier to make than a satisfactory compression test, it is quite usual to rely on tension test as an index of mechanical properties for nearly all metals.

For stresses beyond the elastic limit, hard or brittle materials, including stone and brick, under compression generally fracture by shearing across some plane oblique to the direct compressive stress; more plastic materials, on the other hand, shorten almost without limit, expanding laterally at the same time, and so increasing the area of cross-section as to require higher loads to effect further compressive strain. An ultimate crushing strength is therefore difficult to specify clearly. Typical compressive stress-strain curves are shown in Fig. 29. If the metal reached a state of perfect plasticity the actual stress intensity under which the material "flows" would be constant. Then, assuming no change of volume, if $l$ = original length of a bar, $l_1$ = reduced length.

A = original area of section, and $A_1$ = increased area of section. $A_1 l_1 = Al$ (see Art. 27).

$$\text{Actual final intensity of stress} = \frac{\text{load}}{A_1} = \frac{\text{load}}{A \times \frac{l}{l_1}} = \frac{\text{load} \times l_1}{Al}$$

$$= \frac{\text{load} \times l_1}{\text{volume of bar}}$$

$$\text{or,} \quad \left.\begin{matrix}\text{constant pressure intensity} \\ \text{of plastic flow}\end{matrix}\right\} = \frac{\text{load}(l - \text{reduction in } l)}{\text{volume of bar}}$$

Hence the loads (or the nominal intensity of stress), when plotted as ordinates against the compressive strains as abscissæ, would give a rectangular hyperbola, since their product is a constant. The asymptotes of the hyperbola are the axis along which strains are measured, and a line perpendicular to it corresponding to a position of unit strain.

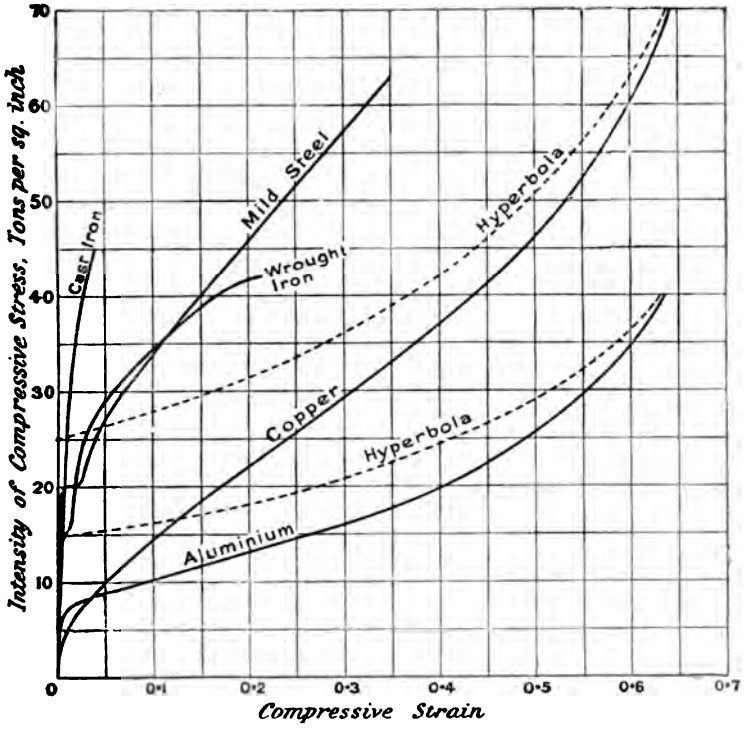



FIG. 29.—Compressive stress-strain curves.

Fig. 29 shows the manner in which the stress-strain curves for such plastic materials as copper and aluminium approach to a hyperbola, *i.e.* how nearly the materials reach to a condition of perfect plasticity, in which the metals flow continuously without increase of the actual intensity of pressure; the pressure intensity then reached is called the pressure of fluidity.

**30. Effect of Temperature on Mechanical Properties.**—The tenacity, ductility, and elasticity of the most important metals do not vary to any serious extent within the limits of ordinary atmospheric temperatures; but it is, of course, well known that the strength of many metals is greatly reduced at "white hot" temperatures.

Experiments show the following effects in statical tests for wrought iron and steel at high temperatures:—

(1) *The tenacity* (*a*) at ordinary temperatures falls off with increased temperatures until between 200° and 300° F., when it is something of the order of 5 per cent. less than at 60° F. (*b*) It rises from this temperature to a maximum value at some temperature between 400° and 600° F., when it is something of the order of 15 per cent. more than at 60° F. (*c*) It falls continuously with further increase of temperature.

(2) *The elastic limit* falls continuously with increase of temperature.

(3) *The elongation* (*a*) falls with increase of temperature above the normal to a minimum value in the neighbourhood of 300° F., and then (*b*) rises again continuously with increase of temperature.

The elongation under tension between 200° and 400° F. does not take place steadily, but at intervals during the application of the load. When the stress and strain are plotted they present a serrated curve nstead of a smooth one.

(4) *The modulus of direct elasticity* (E) decreases steadily with increase of temperature, metals which give a value of about 13,000 tons per square inch at atmospheric temperature falling to about 12,000 tons per square inch at 500° F.

*Low Temperatures.*—Experiments[1] on a very mild steel at very low temperature show progressive increase of tenacity with decrease of temperature; while the elongation practically vanishes, the material behaving like a very brittle substance. On return to ordinary temperatures no permanent change from the original properties is observed.

**31. Stress due to Change of Temperature.**—It is well known that metals, when free to do so, change their dimensions with change of temperature. If, however, such chance of dimensions is resisted and prevented, stress is induced in the material corresponding to the strain or change of dimension prevented. Thus if a long bar is lengthened by heat, and then its ends firmly held to rigid supports, so as to prevent contraction to its original length, the bar on cooling will be in tension, and will exert a pull on the supports. Numerous applications of this means of applying a pull are to be found, such as tie-bars holding two parallel walls together, and tyres shrunk on to wheels.

The linear expansion under heat is for moderate ranges of temperature closely proportional to the increase of temperature. The proportional extension, or extension per unit of length per degree of temperature, is called the coefficient of linear expansion. Thus if $\alpha$ is the coefficient of expansion, a length $l$ of a bar at $t_1°$ becomes

$$l\{1 + \alpha(t_2 - t_1)\}$$

at a temperature $t_2°$.

If subsequently the bar is cooled to $t_1°$ and contraction is wholly prevented, a proportional strain

$$\alpha(t_2 - t_1)$$

remains, and the corresponding tension and pull on the constraints is

$$E\alpha(t_2 - t_1)$$

[1] See a paper by Hadfield in *Journal of Iron and Steel Inst.*, 1901; or *Engineer*, May 26, 1906; or *Engineering*, May 19, 1906.

per unit area of cross-section of the bar, where E is Young's modulus for the material.

The following are the approximate linear coefficients of expansion for Fahrenheit degrees:—

| | |
|---|---|
| Wrought iron | 0·0000067 |
| Steel | 0·0000062 |
| Copper | 0·000010 |
| Cast iron | 0·0000060 |

For steel the tensile strain per degree Fahrenheit if contraction is prevented will be 0·0000062, and taking the stretch modulus as 13,000 tons per square inch, this corresponds to a stress intensity of

$$13{,}000 \times 0{\cdot}0000062, \text{ or } 0{\cdot}0806 \text{ ton per square inch}$$

Thus the cooling necessary to cause a stress of 1 ton per square inch would be

$$\frac{1}{0{\cdot}0806} \text{ or about } 12^\circ \text{ F.}$$

The different amounts of expansion in different metals in a machine may cause serious stresses to be set up due to temperature changes. Occasionally use is made of the different expansions of two parts.

EXAMPLE 1.—If a bar of steel 1 inch diameter and 10 feet long is heated to 100° F. above the temperature of the atmosphere, and then firmly griped at its ends, find the tension in the bar when cooled to the temperature of the atmosphere if during cooling it pulls the end fastenings $\frac{1}{40}''$ nearer together. Assume that steel expands 0·0000062 of its length per degree Fahrenheit, and that the stretch modulus is 13,000 tons per square inch.

The final proportional strain of the bar is

$$0{\cdot}0000062 \times 100 - \tfrac{1}{40} \div 120$$

or

$$0{\cdot}00062 - 0{\cdot}00021 = 0{\cdot}00041$$

$$\text{Intensity of stress} = 13{,}000 \times 0{\cdot}00041$$
$$= 5{\cdot}33 \text{ tons per square inch}$$

and total pull on a bar 1 inch diameter is

$$5{\cdot}33 \times 0{\cdot}7854 = 4{\cdot}18 \text{ tons}$$

**32. Work done in Tensile Straining.**—During the application of a gradually increasing tensile load to a bar, elongation takes place in the direction of the applied force and work is done. If during an indefinitely small extension $\delta x$ inch, the variable stretching force is sensibly constant and equal to F tons, the work done is

$$F \times \delta x \text{ inch-tons}$$

During a total elongation $l$ the work may be conveniently represented by the summation of all such quantities as $F \cdot \delta x$, *i.e.* by

$$\Sigma(F \cdot \delta x) \text{ or } \int^{l} F \cdot dx$$

*Graphical Representation.* — In a load-extension diagram the ordinates represent force and the abscissæ represent the elongation produced, and therefore the area under the curve, viz.

$$\int F \,.\, \delta x$$

represents the work done in stretching. Thus, in Fig. 30 the shaded area represents the work done.

*Scale.*—If the force scale is $p$ tons to 1 inch and the extension scale is $q$ inches to 1 inch, 1 square inch of area on the diagram represents $p \,.\, q$ inch-tons, which is the scale of the work diagram.

In ductile metals the whole work done up to fracture may be taken as roughly equal to the product of the total extension and the yield-point load plus $\frac{2}{3}$ of the product of the extension and the excess of the maximum load over the yield-point load. In other words, the average load is

$$\text{yield load} + \tfrac{2}{3}(\text{maximum load} - \text{yield load})$$

This approximation is equivalent to neglecting the strain up to yield point, and taking the remainder of the stress-strain curve as parabolic.

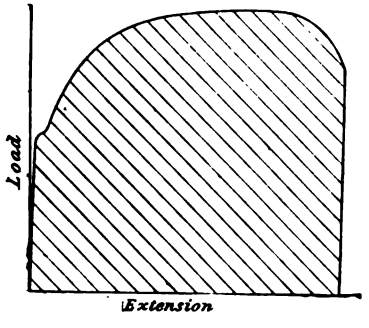



FIG. 30.

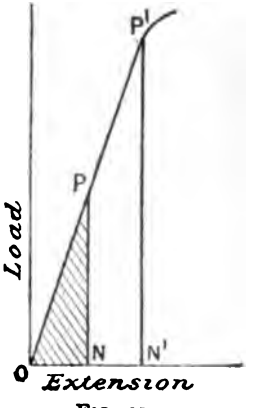



FIG. 31.

**33. Elastic Strain Energy.**—The work done in producing an elastic strain is stored as *strain energy* in the strained material and reappears in the removal of the load. On the other hand, the work done during non-elastic strain is spent in overcoming the cohesion of the particles of the material and causing them to slide one over another, and appears as heat in the material strained. In materials which follow Hooke's Law, the elastic portion of the load-extension diagram being a straight line, the amount of work stored as strain energy for loads not exceeding the elastic limit in tensile straining is equal to

$$\tfrac{1}{2} \,.\, \text{load} \times \text{extension}$$

In Fig. 31 the work stored when the load reaches an amount PN

is represented by the shaded area OPN, or by $\frac{1}{2}$. PN . ON, which is proportional to

$$\tfrac{1}{2}\text{. load} \times \text{extension}$$

**34. Resilience.**—Colloquially, resilience is understood to mean the power of a strained body to spring back on the removal of the straining forces, but technically the term is slightly modified and restricted to the amount of energy restored by the strained body. Within the elastic limit this is generally, as above for tensile straining, the product of half the load and the extension.

In a piece of metal under *uniform* intensity of tensile stress $p$, below the elastic limit, if A is the area of cross-section and $l$ the length, the load is

$$p\,.\,\mathrm{A}$$

and the extension is

$$l \times \text{proportional strain, or } l \times \frac{p}{\mathrm{E}} \text{ (Art. 9)}$$

where E is the stretch modulus. Hence the resilience is

$$\tfrac{1}{2}\,.p\mathrm{A}\,.\,l\frac{p}{\mathrm{E}} = \tfrac{1}{2}\,.\frac{p^2}{\mathrm{E}}\,.\,l\mathrm{A} = \tfrac{1}{2}\frac{p^2}{\mathrm{E}} \times \text{volume of piece}$$

$$\text{or the resilience is } \tfrac{1}{2}\frac{p^2}{\mathrm{E}}$$

per unit volume of the material. Where the tension is not uniform the expression is of similar form, but the factor is less than $\frac{1}{2}$ if $p$ is the maximum intensity of stress. Some particular cases will be noticed later.

*Proof Resilience.*—The greatest strain energy which can be stored in a piece of material without permanent strain is called its proof resilience. If $f$ is the (uniform) intensity of stress at the elastic limit or proof stress, the proof resilience is then

$$\tfrac{1}{2}\frac{f^2}{\mathrm{E}} \times \text{volume}$$

This is represented in Fig. 31 by the area OP'N' for a material obeying Hooke's Law.

The proof resilience is often stated as a property of a material, and is then stated per unit volume, viz.

$$\tfrac{1}{2}\frac{f^2}{\mathrm{E}}$$

**35. Live Tensile Loads within the Elastic Limit.**—If a tensile load is suddenly applied to a bar and does not cause a stress beyond the limit of elasticity, the bar behaves like any other perfect spring, and makes oscillations in the tension, the amplitude on either side of the equilibrium position being equal to the extension which would be produced by the same load gradually applied. Hence the maximum instantaneous strain produced is double that which would be produced by the same load applied gradually.

Suppose, for example, that a tensile load W is suddenly applied to a bar of cross-sectional area A. The instantaneous strain produced is

$$e = 2\frac{W}{A} \div E$$

and the instantaneous intensity of stress produced is

$$p = Ee = 2 \cdot \frac{W}{A}$$

which is twice that for a static or gradually applied load W. It is here assumed that the stress-strain curve (or value of Young's modulus) within the elastic limit is independent of the rate of loading, which is probably nearly true.

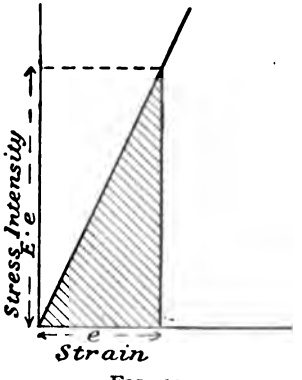


FIG. 32.

The instantaneous stress-strain diagram is shown in Fig. 32. Its area is proportional to

$$\tfrac{1}{2}Ee^2 \text{ or } \tfrac{1}{2}\frac{(Ee)^2}{E}$$

which is the work for unit volume of material.

If a bar already carries, say, a "dead" tensile load $W_0$, and another "live" load W of the same kind is applied, the greatest stress reached, provided the elastic limit is not exceeded, will be

$$\frac{W_0}{A} + \frac{2W}{A}$$

or,

$$\frac{W_0 + W}{A} + \frac{\text{change of load}}{A}$$

If, on the other hand, the live load W causes a stress of opposite kind (say compressive) to that already operating, the instantaneous stress would be

$$\frac{W_0}{A} - \frac{2W}{A}$$

or,

$$\frac{W_0 - W}{A} - \frac{\text{change in load}}{A}$$

EXAMPLE.—Find the statical load which would produce the same maximum stresses as (*a*) a tensile dead load of 40 tons and a tensile live load of 10 tons; (*b*) a tensile dead load of 20 tons and a compressive live load of 30 tons.

(*a*) Equivalent static load = 50 + 10 = 60 tons tension.

(*b*) Equivalent static load = 20 − 30 − 30 = − 40 tons, *i.e.* 40 tons compression.

**36. Impacts producing Tension.**—If an impulsive tensile load, such as that of a heavy falling weight, is applied axially to a light bar and the limits of proportionality of stress to strain are not exceeded, the strain energy instantaneously taken up by the bar is nearly equal to the kinetic energy lost by the falling weight if all the connections except the bar are *infinitely rigid.*

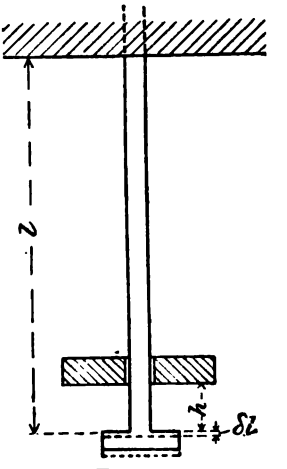


FIG. 33.

If a heavy weight W lbs. (Fig. 33) falls through a height $h$ inches on to a stop in such a way as to bring a purely axial tensile stress on a bar of length $l$ inches and cross-section A square inches, causing a stretch $\delta l = \frac{pl}{E}$, strain $e$, and an instantaneous tensile stress of intensity $p$, then, if the stop, the falling weight, and the supports of the bar be supposed infinitely rigid, neglecting the small loss in impact,

$$W(h + \delta l) = \tfrac{1}{2}Ee \times A \times \delta l \text{ or } \tfrac{1}{2}F \,.\, \delta l$$

where F is the equivalent statical load on the bar in pounds, and E is the stretch modulus of elasticity in pounds per square inch; hence

$$W(h + \delta l) = \tfrac{1}{2}Ee \times A \times el$$
$$= \tfrac{1}{2}Ee^2 \times \text{volume of bar}$$

or, since, $\frac{p}{E} = e$, $\qquad W(h + \delta l) = \tfrac{1}{2}\frac{p^2}{E} \times$ volume of bar

and
$$p^2 = \frac{2E \times W(h + \delta l)}{\text{volume of bar}} \text{ or } \frac{2EWh}{\text{volume}}$$

approximately when $\delta l$ is very small compared to the fall $h$.

From this $p$ may be calculated if E is known. If, as a particular case, we take $h = 0$, the equation

$$p^2 = \frac{2EW(h + \delta l)}{\text{volume of bar}}$$

becomes $p^2 = \frac{2EW\delta l}{A \,.\, l} = 2 \,.\, \frac{W}{A} \,.\, E \,.\, \frac{\delta l}{l} = 2 \,.\, \frac{W}{A} \,.\, p$, and $p = 2 \,.\, \frac{W}{A}$ as in the previous article.

Taking account of the loss of energy at impact consequent on the inertia of the bar, from the principle of the conservation of momentum, the velocity $v$ of the weight W and the free end of the bar immediately after impart may be found by assuming the stretch to be distributed as for a static load W, as if the tension were to spread instantaneously throughout the length. Thus if $w$ = weight of bar,

$$W \,.\, \sqrt{2gh} = Wv + \frac{w}{l}\int_0^l \frac{x}{l} v\,dx = (W + \tfrac{1}{2}w)v, \; v = \sqrt{2gh}\,\frac{W}{W + \frac{1}{2}w}$$

The total kinetic energy after impact is

$$\text{K.E.} = \tfrac{1}{2}Wv^2 + \frac{1}{2g}\cdot\frac{w}{l}\int_0^l\left(\frac{x}{l}v\right)^2 dx = \frac{1}{2g}(W+\tfrac{1}{3}w)v^2 = \frac{(W+\frac{1}{3}w)W^2}{(W+\frac{1}{2}w)^2}\cdot h$$

Then equating this kinetic energy plus the gravitational work done by W and $w$ to the gain in strain energy,

$$\text{K.E.} + W\cdot\frac{p}{E}\cdot l + \frac{w}{l}\cdot\frac{p}{E}\int_0^l x\,dx = \frac{1}{2}\cdot\frac{p^2Al}{E} + \frac{w}{l}\cdot\frac{p}{E}\int_0^l x\,dx$$

$$p^2 - 2p\cdot\frac{W}{A} - \frac{2E}{Al}\cdot\frac{(W+\frac{1}{3}w)}{(W+\frac{1}{2}w)^2}\cdot h = 0$$

$$p = \frac{W}{A}\left\{1+\sqrt{1+\frac{2AE(W+\frac{1}{3}w)}{l(W+\frac{1}{2}w)^2}\cdot h}\right.$$

If $h = 0$, $p = 2\dfrac{W}{A}$ as in the previous article and above. If $h$ is large compared to the extension $\delta l$ the term in $p$ vanishes, and

$$p = W\sqrt{\frac{2E}{lA}\cdot\frac{(W+\frac{1}{3}w)}{(W+\frac{1}{2}w)^2}\cdot h}$$

If $w$ is fixed in magnitude and W is variable, it is instructive to notice that the *ratio* of the impact stress $p - \dfrac{W}{A}$ to the stress $\dfrac{W}{A}$ is

$$\sqrt{1+\frac{2AE}{l}\cdot\frac{(W+\frac{1}{3}w)}{(W+\frac{1}{2}w)^2}\cdot h}$$

which *decreases* with increase in W.

It is not unusual to assume that the stretch modulus is the same for impulsive or very quick loadings as in a static test, although this is not certain. In this connection reference may be made[1] to impact experiments by Prof. B. Hopkinson, showing purely elastic stresses and strains much beyond those usually associated with the limits of elasticity, provided the time during which the stresses exceeded this limit is of the order of $\frac{1}{1000}$ second or less. Whether the relation of stress to strain is the same for such quick rates of applying stress as for rates several thousand times slower, is unknown.

**37. Fatigue of Metals.**—It has been found by experience that metals used in construction ultimately fracture under frequently repeated stresses very much lower than their ultimate statical strength. Further, that if the stresses are not merely repeated, but reversed, that is, the material is subjected to repeated stresses of opposite kinds, the resistance to fracture is less than if the same intensity of only one kind of stress were repeated. In such cases the material is often said to have become "fatigued." Since the cause of failure under varying stress is still imperfectly understood, it is doubtful whether the term "fatigue of the whole of the metal" gives a correct idea of what occurs to the material.

[1] *Proc. Roy. Soc.*, Feb., 1905.

It may be pointed out that the treatment to which metals are subjected in slowly or quickly repeated variations of stress is quite distinct from the blows or impacts mentioned in the previous articles.

Since 1864, when Fairbairn published in the *Philosophical Transactions of the Royal Society* results of some experiments on this subject, many important researches upon it have been carried out, and others are at present in progress. Some mention of the most important will now be made; the term "varying" stresses must be understood to mean stresses of the same kind, fluctuating between a maximum and a minimum value, whilst the term "reversed" stresses will be reserved for fluctuations from one kind of stress to that of an opposite kind, *e.g.* from tension to compression.

**38. Wöhler's Experiments.**[1]—Much light is thrown on the behaviour of iron and steel under fluctuating stresses by the lengthy researches of Wöhler. The experiments included torsional, bending, and simple direct stresses. The most important deductions from these experiments are: (1) That the resistance to fracture under fluctuating stresses depends within certain limits on the range of fluctuation of stress, *i.e.* upon the algebraic difference between the maximum and minimum stress, rather than upon the maximum stress; and (2) that reversed stresses (tensile and compressive) much below the static breaking stress, and even well within the ordinary elastic limit, are sufficient to cause fracture if repeated a great number of times.

The second point may be illustrated by the following Table I. and Fig. 34. The material selected is an axle-iron made by the Phœnix Co., and subjected to equal and opposite tension and compression produced by bending action on a rotating bar. The ultimate strength of this material, as determined by ordinary statical tension tests, was about 23 tons per square inch, and the elongation about 20 per cent.

TABLE I.

(STRESSES IN TONS PER SQUARE INCH.)

| Maximum stress (tension). | Minimum stress (compression). | Range of stress. | Number of repetitions before fracture. |
|---|---|---|---|
| + 15·3 | (−) 15·3 | 30·6 | 56,430 |
| 14·3 | 14·3 | 28·6 | 99,000 |
| 13·4 | 13·4 | 26·8 | 183,145 |
| 12·4 | 12·4 | 24·8 | 479,490 |
| 11·5 | 11·5 | 23·0 | 909,840 |
| 10·5 | 10·5 | 21·0 | 3,632,588 |
| 9·6 | 9·6 | 19·2 | 4,917,992 |
| 8·6 | 8·6 | 17·2 | 19,186,791 |
| 7·6 | 7·6 | 15·2 | 132,250,000 (not broken) |

[1] A full description is given in *Engineering*, vol. xi., 1871. Also a good account with numerous results and discussion is given in Unwin's "Testing of Materials" (Longmans); also in *Brit. Assoc. Report*, 1887, p. 424.

Fig. 34 shows the ranges of stress plotted as ordinates against the repetitions necessary to cause fracture as abscissæ. For an indefinitely great number of repetitions the curve approaches a value of about 15·2 tons per square inch range, corresponding to a maximum tensile or compressive stress of about 7·6 tons per square inch, a value probably

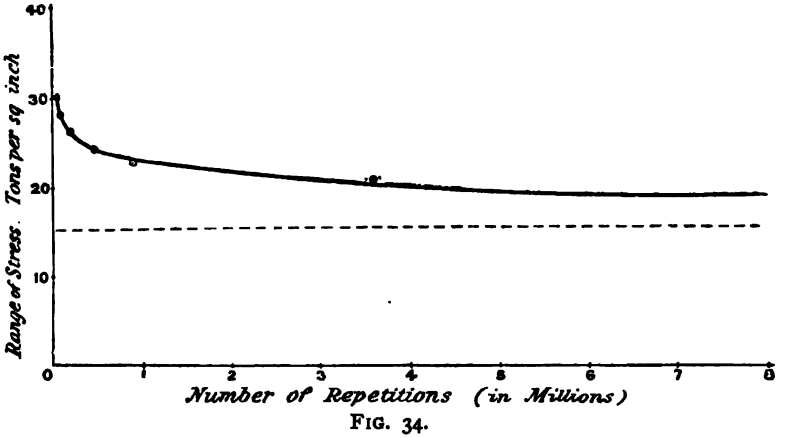



FIG. 34.

well below the ordinary elastic limit of the material. The range is called the "limiting range of stress," for which the number of repetitions necessary to cause fracture becomes infinite.

These results, although rather more regular than some others, may be regarded as typical in character of those for wrought irons and steels of various strength. The harder high carbon steels show a higher limiting range of stress than the softer or milder steels.

The dependence of endurance under fluctuating stress upon the range of stress may be illustrated by the following table (II.) of results of pure tension tests of the above metal:—

TABLE II.

(STRESSES IN TONS PER SQUARE INCH.)

| Maximum stress. | Minimum stress. | Range of stress. | Number of repetitions before fracture. |
|---|---|---|---|
| 22·92 | 0 | 22·92 | 800 |
| 21·01 | 0 | 21·01 | 106,910 |
| 19·10 | 0 | 19·10 | 340,853 |
| 17·19 | 0 | 17·19 | 409,481 |
| 17·19 | 0 | 17·19 | 480,852 |
| 15·28 | 0 | 15·28 | 10,141,645 |
| +21·01 | +9·55 | 11·46 | 2,373,424 |
| +21·01 | +11·46 | 9·55 | 4,000,000 (not broken) |

Here the limiting maximum stress for repeated stresses is about 15·28 tons per square inch with application and complete removal of the load, and about 21 tons per square inch when only about half the load is removed. Thus the limiting maximum stress for the three types of fluctuating load are somewhat as follows :—

| Kind of repeated load. | Limiting maximum stress. | Limiting range. |
|---|---|---|
| Completely reversed . . . . . | 7·6 | 15·2 |
| From maximum to zero . . . . | 15·28 | 15·28 |
| From maximum to half load . . . | 21·01 | about 10 |

From these figures it is evident that in such tests the question of endurance or failure under fluctuating stress depends more upon the range than upon the maximum stress imposed.

No evidence exists that any similar conclusion would hold for other than limiting stresses, *e.g.* there is no evidence that the above material, if subjected to repeated stresses of the same kind and having a maximum intensity of, say, 7 tons per square inch, would be safer if the minimum stress were 5 tons per square inch than if it were 2 tons per square inch or zero, provided that by stress is understood the resisting force exerted by the material, and not simply the force calculated from the external load, independent of whether the increase of load is applied suddenly or otherwise.

Spangenberg continued Wöhler's experiments on the same machines, and obtained similar results for iron and steel and copper alloys. Extensive results of the same kind have been published by Bauschinger[1] and by Sir B. Baker[1] for iron and steel. A few results are quoted for various irons and steels in Table III. These are selected from more extensive tables to be found in Unwin's "Testing of Materials," all except the first being from Bauschinger's experiments. The stresses stated in tons per square inch are those which the metals withstood for over two million times before fracture.

Table III. shows that the "complete reversal" limit of stress varies from about $\frac{1}{4}$ in harder steels to $\frac{1}{3}$ in the most ductile irons and steels, of the ultimate statical strength of the material. Also that the *repetition* limit varies from 40 to 60 per cent. of the ultimate strength, being between 55 and 60 per cent. for the ductile irons and mild steels. Further, that the reversal and repetition limits in the high tenacity steels (high carbon values) are higher than in the milder and more ductile material, although *not* so large a proportion of the ultimate statical strengths.

[1] See summaries in Unwin's "Testing of Materials," and in *Brit. Assoc. Report*, 1887, p. 424.

TABLE III.

| Material and tenacity. | Minimum stress (limiting). | Maximum stress (limiting). | Limiting range of stress. | Ratio of maximum to tenacity. |
|---|---|---|---|---|
| Krupp axle steel, 52 tons . . . | −14·05 | +14·05 | 28·1 | 0·27 |
| | 0 | 20·5 | 20·5 | 0·39 |
| | 17·5 | 37·75 | 20·25 | 0·73 |
| Wrought-iron plate, 22·8 tons . | −7·15 | +7·15 | 14·30 | 0·31 |
| | 0 | 13·10 | 13·10 | 0·57 |
| | 11·4 | 19·2 | 7·8 | 0·84 |
| Bessemer steel, 28·6 tons . . . | −8·55 | +8·55 | 17·10 | 0·30 |
| | 0 | 15·7 | 15·7 | 0·55 |
| | 14·3 | 23·8 | 9·5 | 0·83 |
| Steel rail, 39 tons . . . . . | −9·7 | +9·7 | 19·4 | 0·25 |
| | 0 | 18·4 | 18·4 | 0·47 |
| | 19·5 | 30·85 | 11·39 | 0·79 |
| Mild steel boiler plate, 26·6 tons | −8·65 | +8·65 | 17·3 | 0·33 |
| | 0 | 15·8 | 15·8 | 0·59 |
| | 13·3 | 22·55 | 9·25 | 0·85 |

**39. Other Experiments on Fluctuations of Stress.**—In 1902 Dr. J. H. Smith published,[1] in conjunction with Professor Osborne Reynolds, the results of a long research on reversals of stress in various materials applied by means of the inertia forces of a reciprocating weight. The opposite simple tensile and compressive stresses were of approximately equal magnitudes, and the rapidity of reversal, which in Wöhler's experiments had been from about 60 to 80 fluctuations per minute, was much higher than in any previous work, being from 1300 to 2500 per minute.

The most striking result of these experiments was to show that for reversals of stress the "limiting range of stress," and the number of reversals necessary to cause actual rupture with any fixed stress, are much smaller at these high speeds, and between the speeds stated diminish with increase of speed. Fig. 35 is plotted from Smith's results for mild steel of tenacity 26 tons per square inch and elongation 29 per cent., and shows clearly the smaller range of stress for any fixed number or for an indefinitely large number of reversals at higher speeds. The speeds are not constant for either curve, as variable speed was one method employed for varying the range of stress, which was simply proportional to the square of the speed of reversal when the reciprocating weight was not varied.

Another interesting result was that cast steel of about 58 tons per square inch tenacity and 2·5 per cent. elongation in a static test did not, at these high speeds, show a higher reversal limit of stress than the 26-ton mild steel. In previous experiments at lower rates of reversal the steels of higher tenacity and lower ductility showed greater reversal limits, but these were, even in Wöhler's experiments, a smaller proportion of the higher tenacities (see last column of Table III., Art. 38).

[1] *Phil. Trans. Roy. Soc.*, 1902, p. 265.

Professor J. H. Smith[1] has measured the elastic deformation of a test piece under increasing alternating stresses and finds some evidence that the strain ceases to be proportional to the stress when the range of stress is approximately such as will ultimately cause rupture.

*Stanton and Bairstow's Experiments.*[2]—These were made at the National Physical Laboratory upon irons and steels in common use

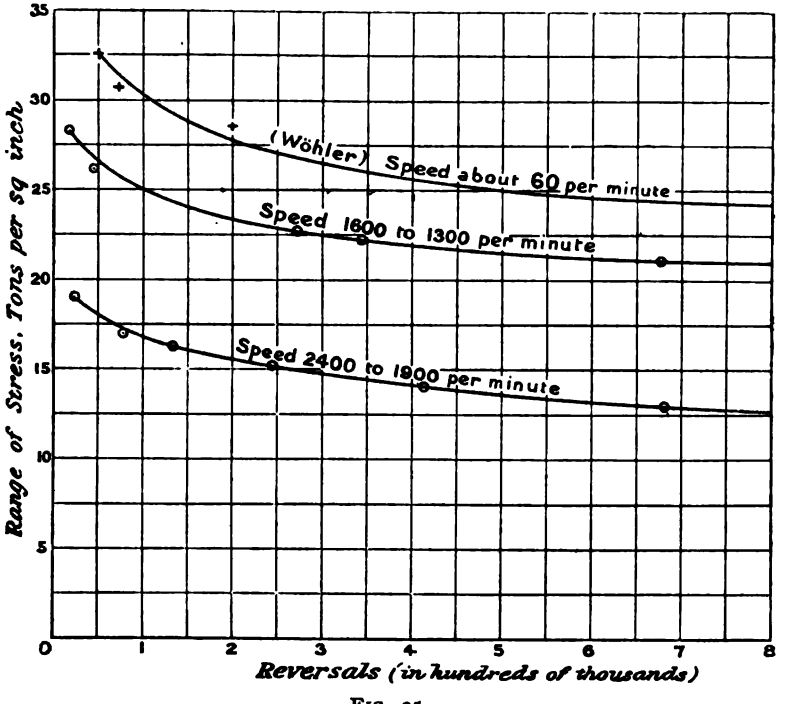



FIG. 35.

by means of a throw-testing machine acting upon the same principle as that of Reynolds and Smith, but taking four specimens simultaneously. The reversals were at the rate of 800 cycles per minute, and the ratio of tensile to compressive stress varied from 1·4 to 0·72 ($\frac{7}{5}$ to $\frac{5}{7}$) with two intermediate (reciprocal) proportions.

The inquiry also included the relative limiting resistance to fracture under reversed stresses of material in various forms, some of which had rather abrupt changes of section, such as in screw threads, etc.

Perhaps the most surprising result of these experiments was that

[1] *Journal of Iron and Steel Institute*, 1910.

[2] "On the Resistance of Iron and Steel to Reversals of Direct Stress," *Proc. Inst. C.E.*, 1906, vol. clxvi. p. 78. See also "Elastic Limits of Iron and Steel under Cyclical Variation of Stress," by L. Bairstow, *Proc. Roy. Soc.*, vol. A. 82, p. 483.

the values obtained for the limiting ranges of stress agree, so far as it was possible to compare the materials used, with those of Wöhler and Bauschinger made at about 60 cycles per minute rather than with those of Reynolds and Smith at from 1400 to 2400 per minute; apparently the change in speed from 60 to 800 does not seriously lessen the resistance to reversals of stress. In fact, the reversal limits reckoned for one million reversals were on the whole distinctly higher proportions of the tenacities than in Wöhler's experiments. Also the greater resistance of the harder steels compared to the more ductile ones as found by Wöhler at 60 cycles per minute was maintained at 800 per minute.

Within the considerable limits mentioned above (1·4 to 0·72) the ratio of tensile stress to compressive stress does not seriously affect the limiting range of stress for wrought iron.

The specimens having a sudden change of section showed diminution in the limiting range of stress, the diminution being greater in the harder than in the more ductile metal; the reduction was to 48 per cent. of the maximum for a rather hard Bessemer steel and to 65 per cent. of the maximum for the mildest steel, with intermediate proportions for other materials. Specimens having less abrupt changes of section showed diminished resistance in a smaller degree.

*Eden's Experiments.*—Messrs. Eden, Cunningham and Rose,[1] experimenting on a rotating beam to find the limiting range of stress to withstand a million reversals, were unable to detect any diminution with increase of speed from 250 to 1300 revolutions per minute.

**40. Limiting Stress with Various Ranges of Fluctuation.**—The relation between the limiting values of the maximum stress for different ranges of stress when, as in Wöhler's experiments, the ratio of maximum stress to minimum is varied over a very wide field, may be shown in various ways graphically or algebraically. The three quantities, maximum stress intensity (say tensile) $f_{max.}$, minimum stress intensity $f_{min.}$ (reckoned negative if compressive), and the range of stress $\Delta$, are evidently connected by the equation

$$\Delta = f_{max.} - f_{min.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The relation between these three quantities for practically an infinite number of stress fluctuations may be illustrated by the results of one of Bauschinger's tests of mild steel boiler plate, given in Table III., Art. 38, viz.

| | $f_{max.}$ | $f_{min.}$ | $\Delta$ |
|---|---|---|---|
| (*a*) | 26·6 | 26·6 | 0 |
| (*b*) | 22·55 | 13·3 | 9·25 |
| (*c*) | 15·8 | 0 | 15·8 |
| (*d*) | +8·65 | −8·65 | 17·3 |

Fig. 36 shows these values of $f_{max.}$ and of $\Delta$ plotted as ordinates against the values of $f_{min.}$ as abscissæ. Perhaps the relation between the three quantities is better illustrated by Fig. 37, where both $f_{max.}$ and $f_{min.}$ are measured vertically, and $\Delta$, the range, is the vertical distance

[1] "Endurance of Metals," *Proc. Inst. M.E.*, 1911.

between the two curves. The portions *de* and *d'e'* are mere speculations, but Stanton's results make it appear that about the portion *dd'* of either figure the range is about constant, *i.e.* that *de* and *d'e'* are nearly parallel. Obviously the range must decrease again with higher compressive stress, but experimental evidence is lacking, this portion of the curve being of least practical importance. The shaded area is such that if both maximum and minimum stress fall within it the material will stand unlimited repetitions or reversals of stress, as the case may be, without fracture. Various empirical formulæ have been suggested to express the relations between the quantities $f_{max.}$ and $\Delta$ from

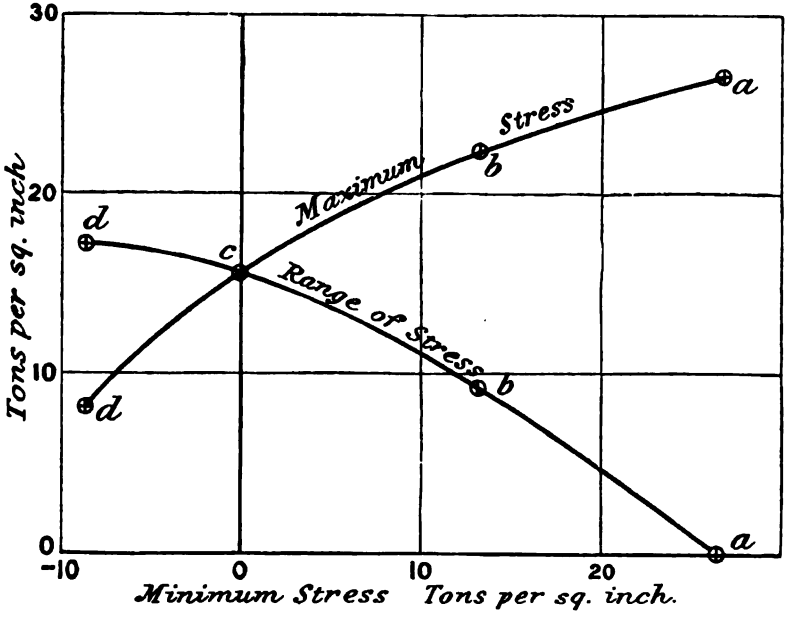



FIG. 36.

the experiments of Wöhler, Bauschinger, Spangenberg, and others. Of these, the best known are the formulæ of Weyrauch[1] and Launhardt, and Gerber's parabolic relation.[2] The last is expressed by the equation

$$f_{max} = \frac{\Delta}{2} + \sqrt{f^2 - n\Delta f} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $f$ is the ultimate static strength or tenacity of the material, and $n$ is a constant to be determined from experimental results. The value of $n$ is found to vary from about 1·4 for ductile metals to above 2 for more brittle ones, its value for ductile metals of construction being

[1] *Proc. Inst. C.E.*, vol. lxiii.
[2] See Unwin's "Elements of Machine Design," vol. i. chap. ii.

generally about 1·5. This value gives a "reversal limit" of $\frac{1}{3}f$, and a repetition limit of $0{\cdot}61f$.

The value 1·53 is the mean value of $n$ deduced from the results for mild steel boiler plate quoted above, and points intermediate between the experimental values at (*a*), (*b*), (*c*), and (*d*) have been calculated in plotting Figs. 36 and 37, from which it will be noticed how closely the empirical relation fits the few observed points. How far such calculated results may be relied upon is doubtful, and in any case values of the maximum limiting stress between *a* and *c* considerably

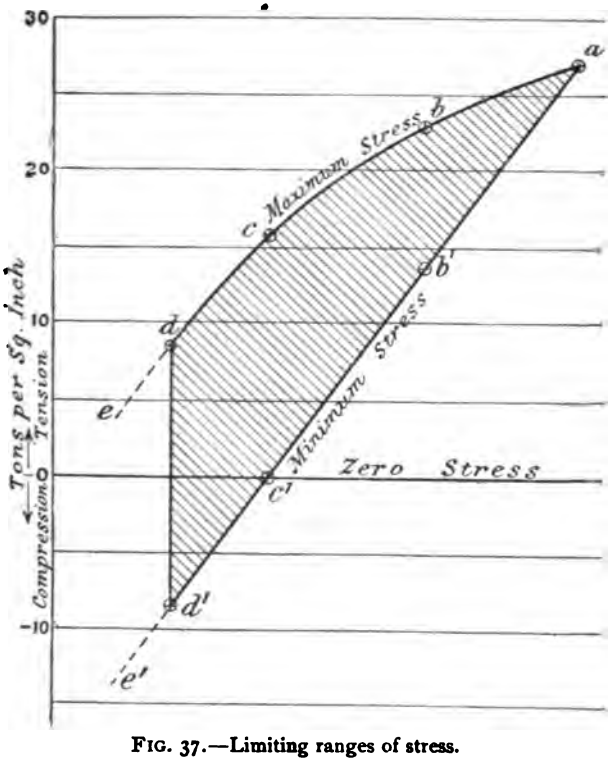



FIG. 37.—Limiting ranges of stress.

exceeding the elastic limit, although of considerable scientific interest, are not of great practical importance, since stresses which would produce considerable strains cannot be used in machines or structures. The most important practical relations, then, are those between the repetition limit (minimum stress zero) and the reversal limit (equal and opposite tension and compression), shown in the area *cdd'c'*, Figs. 36 and 37, and over this region the variation of the stress is not great. Stanton and Bairstow's experiments seem to show that for wrought iron, for some distance on either side of *dd'* the range $\Delta$ is practically constant.

Launhardt's formula for stresses varying between zero and a maximum value is

$$f_{max.} = f_1 + (f - f_1)\frac{f_{min.}}{f_{max.}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $f_1$ is the repetition limit or value of $f_{max.}$ when the minimum stress is zero and $f$ is the ultimate static strength.

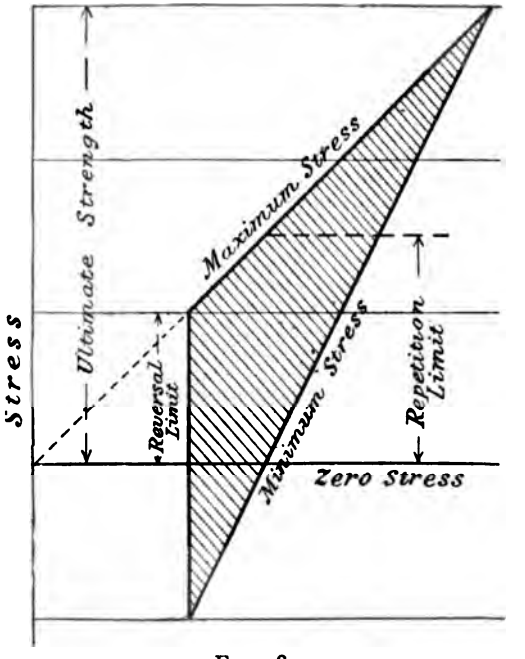



FIG. 38.

For variation between tension and compression Weyrauch proposed the formula

$$f_{max.} = f_1 + (f_1 - f_2)\frac{f_{min.}}{f_{max.}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Where $-f_{min.}$ is the greatest compressive stress and $f_2$ is the reversal limit or value of $f_{max.}$ when the value of $f_{min.}$ is equal to $-f_{max.}$.

*Launhardt-Weyrauch formula.*—The two values (3) and (4) may be taken as equal, and we may write

$$f_{max.} = f_1 + (f - f_1)\frac{f_{min.}}{f_{max.}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

taking account of the negative sign of $f_{min.}$ for a reversed stress. Equating (4) and (5) for the reversal or for the repetition limits shows that (5) involves the supposition that

$$f_1 = \tfrac{1}{2}(f + f_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus if $f_1 = \frac{2}{3}f$ it follows from (5) or (6) that $f_2 = \frac{1}{3}f$. These values are roughly correct for mild steel, and adopting them in (5) the formula becomes

$$f_{max.} = \tfrac{2}{3}f\left(1 + \tfrac{1}{2}\frac{f_{min.}}{f_{max.}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If any of the above empirical formulæ be plotted in the same manner as the relation (1), curves corresponding fairly well with Figs. 36 and 37 will be obtained.

Owing to want of sufficient data the curve of Fig. 37 is sometimes taken as a straight line, with a repetition limit of half and a reversal limit of one-third the statical tenacity, these being average values for a variety of materials. The limiting range is as shown by vertical ordinates of the shaded area in Fig. 38. The divergence of the lines of maximum and minimum stress does not greatly alter the range in the immediate neighbourhood of the reversal limit, and the method at least possesses the merit of simplicity. The relation is algebraically expressed by the equation

$$f_{max.} = f - \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This may also be written

$$f_{max.} = f - (f_{max.} - f_{min.}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or $$2f_{max.} = f + f_{min.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or $$f = 2f_{max.} - f_{min.} = 2f_{max.}\left(1 - \tfrac{1}{2}\frac{f_{min.}}{f_{max.}}\right) \quad . \quad . \quad . \quad (11)$$

hence $$f_{max.} = \frac{\frac{1}{2}f}{\left(1 - \frac{1}{2}\frac{f_{min.}}{f_{max.}}\right)} \text{ or } \frac{f_1}{1 - \frac{1}{2}\frac{f_{min.}}{f_{max.}}} \quad . \quad . \quad . \quad . \quad (12)$$

**41. Working Loads and Stresses.**—The various experiments on fluctuating stress, as well as the results of general experience in the design and use of structures and machines, point to the use of different working stresses according to the nature of the straining action to be endured. If a factor of safety or ratio of ultimate statical strength to maximum working stress of, say, 3 be sufficient for mild steel to cover accidental and uncalculated straining actions, errors of workmanship, for a steady, unvarying or dead load, a similar factor might be applied to the maximum strength as given by the formulæ of Art. 40.

*Launhardt-Weyrauch Formula.*—Thus the Launhardt-Weyrauch formula (7) gives

$$\text{working unit stress} = \tfrac{1}{3}f_{max.} = \tfrac{2}{9}.f.\left(1 + \tfrac{1}{2}\frac{\text{minimum stress}}{\text{maximum stress}}\right)$$

or more generally if $r$ is the factor of safety for an unvarying or dead load (*i.e.* $\frac{f}{r}$ is the allowable working stress where $f$ is the ultimate strength) then for a varying load,

$$\text{working unit stress} = \tfrac{2}{3}\frac{f}{r}\left(1 + \tfrac{1}{2}\frac{\text{minimum stress}}{\text{maximum stress}}\right) \quad . \quad . \quad (1)$$

where $\frac{2}{3}$ of the working stress for dead loads may be written instead of $\frac{2}{3}\frac{f}{r}$

The ratio $$\frac{\text{minimum stress}}{\text{maximum stress}}$$

is very frequently known before the actual values of the stresses are determined, for their ratio may be the same as the ratio of the loads (or other straining action such as bending moment) producing the stresses.

The highest value of (1) occurs for a dead load, *i.e.* when minimum stress = maximum stress. If this is taken at $6\frac{1}{2}$ tons per square inch for tension in steel, the Board of Trade allowance for structures in the United Kingdom, the formula becomes

$$\text{working unit stress} = 4{\cdot}33\left(1+\tfrac{1}{2}\frac{\text{min. load}}{\text{max. load}}\right)\text{tons per sq. inch} \quad . \quad (2)$$

which allows 4·33 tons per square inch for a wholly live load and 2·17 tons per square inch for a wholly reversible load, *i.e.* when minimum load equals minus maximum load. A more usual value of the constant gives

$$\text{working unit stress} = 5\left(1+\tfrac{1}{2}\frac{\text{min. load}}{\text{max. load}}\right)\text{tons per sq. inch} \quad . \quad (3)$$

and with some such equivalent constant this formula has been widely used in America and on the Continent of Europe. Another empirical variation, known as Professor Cain's formula, for loads which do not change direction, is

$$\text{working unit stress} = \tfrac{1}{2}\text{ unit stress for dead loads} \times \left(1+\frac{\text{min. load}}{\text{max. load}}\right) \quad (1A)$$

*Dynamic Formula.*—Similarly with a factor of safety which is $r$ for wholly dead load, (12) Art. 40, for varying loads gives

$$\text{working stress} = \frac{\frac{f}{2r}}{1-\tfrac{1}{2}\frac{\text{minimum load}}{\text{maximum load}}} \quad . \quad . \quad . \quad (4)$$

With a stress of $6\frac{1}{2}$ tons per square inch for a wholly dead tensile load this becomes

$$\text{working stress} = \frac{3{\cdot}25}{1-\tfrac{1}{2}\frac{\text{minimum load}}{\text{maximum load}}}\text{ tons per sq. inch} \quad . \quad (5)$$

which gives 3·25 tons per square inch for a wholly live load and 2·17 tons per square inch for a wholly reversible load, when a minimum load equals minus maximum load. The formula (5) may in general be written

$$\text{working unit stress} = \frac{\tfrac{1}{2}(\text{working unit stress for dead loads})}{1-\tfrac{1}{2}\frac{\text{minimum load}}{\text{maximum load}}} \quad . \quad (6)$$

due regard being paid to the proper sign (+ or −) of the minimum load.

*Equivalent Dead Load*—One method of proportioning members of structures is to use a constant working stress, viz. that applicable to a dead load, and to increase the maximum load by some amount to allow for the effect of a live load. The allowance may be the whole or some fraction of the extreme variation or range of load. If there is a sudden fluctuation of the load from the minimum to the maximum, the elastic vibrations would produce the same straining effect as twice that alteration occurring very gradually (see Art. 35), so that in this case

$$\text{Equivalent dead load} = \text{min. load} + 2\,(\text{max. load} - \text{min. load})$$
$$= \text{max. load} + (\text{max. load} - \text{min. load})$$

or

$$\text{max. load} + \text{variation in load} \quad . \quad . \quad (7)$$

This method is parallel with equation (8) of Art. 40, which is for ultimate stresses, and with equation (6) of the present article; equations (6) and (7) will lead to the same dimensions for a member.

*Impact Allowances.*—Sometimes the equivalent dead load is taken as the dead load and the live load plus some fraction of the variation in load, so that

$$\text{Equivalent dead load} = \text{max. load} + k\,(\text{range or variation in load}) \quad (8)$$

where $k$ is a coefficient dependent on circumstances such as the suddenness or otherwise of the change in stress. For example, in a girder traversed by a moving load the change of stress is not sudden, but occurs much more quickly in some parts than in others, as is evident from Arts. 88, 89, and 90.

The formulæ (6) and (7) are equivalent to taking $k = 1$ in (8) or to taking an equivalent dead load equal to minimum load plus twice the range of load; this is a sufficient allowance for both fatigue (if any) due to repetitions of load and impact or dynamical action in producing a stress higher than that resulting from a static load. They represent something like ordinary British practice although the actual empirical formulæ used vary in form.

In America an allowance for impact used with a constant unit stress is common in bridge work, well-known values of $k$ in (8) being Prichard's value[1] for load of constant sign,

$$k = \frac{\text{live load}}{\text{maximum load (pull or thrust)}} \quad . \quad . \quad . \quad . \quad (9)$$

or, to include reversals of stress,

$$k = \frac{\text{range of load}}{\text{maximum load}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9A)$$

[1] First used in 1895; this gives results in accord with good practice. It has been abandoned by Mr. Prichard in favour of (9B) and (9C) as being contrary to theory in that it gives an increase instead of decrease in the ratio of impact stress to live load stress with increased live load (*Proc. of Engineers Society of Pennsylvania*, vol. xxiii. p. 603, Dec. 1907). Also given by Mr. E. H. Stone (*Proc. Am. Soc. Civ. Engineers*, vol. xli. pp. 491, 492.

Prichard's later values are from L = 0 to 125'.

$$k = 1 - 0{\cdot}004\,L - 0{\cdot}1\,(N - 1) \quad . \quad . \quad . \quad . \quad (9B)$$

and from L = 125' to 500',

$$k = 0{\cdot}5 - 0{\cdot}1\,(N - 1) \quad . \quad . \quad . \quad . \quad . \quad (9C)$$

where N = number of loaded tracks and L = loaded length of span.

The American Railway Engineering and Maintenance of Way Association's specification (1905) is

$$k = \frac{300}{L + 300} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where L is the loaded length of the track in feet producing the maximum stress; the value (10) varies for chord members from 1 to 0·5 as the span increases from 0 to 300 feet. These are generally considered a sufficient margin for both fatigue and impact. Whether the two effects should be included in a common allowance or be treated separately is a matter on which opinions differ. The allowance for impact is based upon the fact shown by actual deflection measurements that rolling loads (including imperfectly balanced wheels) produce greater effects than stationary ones and effects which increase with increased speeds. The dynamic effect is naturally greater in short spans than in long ones where the greater inertia of the structure permits of smaller strains from a given impact, provided such effects do not accumulate due to a coincidence of vibration periods with a periodic disturbance.

Many valuable experiments on the effect of live loads on bridges have been made and reported[1] by a committee of the American Railway Engineering and Maintenance of Way Association, as a result of which it gives as the maximum probable impact, as a percentage of the live load

$$\frac{100}{1 + \frac{l^2}{20{,}000}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $l$ is the span in feet. This varies from 100 to 50 per cent. while the span increases from 0 to 142 feet. Two other conclusions from these experiments are of great interest: (1) That the chief cause of impact stresses is found in unbalanced driving wheels of locomotives; (2) serious impact stresses arise principally from cumulative vibration resulting from the near approach of the period of rotation of driving wheels at a critical speed to the frequency of vibration of the loaded structure.

*The Alternative Methods.*—The reader should realize that the alternatives of using a variable unit stress in connection with the maximum straining action or using a constant (or dead load) unit stress

[1] For an abstract of the sub-committee's report see *Engineering News*, No. 13, vol. lxiii. p. 385, and No. 14, vol. lxiii. p. 407, March 31 and April 7, 1910.

with a dynamically increased equivalent dead load is largely a matter of individual choice, and that one system can always be expressed in terms of the other, and that in either case the rules in common use are empirical. Thus, illustrating from the simple case of tension, let M = maximum load, R = range of load, then for a tie bar

$$\text{sectional area} = \frac{\text{M}}{\text{variable unit stress}} = \frac{\text{M} + k\text{R}}{\text{dead load unit stress}} \quad (12)$$

hence

$$\text{variable unit stress} = \frac{\text{dead load unit stress}}{1 + \frac{k\text{R}}{\text{M}}} \quad . \quad . \quad (13)$$

Thus writing $k = 1$ we get formula (6).

And by putting $k = \frac{\text{R}}{\text{M}}$ as in (9A)

$$\text{variable unit stress} = \frac{\text{dead load unit stress}}{1 + \frac{\text{R}^2}{\text{M}^2}} \quad . \quad . \quad (14)$$

A nearly equivalent form has been proposed by Mr. E. H. Stone,[1] viz. approximately,

$$\text{dead load unit stress} \times \left(1 - \tfrac{1}{2}\frac{\text{R}^2}{\text{M}^2}\right) \quad . \quad . \quad . \quad (14\text{A})$$

He recommends for steel bridges the value

$$9 - 5\left(\frac{\text{R}}{\text{M}}\right)^2 \text{ tons per square inch}$$

Or again, inversely, from (12) we can obtain the value of the impact coefficient $k$ corresponding to variable unit stress rules, for

$$k = \frac{\text{M}}{\text{R}}\left(\frac{\text{unit stress for dead loads}}{\text{variable unit stress}} - 1\right) \quad . \quad . \quad (15)$$

Thus, using the Launhardt formula, (1) gives

$$k = \frac{\text{M}}{3\text{M} - \text{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and using Cain's formula, (1A) gives

$$k = \frac{\text{M}}{2\text{M} - \text{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

An example will make clear the use of the formulæ of the present article.

EXAMPLE.—Taking a dead load stress of 6 tons per square inch find the cross-sectional area for a tie bar subject to a tensile dead load

[1] *Trans. Am. Soc. C.E.*, vol. xli.

of 4 tons and live loads which vary from 1 ton compression to 2 tons tension.

maximum tensile $= 4 + 2 = 6$ tons
minimum load $= 4 - 1 = 3$ tons. Range of load $= 3$ tons
ratio $\frac{\text{minimum}}{\text{maximum}} = \frac{3}{6} = \frac{1}{2}$

Launhardt-Weyrauch formula (1) gives

working stress $= 4\,(1 + \frac{1}{2} \times \frac{1}{2}) = 5$ tons per square inch
area required $= \frac{6}{5} = 1{\cdot}2$ square inch

or from (16)

$$k = \frac{6}{18 - 3} = \frac{2}{5}$$

hence from (8)

equivalent dead load $= 6 \times 3 \times 0{\cdot}4 = 7{\cdot}2$ tons
area required $= 7{\cdot}2 \div 6 = 1{\cdot}2$ square inch

Dynamic formula (5) or (6) give

working unit stress $= 3 \div (1 - \frac{1}{2} \times \frac{1}{2}) = 4$ tons per square inch
area required $= 6 \div 4 = 1{\cdot}5$ square inch

or putting $k = 1$ in (8) or from (7)

equivalent dead load $= 6 + 3 = 9$ tons
area $= 9 \div 6 = 1{\cdot}5$ square inch

Cain's formula (1A) gives

working unit stress $= 3\,(1 + \frac{1}{2}) = 4{\cdot}5$ tons per square inch
area required $= 6 \div 4{\cdot}5 = 1{\cdot}33$ square inch

or from (17)

$$k = \frac{6}{12 - 3} = \frac{2}{3}$$

and from (8)

equivalent dead load $= 6 + \frac{2}{3} \times 3 = 8$ tons
area required $= 8 \div 6 = 1{\cdot}33$ square inch

Prichard's Impact Coefficient (9) or (9A) gives $k = 0{\cdot}5$, hence from (8)

equivalent dead load $= 6 + (0{\cdot}5 \times 3) = 7{\cdot}5$ tons
area required $= 7{\cdot}5 \div 6 = 1{\cdot}25$ square inch

or from (14)

working unit stress $= 6 \div 1{\cdot}25 = 4{\cdot}8$ tons per square inch
area required $= 6 \div 4{\cdot}8 = 1{\cdot}25$ square inch

Fig. 39 illustrates the relation of the working unit stresses in the four well-known systems explained above, taking steel of 30 tons per square inch and a factor of safety of 4 for dead loads. Fig. 40 shows the so-called factors of safety on the same working unit stresses.

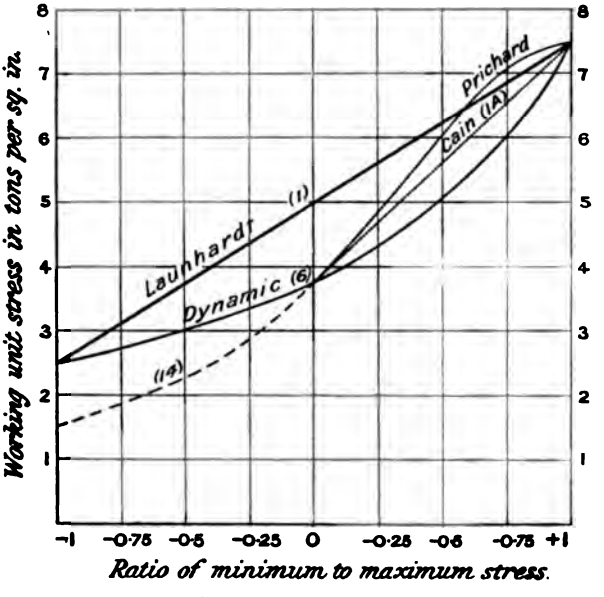



FIG. 39.—Relation of working unit stresses.

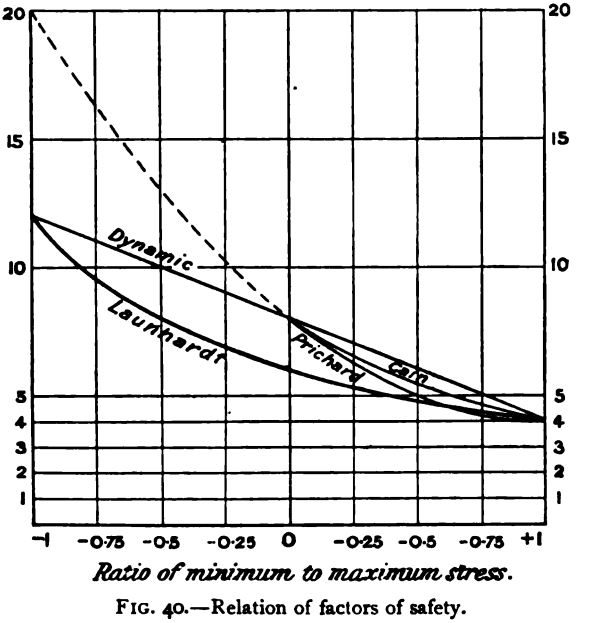



FIG. 40.—Relation of factors of safety.

Unwin gives the following table of factors of safety for different materials and circumstances:—

TABLE OF FACTORS OF SAFETY.

| Material. | Factors of safety for | | | |
|---|---|---|---|---|
| | Dead load. | Live or varying load. | | Structure subject to shock. |
| | | Stress of one kind only. | Reversed stresses. | |
| Cast iron . . . . | 4 | 6 | 10 | 15 |
| Wrought iron and steel | 3 | 5 | 8 | 12 |
| Timber . . . . . | 7 | 10 | 15 | 20 |
| Brickwork and masonry | 20 | 30 | — | — |

TABLE OF ULTIMATE STRENGTHS.

(*The following are average and not extreme values.*)

| Material. | Tenacity in tons per square inch. | Shearing strength in tons per square inch. |
|---|---|---|
| Cast iron . . . . . | 7 to 10 | 9 to 11 |
| Wrought-iron bars . . . . | 20 to 24 | 15 to 18 |
| ,, plates (with fibre) . . . | 21 | 16 |
| ,, ,, (across fibre) . . . | 19 | 14 |
| [1]Steel, mild structural . . . . . | 28 to 32 | 21 to 24 |
| ,, ,, for rivets . . . . . | 26 to 29 | — |
| ,, for rails . . . . . | 30 to 40 | — |
| ,, castings and forgings . . . . | 25 to 35 | — |
| ,, wire . . . . . | 70 to 90 | — |
| Tool steel (carbon, hardened) . . . . | 70 | 45 |
| Copper, cast . . . . . | 9 | — |
| ,, hard drawn . . . . . | 20 | — |
| ,, annealed . . . . . | 13 | — |
| Brass . . . . . | 8 | 8 to 10 |
| Gun-metal . . . . . | 14 to 17 | 15 |
| Phosphor-bronze . . . . . | 26 | 24 |
| Manganese-bronze . . . . . | 35 | — |
| Aluminium, cast . . . . . | 3 to 5 | — |
| ,, rolled . . . . . | 7 to 10 | 6 |
| Aluminium-bronze (10 per cent. copper) . | 40 | 25 |
| Oak (British) . . . . . | 4 to 8 | $\frac{1}{5}$ |
| Ash . . . . . | 2 to 7 | $\frac{1}{4}$ |
| Elm . . . . . | 2 to 6 | Along the grain. |
| Teak . . . . . | 2 to 7 | |
| Yellow pine . . . . . | 1 to 2 | $\frac{1}{8}$ |
| Red pine . . . . . | 2 to 6 | |
| Spruce . . . . . | 2 to 3 | $\frac{1}{6}$ |

[1] See table in Art. 28.

TABLE OF ULTIMATE COMPRESSION OR CRUSHING STRENGTH.

| Material. | Breaking strength in tons per square inch. | |
|---|---|---|
| Cast iron | 40 to 50 | |
| Brass | 5 | |
| Copper (cast) | 20 | |
| Brick | 1 to 3 | |
| Granite | 10 | |
| Sandstone | 3 to 4 | |
| Oak | 2 to 4 | Along the grain. |
| Ash | 4 | |
| Yellow pine | 2 to 2½ | |
| Red pine | 4 to 5 | |

TABLE OF COEFFICIENTS OF ELASTICITY.

| Material. | Stretch, direct, or Young's modulus (E) in tons per square inch. | Transverse or shearing modulus or modulus of rigidity (N, C, or G) in tons per square inch. |
|---|---|---|
| Wrought iron | 12,000 to 13,000 | 5000 to 6000 |
| Steel | 13,000 to 14,000 | 5500 to 6500 |
| Cast iron | 6,000 to 9,000 | 2500 to 3500 |
| Copper | 6,000 to 7,000 | 2000 to 3000 |
| Brass | 5,000 to 6,000 | 2000 to 3000 |
| Gun-metal | 5,000 to 6,000 | 2000 to 3000 |
| Aluminium | 4,000 to 5,000 | — |
| Aluminium-bronze | 7500 | — |
| Oak | — | 650 |
| Ash | — | 700 |
| Elm | — | 500 |
| Teak | — | 1000 |
| Yellow pine | — | 700 |
| Red pine | — | 700 |
| Spruce | — | 700 |

TABLE OF APPROXIMATE WORKING STRESSES FOR DEAD LOADS.

| Material. | Kind of stress. | Magnitude of allowable stress. |
|---|---|---|
| Structural steel (Board of Trade allowance) | Tension | 6·5 tons per square inch. |
| Structural steel (Board of Trade allowance) | Compression | 6·5 ,, ,, ,, |
| Rivet steel | Shearing | 5 ,, ,, ,, |
| Wrought iron (Board of Trade) | Tension | 5 ,, ,, ,, |
| ,, ,, ,, ,, | Compression | 5 ,, ,, ,, |
| ,, ,, ,, ,, | Shearing | 4 ,, ,, ,, |
| Cast iron | Tension | 2 ,, ,, ,, |
| ,, ,, | Compression | 4 ,, ,, ,, |
| ,, ,, | Shearing | 1·5 ,, ,, ,, |
| Portland cement concrete, 5 to 1 | Compression | 15 ,, ,, foot |
| Bricks in mortar | Compression | 4 ,, ,, ,, |
| Granite | Compression | 70 ,, ,, ,, |
| Sandstone | Compression | 35 ,, ,, ,, |

## EXAMPLES II.

1. The following figures give the observations from a tensile test of a round piece of mild steel 1 inch diameter and 10 inches between the gauge points :—

| Load in tons | 5 | 10 | 15 | 16 | 17 | 18 | 19 | 20 | 20·5 | 21 | 21·5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Extension in inches . | 0·0047 | 0·0096 | 0·0145 | 0·0155 | 0·16 | 0·21 | 0·26 | 0·32 | 0·36 | 0·39 | 0·43 |

| Load in tons | 22 | 22·5 | 23 | 23·5 | 24 | 24·5 | 25 | 25·45 | 25·1 | 23·1 | 21·7 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Extension in inches . | 0·49 | 0·53 | 0·60 | 0·69 | 0·78 | 0·89 | 1·08 | 2·13 | 2·13 | 2·30 | 2·35 |

Plot separate stress-strain diagrams for the elastic and ductile extensions and find the ultimate tensile strength, intensity of stress at yield point, the percentage elongation on 10 inches, and the stretch modulus for the metal.

2. Two parallel walls, 25 feet apart are stayed together by a steel bar 1 inch diameter, passing through metal plates and nuts at each end. The nuts are screwed up to the plates while the bar is at a temperature of 300° F. Find the pull exerted by the bar after it has cooled to 60° (*a*) if the ends do not yield ; (*b*) if the total yielding at the two ends is $\frac{1}{4}$ inch. Steel expands 0·0000062 of its length per degree Fahrenheit, and E = 13,500 tons per square inch.

3. Find the work done per cubic inch of material in the static test to fracture given in question 1, Examples II.

4. Find the total elastic strain energy or resilience of a bar of mild steel 1 inch diameter and 10 feet long, carrying a tensile load of 7 tons. E = 13,500 tons per square inch.

5. Find the total proof resilience of a bar of steel $1\frac{1}{2}$ inch diameter and 8 feet long, the tensile elastic limit being 14 tons per square inch and the stretch modulus (E) 13,500 tons per square inch. Find also the proof resilience per cubic inch.

6. Find the intensity of stress and extension produced in a bar 10 feet long and 1·5 square inch in section, by the sudden application of a tensile load of 6 tons. What suddenly applied load would produce an extension of $\frac{1}{20}$ of an inch? Take E = 13,000 tons per square inch.

7. Estimate the dead loads equivalent to the following : (*a*) A dead load (tensile) of 15 tons and a live load of 20 tons. (*b*) A dead load (compressive) of 15 tons and a live tensile load of 20 tons. If the strain is not to exceed 0·001, find the area of section required in each case, E being 13,500 tons per square inch.

8. A load of 560 lbs. falls through $\frac{1}{2}$ inch on to a stop at the lower end of a vertical bar 10 feet long and 1 square inch in section. If the stretch modulus (E) is 13,000 tons per square inch, find the stress produced in the bar.

9. Find the greatest height from which the load in question 6 may fall before beginning to stretch the bar in order not to produce a greater stress than 14 tons per square inch.

10. What is a suitable value for the working stress for a bar carrying a dead load of 7 tons tension and subject also to a load which fluctuates between 3 tons tension and 2 tons compression if the safe stress for dead loads is 5 tons per square inch.

11. Find a suitable area of cross-section for a tie bar the tension in which varies from 10 tons to 8 tons, the safe stress for a dead load being 7 tons per square inch.

# CHAPTER III

## *STATICS*

**42. Systems of Forces in Equilibrium.**—In estimating the stresses on parts of a structure, it will frequently be necessary to consider the equilibrium of the structure regarded as a rigid body under the action of a system of forces, some of which (the loads) may be known, and others, the supporting forces or reactions, may be unknown and require to be found by the principles of statics. Or again, it is often necessary to consider in the same way a portion of a structure, and very frequently the equilibrium of the system of concurrent forces meeting at some point in the structure. It is convenient here to deal with the relations between forces forming a system in equilibrium; mainly graphical methods will be used, and the corresponding algebraic theorems will be briefly indicated. The foundation of graphical statics is the principle of geometrical or vector addition of forces. It happens that the rules relating to coplanar systems are frequently sufficient for estimating the forces on a structure, but extensions to other cases are occasionally useful and will be mentioned. In any case a force is completely specified by its magnitude, direction, and position.

**43. Graphical Methods.**—When statical problems are solved by graphical methods, it is usually necessary to first draw out a diagram, showing correctly the inclinations of the lines of action of the various known forces to one another, and, to some scale, their relative positions. Such a diagram is called a diagram of positions, or *space diagram;* this is not to be confused with the vector diagram of forces, which gives magnitudes and directions, but not positions of forces.

**Bow's Notation.**—In this notation the lines of action of each force in the *space diagram* are denoted by two letters placed one on each side of its line of action. Thus the spaces rather than the lines or intersections have letters assigned to them, but the limits of a space having a particular letter to denote it may be different for different forces.

The corresponding force in the *vector diagram* has the same two letters at its ends as are given to the spaces separated by its line of action in the space diagram. We shall use capital letters in the space diagram, and the corresponding small letters to indicate a force in the vector diagram. The notation will be best understood by reference to an example such as that in the following article.

**44. Concurrent Forces.**—If several forces, AB, BC, CD, DE (Fig. 41), all acting in one plane at a point X are given in magnitude and direction, their resultant may be found by starting from any point $a$, and drawing an open vector polygon $abcde$ with sides $ab$, $bc$, $cd$, and $de$ placed end to end, in the direction of, and proportional to, the given forces AB, BC, CD, and DE respectively, and closing the open polygon by joining $a$ to $e$; the vector $ae$ gives the magnitude and direction of the resultant force which must act through X. If an additional equilibrant or force represented by the vector $ea$ in the line EA through X be added the system would be in equilibrium. Thus if $n$ coplanar forces are in equilibrium and $n - 1$ are known completely, the vector polygon gives the $n$th force in magnitude and direction and its line of action passes through the point of concurrency of the given forces. If $n - 2$ of the forces are given completely, and two by their directions, the magnitude of these may be found by closing the open vector polygon by two sides parallel to the two given directions; this is equivalent to resolving EA into two components in assigned directions. Or again if $n - 2$ of the forces are given completely and two by their magnitudes, or one by its magnitude and one by its direction, it is easy (provided the data are not inconsistent) to close the open vector polygon of $n - 2$ sides and so find completely the two remaining forces. In any case $n - 2$ conditions being given, the number of conditions found by drawing the vector polygon will be two of the total $2n$ conditions, viz. magnitude and direction of a single equilibrant, magnitude of one and direction of another force, direction of two forces, or magnitude of two forces.

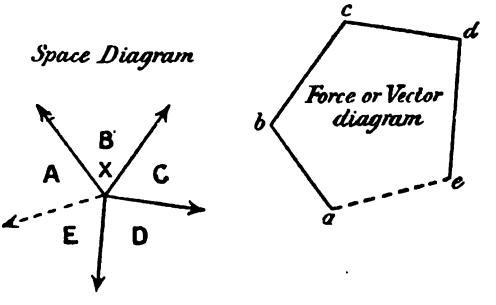



FIG. 41.—Concurrent forces.

If all the concurrent forces are not in one plane the same method holds good, and if $3n - 3$ conditions are given the vector polygon may be drawn in plan and elevation, its completion determining three conditions of magnitude or direction. The plan and elevation of each vector evidently involve only three independent conditions of length and directions (not four) and the total number of conditions is therefore $3n$.

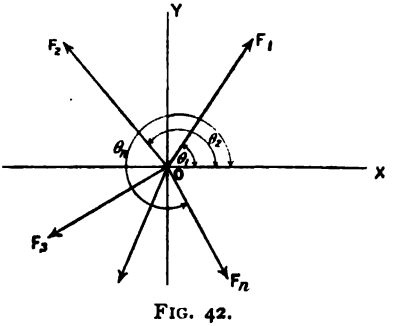



FIG. 42.

*Algebraic Method.*—The conditions to be satisfied in order that $n$ concurrent forces $F_1, F_2, F_3 \ldots F_n$ (Fig. 42) in one plane shall be in equilibrium are that the algebraic sum of the components in each of two independent directions shall be zero. Taking two rectangular axes OX and OY through the point of concurrency O, the total (horizontal) component in the direction OX is

$$X = F_1 \cos \theta_1 + F_2 \cos \theta_2 + F_3 \cos \theta_3 + \ldots + F_n \cos \theta_n$$

or $$\Sigma(F \cos \theta)$$

where the forces make angles $\theta_1, \theta_2, \theta_3 \ldots \theta_n$ with OX.

And the total (vertical) component in the direction OY is

$$Y = F_1 \sin \theta_1 + F_2 \sin \theta_2 + F_3 \sin \theta_3 + \ldots$$

or $$\Sigma(F \sin \theta)$$

The two conditions of equilibrium are

$$X \text{ or } \Sigma(F \cos \theta) = 0 \quad \ldots\ldots \quad (1)$$
$$Y \text{ or } \Sigma(F \sin \theta) = 0 \quad \ldots\ldots \quad (2)$$

These two equations enable two of the quantities $F_1, F_2, F_3 \ldots F_n$ and $\theta_1, \theta_2, \theta_3 \ldots \theta_n$ to be found if $2n - 2$ are given. If the system is not in equilibrium, the resultant R is given by

$$R = \sqrt{X^2 + Y^2}$$

and its inclination to OX by

$$\tan \theta = \frac{Y}{X}$$

If all the forces are not in one plane, they may be resolved into three mutually perpendicular directions OX, OY, OZ, then if X, Y, and Z are the algebraic sums of the components in the directions OX, OY, and OZ respectively, viz.

$$X = F_1 l_1 + F_2 l_2 + F_3 l_3 + \ldots + F_n l_n = \Sigma(F \cdot l)$$
$$Y = F_1 m_1 + F_2 m_2 + F_3 m_3 + \ldots\ldots = \Sigma(F \cdot m)$$
$$Z = F_1 n_1 + F_2 n_2 + F_3 n_3 + \ldots\ldots = \Sigma(F \cdot n)$$

where $l_1, m_1, n_1$, etc., are the appropriate direction cosines and the three conditions of equilibrium are

$$X = 0 \ldots (3) \qquad Y = 0 \ldots (4) \qquad Z = 0 \ldots (5)$$

which equations serve to determine three unknown conditions when the remaining three are given.

**45. Non-concurrent Forces; Funicular or Link Polygon.**—To find graphically the resultant or equilibrant of several non-concurrent coplanar forces AB, BC, CD, DE (Fig. 43), we may proceed as for concurrent forces to draw the open vector or force polygon *abcde* to represent by its sides, the magnitude and direction of the several forces. Then as before *ae* represents the resultant (or *ea* the single equilibrant) in magnitude and direction. But as the forces do not all pass through one point the position of the resultant remains to be determined. This

may be found by replacing two such forces as AB and BC by a single force AC (represented in magnitude and direction by *ac*) through the intersection Q of their lines of action AB and BC, and then similarly adding a third force CD to the resultant at the intersection S of AC and CD giving a force represented by *ad* through S and so on until the last force DE is added at the intersection T, which is a point on the resultant *ae* or the single equilibrant *ea*. But this fails for the important case of parallel forces, and is very inconvenient for forces which are nearly parallel on account of the acute intersection of the lines of action, and the following alternative is then adopted. Any pole *o*, Fig. 44, is chosen in or about the vector polygon and joined to each vertex *a*, *b*, *c*, *d* and *e*, and then from any point P, say on the line of

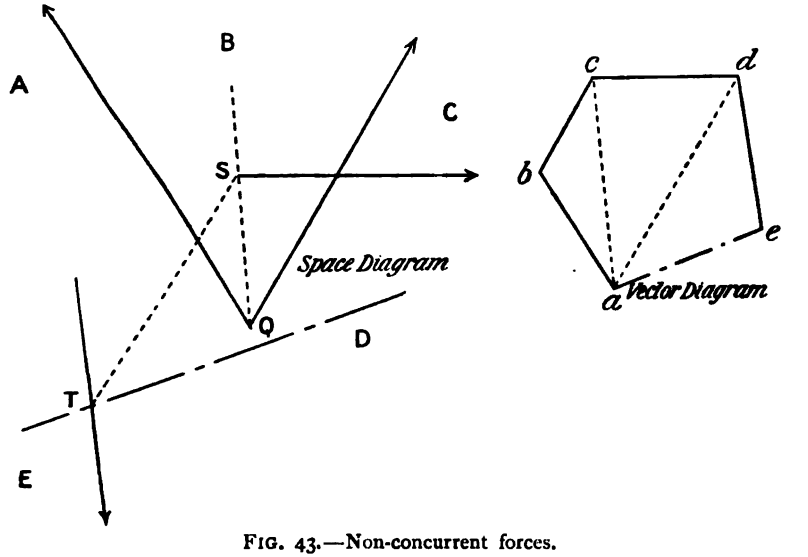



FIG. 43.—Non-concurrent forces.

action of AB, a line PT called AO is drawn parallel to *ao* across the space A, from P a line (BO) drawn across the space B parallel to *bo* to meet the line of action of BC in Q, and from Q a line is drawn across the space C parallel to *oc* to meet CD in R, and this process is continued until finally a line EO is drawn from the intersection S parallel to *oe* across the space E to meet the line AO in T. Then T is a point in the line of action of the resultant, the direction of which is given by *ae* in the vector diagram. Hence the equilibrant EA or the resultant AE is completely determined. The closed polygon PQRST, having its vertices on the lines of action of the forces, is called a *funicular* or *link polygon*. That T must be a point on the line of action of the resultant is evident from the following considerations. Any force may be resolved into two components along any two lines which

intersect on its line of action, for it is only necessary for the force to be the geometric sum of the components. Let each force, AB, BC, CD, and DE, be resolved along the two sides of the funicular polygon which meet on its line of action, viz. AB along TP and QP, BC along PQ and RQ, and so on. The magnitude of the two components is given by the corresponding sides of the triangle of forces in the vector diagram, *e.g.* AB may be replaced by components in the lines AO and BO (or TP and QP), represented in magnitude by the lengths of the vectors *ao* and *ob* respectively. Similarly, CD is replaced by components in the lines CO and OD represented by *co* and *od* respectively. When this process is complete, all the forces AB, BC, CD, and DE are replaced by components, the lines of action of which are the sides TP, PQ, QR, etc., of the funicular polygon. Of these component forces, those in the line PQ or BO are represented by the vectors *ob*

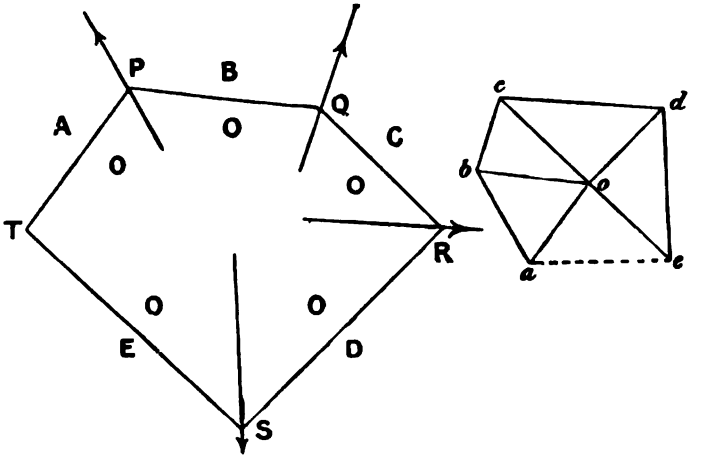



FIG. 44.—Funicular or link polygon.

and *bo*, and therefore have a resultant nil. Similarly, all the other components balance in pairs, being equal and opposite in the same straight line, except those in the lines TP and TS, represented by *ao* and *oe* respectively. These two have a resultant represented by *ae*, which acts through the point of intersection T of their lines of action. Hence finally the resultant of the whole system acts through T, and is represented in magnitude and direction by the line *ae*; the equilibrant is equal and opposite in the same straight line.

*Choice of Pole.*—In drawing the funicular polygon, the pole *o* (Fig. 44) was chosen in any arbitrary position, and the first side of the funicular polygon was drawn from *any* point P in the line AB. If the side AO had been drawn from any point in AB other than P, the funicular polygon would have been a similar and similarly situated figure to PQRST.

The choice of a different pole would give a different shaped

funicular polygon, but the points in the line of action of the unknown equilibrant obtained from the use of different poles would all lie in a straight line. The choice of a suitable pole will generally lead to a well-shaped link polygon, *i.e.* one in which the intersections of successive sides are not at very acute angles. A badly chosen pole will give an ill-conditioned link polygon with intersections so acute as to make the vertices difficult to locate exactly, or it may be, outside the limits of a sheet of drawing paper. The method shown in Fig. 43 is a particular case of the link polygon in which the pole is at an intersection of two sides of the force polygon.

*Algebraic Method.*—In the notation of Art. 44 the total horizontal component X of the resultant is

$$X = \Sigma(F \cdot \cos\theta)$$

and the total vertical component is

$$Y = \Sigma(F \cdot \sin\theta)$$

hence the resultant R is given by

$$R = \sqrt{(X^2 + Y^2)}$$

and its inclination $\theta$ to the axis OX is given by

$$\tan\theta = \frac{Y}{X}$$

The position of R may be specified in various ways such as by its perpendicular distance $r$ from the origin, given by equating the moments

$$R \times r = \Sigma(y \cdot F \cdot \cos\theta - x \cdot F \cdot \sin\theta)$$

reckoning R and clockwise moments positive for the usual directions of the axes OX and OY.

**46. Conditions of Equilibrium.**—If we include the equilibrant EA (Fig. 44, Art. 45) with the other four forces, we have five coplanar forces in equilibrium, and (1) the forces or vector polygon *abcde* is closed; and (2) the funicular polygon PQRST is a closed figure. Further, if the force polygon is *not* closed, the system reduces to a single resultant, which may be found by the method just described (Art. 45).

It may happen that the force polygon is a closed figure, and that the funicular polygon is not. Take, for example, a diagram (Fig. 45) similar to the previous one, and let the forces of the system be AB, BC, CD, DE, and EA, the force EA *not* passing through the point T found in Fig. 44, but through a point V (Fig. 45), in the line TS. If we draw a line, VW, parallel to *oa* through V, it will *not* intersect the line TP parallel to *ao*, for TP and VW are then parallel. Replacing the original forces by components, the lines of action of which are in the sides of the funicular polygon, we are left with two parallel unbalanced components represented by *ao* and *oa* in the lines TP and VW respectively. These form a couple, and such a system is not in equilibrium nor reducible to a single resultant. The magnitude of the

couple is equal to the component represented by *oa* multiplied by the length represented by the perpendicular distance between the lines TP and VW. It is also equal to the force EA represented by *ea*, multiplied by the distance represented by the perpendicular from T on the line VW. Or the resultant of the forces in the lines AB, BC, CD, and DE is a force represented by *ae* acting through the point T; this with the force through V, and represented by *ea*, forms a couple.

Hence for the equilibrium of a system of non-concurrent forces all in one plane it is essential that (1) the polygon of forces is a closed figure; (2) that the link or funicular polygon is a closed figure, these require the forces to satisfy three conditions of magnitude, direction, or position, and if of $n$ non-concurrent coplanar forces in equilibrium (completely specified by $3n$ such conditions) $3n - 3$ conditions are given, the remaining three can generally be found by the vector and link polygons. Thus in Art. 45 the three conditions one magnitude, one direction,

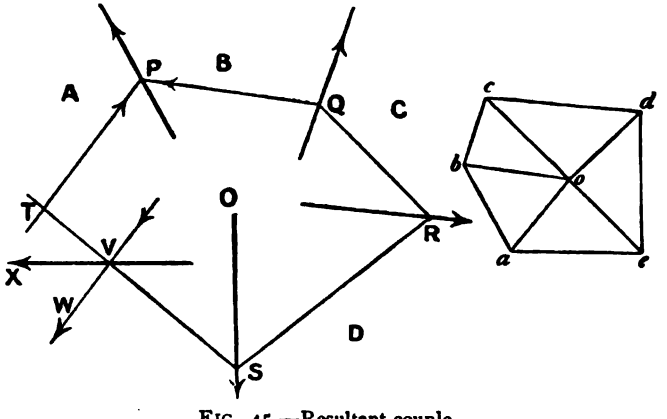



FIG. 45.—Resultant couple.

and one position all relating to a single equilibrant were determined; other cases having useful applications will be treated later in the chapter (Arts. 47, 48, 51).

If $n$ non-concurrent forces in equilibrium are not all in one plane the vector polygon of forces and the funicular polygon must both close, but this requires the fulfilment of six independent conditions represented graphically by the closing of both polygons in plan and in elevation.

*Algebraic Method.*—For $n$ non-concurrent forces coplanar in equilibrium the three conditions to be fulfilled are equations (1) and (2) of Art. 44, and that the resultant moment about one point such as the origin O shall be zero, *i.e.*

$$\Sigma(y \,.\, \mathrm{F} \cos\theta - x \,.\, \mathrm{F} \sin\theta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $x_1$, $x_2$, $x_3$, etc., represent the horizontal distances of the vertical components $\mathrm{F}_1 \,.\, \sin\theta_1$, $\mathrm{F}_2 \,.\, \sin\theta_2$, etc., from O, and $y_1$, $y_2$, etc., represent the vertical distance of the horizontal components $\mathrm{F}_1 \,.\, \cos\theta_1$, $\mathrm{F}_2 \,.\, \cos\theta_2$,

etc., from O. It may also be shown that the conditions of equilibrium are fulfilled if equations such as (1) hold for three points in the plane of the forces.

For $n$ non-concurrent forces not all in the same plane the six conditions of equilibrium are equations (3), (4), and (5) of Art. 44, together with three expressing that the moments about three independent axes are each zero. To fulfil the six conditions an unbalanced system will generally require at least two equilibrants.

**47. Two and Three Equilibrants by the Link Polygon.**—It was shown in Art. 45 how to find by the link polygon a single equilibrant to a system of non-concurrent forces all in the same plane. The system may be balanced by two or by three equilibrants to comply with three conditions, and two important cases will now be explained.

(1) *Two Equilibrants.*—Of $n$ non-concurrent coplanar forces in equilibrium given $n - 2$ completely, one by its line of action (*i.e.* position

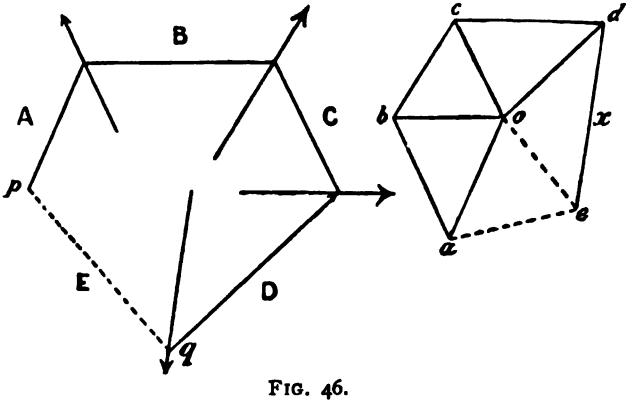



FIG. 46.

and direction) and another by a point on its line of action (position), to find completely the $n$ forces.

Let AB, BC, and CD (Fig. 46) be the lines of action of given forces represented in magnitude by *ab*, *bc*, and *cd* respectively in the vector polygon. Let ED be the line of action of one equilibrant, and $p$ a point in the line of action of the second. Draw a line, *dx*, of indefinite length parallel to DE. Choose a pole, *o*, and draw in the funicular polygon corresponding to it, *but drawing first the side AO through the given point p*. Let the last side DO cut ED in *q*. Then, since the complete funicular polygon is to be a closed figure, join *pq*. Then the vector *oe* is found by drawing a line, *oe*, through *o* parallel to *pq* to meet *dx* in *e*. The magnitude of the equilibrating force in the line DE is represented by the length *de*, and the magnitude and direction of the equilibrant EA through $p$ is given by the length and direction of *ea*.

*Algebraic Method.*—Find the magnitude of DE by equating its moment about $p$ to that of the known forces. Then including DE

in the known forces find the magnitude and direction of EA as for concurrent forces, Art. 44, viz. (for the equilibrant or R reversed)

$$EA = \sqrt{X^2 + Y^2} \qquad \tan\theta = -\frac{Y}{X}$$

(2) *Three Equilibrants.*—If three of the $n$ forces are given by their lines of action, produce two of them to meet and treat their intersection

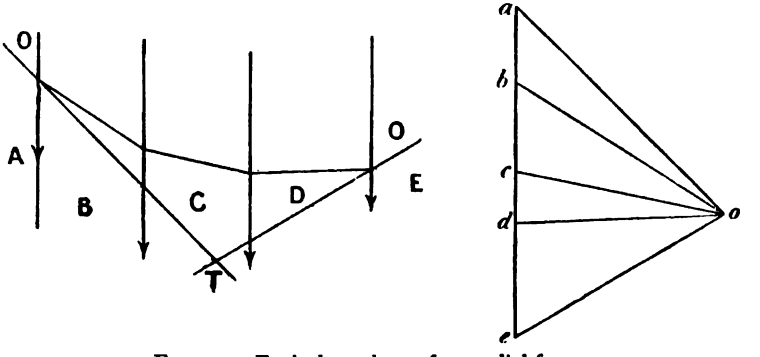



FIG. 47.—Funicular polygon for parallel forces.

as the point $p$ in Fig. 46, finally replacing the force through this point by two components along the lines which intersect there.

**48. Funicular Polygon for Parallel Forces.**—To find the equilibrant or resultant of several parallel forces the procedure is exactly the same as for non-parallel forces, but the vector polygon of forces has

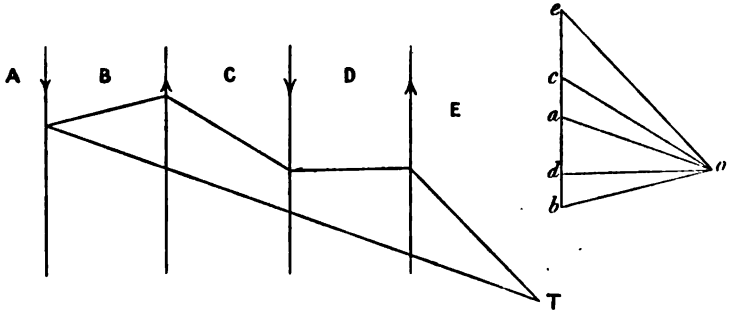



FIG. 48.—Funicular polygon for parallel forces.

its sides all in the same straight line; it is "closed" if after drawing the various consecutive vectors end to end the last one terminates at the starting point of the first one.

In Fig. 47 four forces AB, BC, CD, DE are given and their resultant is required, *abcde* is the open vector polygon, and the magnitude and direction of the resultant is given by *ae*. A pole is chosen at *o* which is joined to *a*, *b*, *c*, *d* and *e*. The funicular polygon

having sides parallel to *ao*, *bo*, *co*, etc., is then drawn in the space diagram, starting from any arbitrary point. The extreme sides AO and EO (parallel to *ao* and *eo*) intersect in T, and this gives a point in the line of action of the resultant AE and so fixes its position. The equilibrant of the four forces is a force EA given in magnitude and direction by *ea* acting through T.

Fig. 48 illustrates another case in which some of the forces are in one direction and some are in the opposite direction. The vector polygon *abcde* is set off as before; starting from *a* the equilibrant acts downward through the point T in which the extreme sides AO and EO (parallel to *ao* and *eo* respectively) intersect.

*Algebraic Method.*—The resultant R is equal to the algebraic sum of the several forces, hence the distance of the resultant from any point is

$$\frac{\Sigma(\text{moments about the point})}{\Sigma(\text{forces})}$$

both summations being merely algebraic.

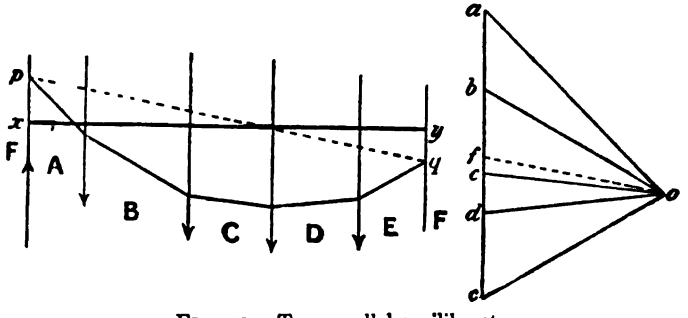



FIG. 49.—Two parallel equilibrants.

*Two Equilibrants.*—If two parallel equilibrants to the given (vertical) forces AB, BC, CD, DE are required through two given points $x$ and $y$ (Fig. 49), choose a pole, $o$, as before, and draw in the funicular polygon with side AO, BO, CO, DO, and EO respectively parallel to *ao*, *bo*, *co*, *do*, and *eo* in the vector diagram. Let AO meet the line FA (*i.e.* the vertical through $x$) in $p$, and let $q$ be the point in which EO meets the line EF (*i.e.* the vertical through $y$). Join $pq$, and from $o$ draw a parallel line $of$ to meet the line *abcde* in $f$. The magnitude of the upward equilibrant or supporting force in the line EF is represented by $ef$, and the other reaction in the line FA is represented by the vector $fa$. This may be proved in the same way as the proposition in Art. 45.

Another case is illustrated in Fig. 50 in which the two equilibrants FG and GA are not the extreme outside forces of the system; this presents no difficulty if the forces *ab*, *bc*, *cd*, *de*, *df* are set off continuously on the vector diagram and the spaces lettered accordingly.

Thus, the spaces C, E, and G extend as shown, from the lines BC to CD, DE to EF and FG to GA respectively.

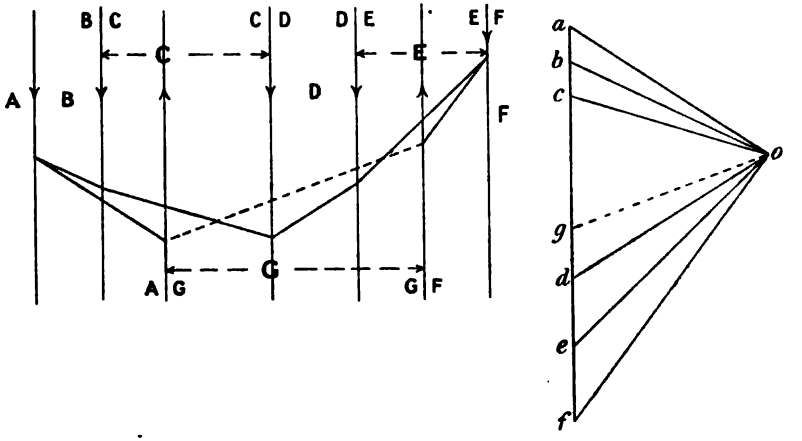



FIG. 50.—Two parallel equilibrants.

**49. Moments from Funicular Polygon.**—When a system of coplanar forces reduces to a couple it was shown in Art. 46 how the magnitude of the couple was found from a funicular polygon, viz. in Fig. 45 the

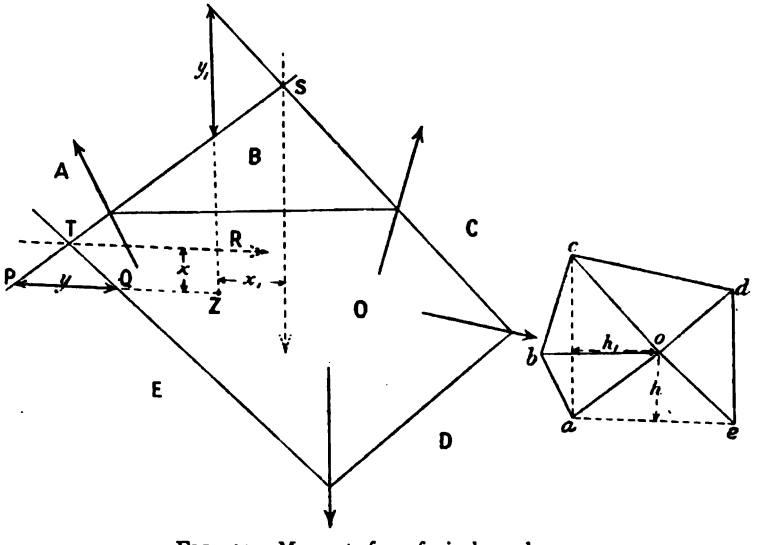



FIG. 51.—Moments from funicular polygon.

magnitude of the couple was given by *ao* multiplied by the distance between the lines TP and VW or by *ea* multiplied by the perpendicular

distance of VX from T. But if the forces are not equivalent to a couple or in equilibrium we may find their moment about any point from a funicular polygon. Thus, in Fig. 51 the four forces AB, BC, CD, DE reduce to a force R (represented by *ac*) through T, the intersection of the extreme sides AO and EO of the funicular polygon as in Art. 45 and Fig. 44. To find the moment of these forces about any point Z in their plane, draw a line PZ through Z parallel to the resultant *ae* to meet the extreme sides AO and EO of the funicular polygon in P and Q, then PQ or $y$ represents the moment of AB, BC, CD, and DE (or of R), about Z for the moment is $R \times x$ where $x$ is the perpendicular distance of R from Z. But since the triangles *aeo* and PQT are similar

$$\frac{PQ}{x} \text{ or } \frac{y}{x} = \frac{ae}{h} \quad \text{(see Fig. 51)}$$

where $h$ is the perpendicular distance of the pole $o$ from *ae*, hence

$$y = \frac{ae \times x}{h}$$

which is proportional to $R \times x$, or $y$ represents $R \times x$ on a scale dependent on $h$ and the scales of force and distance used in constructing the space and funicular polygons. If the force scale is $p$ lbs. to one inch and the distance scale is $q$ feet represented by one inch, and if $h$ measures $h$ inches, the scale on which $y$ represents the moment about Z is

$pqh$ lb.-ft. to one inch.

Similarly, to find the moments of, say, AB and BC about Z the extreme sides AO and CO are produced and a line drawn through Z intercepts a length $y_1$ between AO and CO, then

$$\frac{y_1}{x} = \frac{ae}{h_1}$$

and $y_1$ represents the moment of AB and BC (or *ac* acting through S) on a scale $pqh_1$ lb.-ft. to one inch.

Fig. 52 illustrates the same points as Fig. 51 but for a differently arranged set of forces, the notation being as in Fig. 51.

**50. Moments of Parallel Forces from Funicular Polygon.**—The case of parallel forces is specially simple, and very important, and is therefore treated separately in this article. Let AB, BC, CD, DE, EF (or $W_1$, $W_2$, $W_3$, $W_4$, and $W_5$) (Fig. 53) be five parallel (vertical) forces balanced by two equilibrants *fg* and *ga* (or $R_1$ and $R_2$). Let the funicular polygon for any pole $o$, starting, say, from $z$, be drawn as directed in Art. 48, *og* being drawn parallel to *zp* or GO, the closing line of the funicular, so that $R_1$, the left-hand equilibrant, is represented by the vector *ga* and $R_2$ by *fg*, while the loads $W_1$, $W_2$, $W_3$, $W_4$, and $W_5$ are represented by the vectors *ab*, *bc*, *cd*, *de*, and *ef* respectively. Consider any vertical line through X, at which the height of the link polygon is *xl*. Produce *xl* and the side *zw* to meet in $y$. Also produce the side *wm* of the funicular polygon to meet *xy* in *n*, and let the next side *mq* of the funicular meet *xy* in *l*. The sides *zw*, *wm*, and *mq* (or

AO, BO, and CO) are parallel to *ao*, *bo*, and *co* respectively. Draw a horizontal line, *zk*, through *z* to meet *xy* in *k*, a horizontal line through *w* to meet *xy* in *r*, and a horizontal *o*H through *o* in the vector polygon

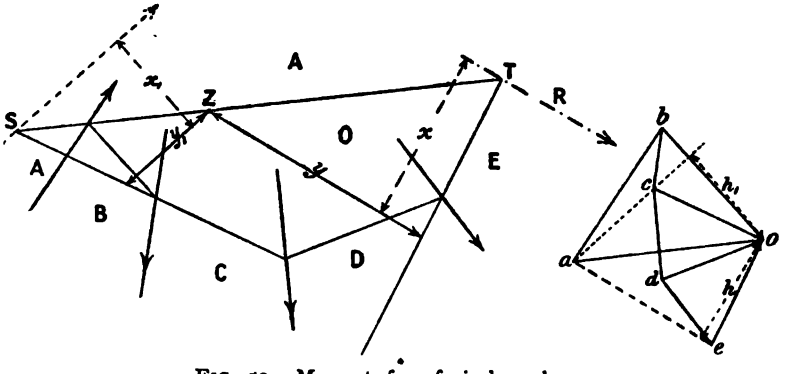



FIG. 52.—Moments from funicular polygon.

to meet the line *abcdef* in H. Then in the two triangles *xyz* and *gao* there are three sides in either parallel respectively to three sides in the other, hence the triangles are similar, and

$$\frac{xy}{ag} = \frac{zk}{oH}, \text{ or } xy \,.\, oH = ag \times zk, \text{ or } xy = \frac{ag \,.\, zk}{oH}$$

Therefore, since *ag* is proportional to $R_1$, and *zk* is equal or proportional to the distance of the line of action of $R_1$ from X, *ag* . *zk* is proportional

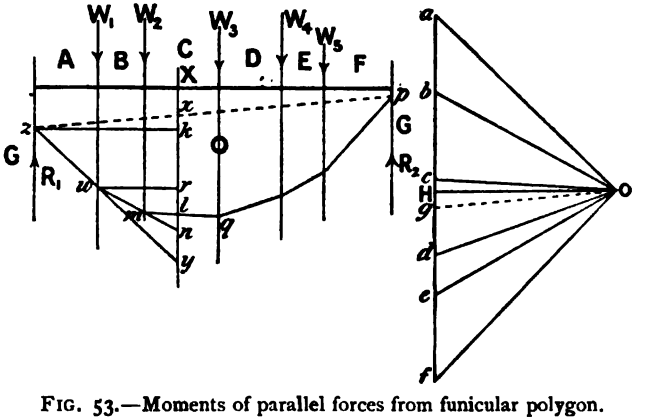



FIG. 53.—Moments of parallel forces from funicular polygon.

to the moment of $R_1$ about X, and *o*H being an arbitrarily fixed constant, *xy* is proportional to the moment of $R_1$ about X.

Similarly

$$yn = \frac{ab \,.\, wr}{oH}$$

and therefore represents the moment of $W_1$ about X to the same scale that $xy$ represents the moment of $R_1$ about X. Hence $xn$ or $xy - ny$ represents to the same scale the moment of the two forces R and W about X (or of their algebraic sum acting at their intersection of $xz$ and $wn$). Similarly $nl$ represents the moment of W about X to the same scale and

$$xl = xy - ny - ln$$

represents the moment about X of the three forces $R_1$, $W_1$ and $W_2$, or of their resultant (the algebraic sum) acting at the intersection of the lines $xz$ and $ln$. For any point in the plane, and for any number of parallel forces the proper intercept between the sides of the funicular polygon represents the moment and always to the same scale, since the distance from $o$ to any side of the vector polygon *abcdefg* is the same, viz. $o$H. For different pole distances ($o$H) the depth of closed link polygon will be inversely proportional to the pole distance.

*Scales.*—If the scale of forces in the vector diagram is

1 inch to $p$ lbs.

and the scale of distance in the space diagram is

1 inch to $q$ feet;

and if $o$H is made $h$ inches long, the scale on which the intercepts $xl$, $xn$, $xy$, $ny$, etc., represent the moments about X is

1 inch to $p.\ q.\ h.$ lb.-feet.

**51. Link Polygon to Given Conditions for Forces in One Plane.**—(*a*) *To pass through two given Points in the Plane.*—Let AB, BC, CD, and DE be given forces (Fig. 54), and let P and Q be any two points through which the link polygon is required to pass. Set out the vector polygon *abcde* and choose any pole $o_1$ and draw the corresponding link polygon, *starting through one of the given points*, say P. Let the last side (EO) parallel to $eo$ meet a line through Q parallel to $ae$ in T. Join PT, and from $o_1$ draw $o_1h$ parallel to TP. Then $eh$ and $ha$ represent parallel equilibrants (as in Art. 48, Fig. 49) through P and Q to the given forces and for all poles, link polygons for the given balanced system of six forces starting from P will have closing sides joining P to the line TQ and parallel to the line adjoining $h$ to the pole. Then in order that the polygon shall pass through Q the closing side must be the line PQ. Hence any polygon having its pole at $o_2$, $o_3$, or $o_4$, etc., on the line $ho_5$ through $h$ parallel to PQ will satisfy the conditions, and its extreme sides OA and OE will meet on the line of the resultant AE. To absolutely fix the polygon through P and Q it may be made to fulfil some additional condition. For instance, a certain side, say the side OD, may be made of a given inclination; the pole would then be at the intersection of a line at the given inclination through $d$ and the line $h$, $o_2$, $o_3$, $o_4$, $o_5$. Or again, the pole distance $ho$ or any link may be a certain specified length. Or the polygon may be made to pass through a third point.

(*b*) *To pass through three given Points in the Plane.*—Let AB, BC, CD, DE, and EF be given forces (Fig. 55), and let P, Q, and R be

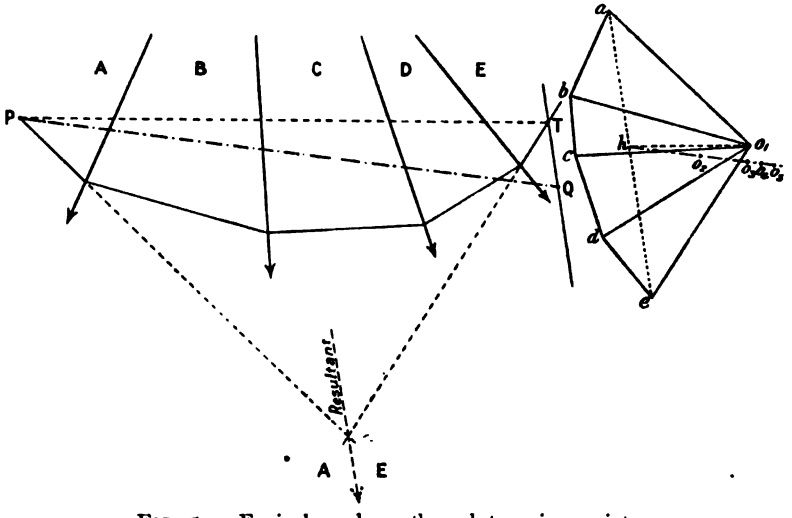



FIG. 54.—Funicular polygon through two given points.

three given points through which the link polygon is to pass. Draw the vector polygon *abcdef*, choose any pole $o_1$ and draw the link polygon, starting, say, the side OD parallel to $o_1d$ *through* P. Through Q draw

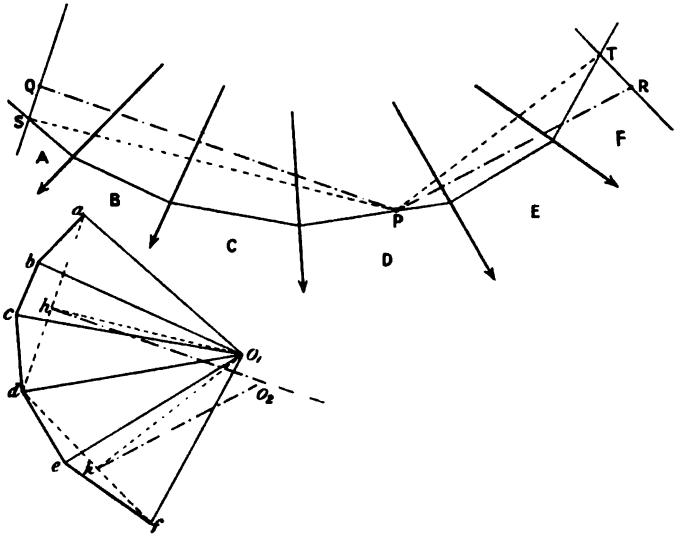



FIG. 55.—Funicular polygon through three given points.

a line QS parallel to *ad*, the resultant of AB, BC, and CD to meet the side AO in S. Join SP and through $o_1$, draw $o_1h$ parallel to SP to meet *ad* in *h*. Then all link polygons started through P and having their poles on the line $ho_2$ parallel to PQ will pass through P and Q as in the previous case. Similarly by drawing a line RT parallel to *df* to meet the side OF in T and a line $o_1k$ parallel to PT a point *k* in *df* is determined. And if through *k* a line $ko_2$ is drawn parallel to PR, all link polygons having their poles on $ko_2$ will pass through R. Hence finally $o_2$ being at the intersection of $ho_2$ and $ko_2$ is the pole of the required link polygon, and if a polygon having $o_2$ as pole be started through P (or R or Q), it will pass through the other two points; that it actually does so forms a check on the graphical work. The required link polygon is not shown on Fig. 55, as it would unnecessarily complicate the figure.

(*c*) *Special Case of Parallel Forces.*—The case of parallel forces (say vertical) presents no special features for the link polygon through two

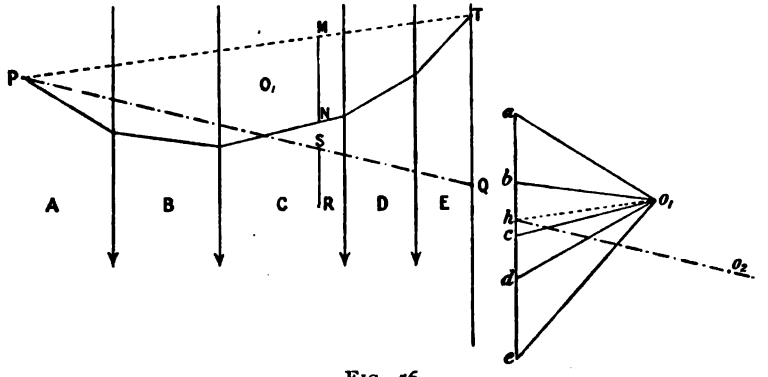



FIG. 56.

given points and satisfying one other condition, and Fig. 56 may be taken in place of Fig. 54. But if the third condition is that the polygon shall also pass through a third point R, there is a simple alternative solution to that given in case (*b*). Let the link polygon be drawn for the pole $o_1$ (Fig. 56). Then by Art. 50 the moment of all forces to right or left of a point in a vertical line through R is represented by MN, the height of the closed polygon in this vertical line, which is inversely proportional to the horizontal distance of $o_1$ from *abcde*. But in the polygon required passing through P, Q, and R the height in the same vertical line must be SR, *i.e.* the vertical distance of R from the line PQ. Hence the horizontal distance of the required pole $o_2$ from *abcde* is $\frac{MN}{SR}$ times the horizontal distance of $o_1$ from *abcde*. And if the line $ho_2$ is determined as before, the required pole $o_2$ is completely determined by the intersection of $ho_2$ with a vertical at the above distance from *abcde*.

**52. Moments, Centroids, and Moments of Inertia of Plane Areas.**—The moment of plane area about a line in its plane is the limit of the sum of products of small elements of the area and their perpendicular distances from that line. If $\delta A$ represents any element of a plane area A distant $y$ from a given line in its plane, the moment of the area is the limiting value of

$$\Sigma(y \cdot \delta A) \text{ or } \int y dA$$

The *centroid* of a plane area (also called the centre of gravity of the area), may be defined as a point in its plane such that the moment of the area about any line in the plane passing through that point is zero. Or for any line in the plane, through the centroid the product sum

$$\Sigma(y\delta A) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

*Central Axis.*—Such a line through the centroid is called a *central axis* of the figure. The distance $\bar{y}$ of the centroid from any other line in the plane is given by the equation

$$\bar{y} = \Sigma(y \cdot \delta A) \div A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The position of the centroid of simple geometrical figures is dealt with in books on elementary mechanics.[1]

*Moment of Inertia or Second Moment of a Plane Area.*—The moment of inertia (I) or second moment of the area about any axis in its plane is defined by the relation

$$I = \Sigma(y^2 \cdot \delta A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where values of $y$ are the distances of elements of area $\delta A$ from the axis about which the quantity I is to be estimated.

The calculation of the quantity I for various simple geometrical figures about various axes will now be briefly considered. The summation denoted by $\Sigma(y^2 \cdot \delta A)$ can often be easily carried out by ordinary integration. If A be the area of any plane figure and I its moment of inertia about an axis in its plane, the radius of gyration ($k$) of the area about that axis is defined by the relation

$$k^2 A = I = \Sigma(y^2 \cdot \delta A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or $k$ is that value of $y$ at which, if the area A were concentrated, the moment of inertia would be the same as that of the actual figure. Two simple theorems are very useful in calculating moments of inertia of plane figures made up of a combination of a number of parts of simple figures such as rectangles and circles.

*Theorem* (1).—The moment of inertia of any plane area about any axis in its plane exceeds that about a parallel line through its centre of gravity (or centroid) by an amount equal to the product of the area and the square of the distance of the centroid from the axis.

Otherwise, if I is the moment of inertia of an area A about any axis in the plane of the figure, and $I_G$ is the moment of inertia about

[1] Such as the Author's "Mechanics for Engineers."

a parallel central axis, *i.e.* a parallel axis through the centroid, and $l$ is the distance between the two axes

$$I = I_G + l^2A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or, dividing each term by A

$$k^2 = k_G^2 + l^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $k$ is the radius of gyration about any axis distance $l$ from the centroid and $k_G$ that about a parallel axis through the centroid. The proof of the theorem may be briefly stated as follows :—

$$\begin{aligned} I &= \Sigma\{(l+y)^2\delta A\} = \Sigma\{(l^2 + 2ly + y^2)\delta A\} \\ &= l^2\Sigma(\delta A) + 2l\Sigma(y\,.\,\delta A) + \Sigma(y^2\delta A) \\ &= l^2\,.\,A + o + I_G \end{aligned}$$

when $y$ is measured from an axis through the centroid.

*Theorem* (2).—The sum of the moments of inertia of any plane figure about two perpendicular axes in its plane is equal to the moment of inertia of the figure about an axis perpendicular to its plane passing through the intersection of the other two axes. Or, if $I_Z$, $I_X$, and $I_Y$ are the moments of inertia about three mutually perpendicular axes OZ, OX, and OY intersecting in O, OX and OY being in the plane of the figure

$$I_Z = I_X + I_Y$$

or $$\Sigma(r^2\,.\,\delta A) = \Sigma(y^2\,.\,\delta A) + \Sigma(x^2\delta A) \text{ or } \Sigma\{(x^2+y^2)\delta A\}$$

where $r$, $y$, and $x$ are the distances of any element of area $\delta A$ from OZ, OX, and OY respectively, since $r^2 = x^2 + y^2$.

*Rectangular Area.*—The moment of inertia of the rectangle ABCD, Fig. 57, about the axis XX may be found as follows, using the notation given in the figure, by taking strip elements of area $b\,.\,dy$ parallel to XX—

$$I_{XX} = \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2\,.\,bdy = \tfrac{1}{3}b\Big[y^3\Big]_{-\frac{d}{2}}^{\frac{d}{2}} = \tfrac{1}{12}bd^3$$

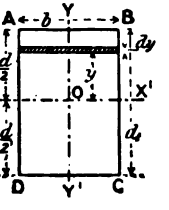


FIG. 57.

Similarly about YY—

$$I_{YY} = \tfrac{1}{12}db^3$$

About DC, by theorem (1) above—

$$I_{DC} = I_{XX} + bd\,.\left(\frac{d}{2}\right)^2 = bd^3(\tfrac{1}{12} + \tfrac{1}{4}) = \tfrac{1}{3}bd^3$$

which might also be obtained by integrating thus—

$$I_{DC} = \int_0^d by^2dy = \tfrac{1}{3}bd^3$$

$y$ being measured from DC.

*Hollow Rectangular Area and Symmetrical* I *Section.*—The moment of inertia about the axes XX of the two areas shown in Fig. 58 are equal, for the difference of distribution of the areas in a direction

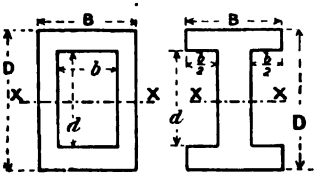



FIG. 58.

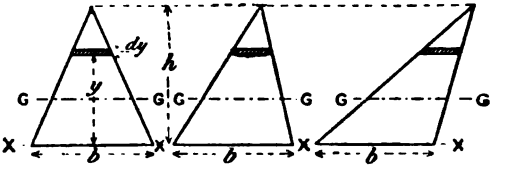



FIG. 59.

parallel to XX does not alter the moment of inertia about that line. In either case

$$I_{XX} = \tfrac{1}{12}(BD^3 - bd^3)$$

*Triangular Area.*—For any of the triangles shown in Fig. 59 about the base $b$

$$I_{XX} = \int_0^h b \times \frac{h-y}{h} y^2 dy = \frac{b}{h}\int_0^h (hy^2 - y^3)dy = \tfrac{1}{12}bh^3$$

and using theorem (1), about a parallel axis GG through the centroid

$$I_{GG} = I_{XX} - \tfrac{1}{2}bh(\tfrac{1}{3}h)^2 = \tfrac{1}{36}bh^3$$

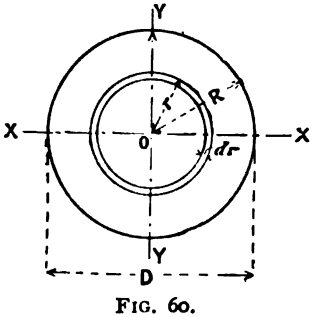



FIG. 60.

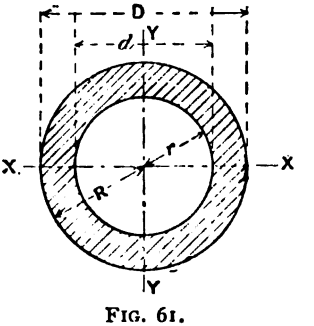



FIG. 61.

*Circular Area.*—The moment of inertia $I_0$ about an axis perpendicular to the circular surface and through its centre (Fig. 60) is found by taking circular strips of radius $r$ and width $dr$.

$$I_0 = \int_0^R r^2 . 2\pi r dr = 2\pi \frac{R^4}{4} = \tfrac{1}{2}\pi R^4 \text{ or } \frac{\pi}{32}D^4$$

Using theorem (2)

$$I_0 = I_{XX} + I_{YY}$$

where $I_{XX}$ and $I_{YY}$ are the moments of inertia about two perpendicular diameters XX and YY; and since by symmetry $I_{XX} = I_{YY}$

$$I_0 = 2I_{XX} = 2I_{YY}$$

and $$I_{XX} = I_{YY} = \tfrac{1}{4}\pi R^4 \text{ or } \frac{\pi}{64}D^4$$

which might easily be established by taking straight strips parallel to XX or YY.

*Circular Ring Area.*—Evidently, from the above result, if $I_0$ is the moment of inertia about a central axis perpendicular to the plane of Fig. 61

$$I_0 = \frac{\pi}{2}(R^4 - r^4) \text{ or } \frac{\pi}{32}(D^4 - d^4)$$

and $$I_{XX} = I_{YY} = \frac{\pi}{4}(R^4 - r^4) \text{ or } \frac{\pi}{64}(D^4 - d^4)$$

**I**-*Shaped Sections.*—The moment of inertia, etc., of a rolled **I** section such as that in Fig. 62 may generally be calculated by dividing

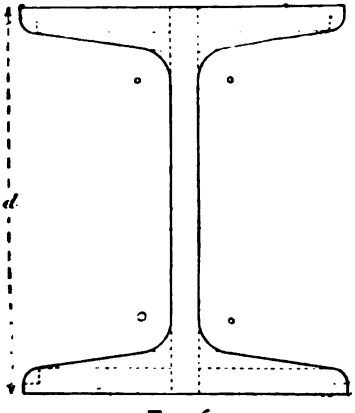



FIG. 62.

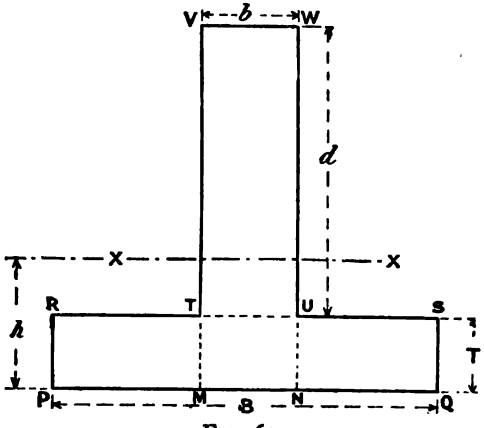



FIG. 63.

it into rectangles, triangles, circular sections, and spandrils as shown, and applying theorem (1), but such a process is very laborious and leads to a result of perhaps needless exactness, for all the dimensions, though specified with great precision, could scarcely be adhered to in manufacture with similar exactness. The moments of inertia of the sections recommended by the Engineering Standards Committee have been worked out by the exact method and tabulated (see Appendix). A graphical method suitable for any kind of section is given in the next article.

**T** *Sections, etc.*—These sections will usually have rounded corners, and if they are known exactly, the moment of inertia may be calculated by division, as in Fig. 62. If, however, the rounding is neglected and

the section regarded as consisting of rectangles, as in Fig. 63, we may proceed as follows. Find the distance $h$ of the centre of gravity or centroid from the edge PQ by the methods of moments, thus

$$h\{(B.T) + (b.d)\} = (B.T.\tfrac{1}{2}T) + (b.d)(T + \tfrac{1}{2}d)$$

from which $h$ can be found.

Then find the moment of inertia $I_{PQ}$ about PQ, taking the rectangles PRSQ and VWUT

$$I_{PQ} = \tfrac{1}{3}B.T^3 + \tfrac{1}{12}b.d^3 + b.d(T + \tfrac{1}{2}d)^2$$

or taking the rectangles VWNM and twice RTMP

$$I_{PQ} = \tfrac{1}{3}(B - b)T^3 + \tfrac{1}{3}b(T + d)^3$$

Having found $I_{PQ}$, apply theorem (1), Art. 66, whence

$$I_{XX} = I_{PQ} - (BT + bd)h^2$$

Another alternative would be to find $I_{XX}$ directly by subdivision into rectangles and application of theorem (1); as $h$ will not generally be so simple a number as the main dimensions, this will generally involve multiplications of rather less simple figures than in the above methods.

Yet another plan would be to find the moment of inertia about VW, thus

$$I_{VW} = \tfrac{1}{3}B(d + T)^3 - \tfrac{1}{3}(B - b)d^3$$

and then apply theorem (1), to find $I_{XX}$.

Precisely similar principles may be applied to find the moment of inertia of any section divisible into rectangles and not symmetrical about the neutral axis, *e.g.* that in Fig. 102.

**53. Graphical Determination of Moments, Centroids, and Moments of Inertia of Areas.**—To determine the moment and moment of inertia (or second moment) of sections which are not made up of simple geometrical figures, some approximate form of estimation must generally be employed, and a graphical method offers a convenient solution. Of the various graphical methods, probably the following is the simplest, a planimeter being used to measure the areas.

To find the moment and moment of inertia of any plane figure APQB (Fig. 64), about any axis XX, and the moment of inertia about a parallel axis through the centroid. Draw any line SS parallel to XX and distance $d$ from it; choose any pole O in XX, preferably the point nearest to the figure APQB. Draw a number of lines, such as PQ and AB across the figure parallel to XX. From the extremities P and Q, etc., project lines perpendicular to SS, meeting it in N and M, etc. Join such points as N and M to O by lines meeting PQ in $P_1$ and $Q_1$, AB in $A_1$ and $B_1$, etc. Through the points so derived, draw in the modified or first derived area $P_1Q_1B_1A_1$. Repeat the process on this figure, projecting $P_1Q_1$ at $N_1M_1$ and obtaining $P_2Q_2$ and a second modified figure or derived area $P_2Q_2B_2A_2$. Then

(First derived area $P_1Q_1B_1A_1) \times d$ = moment of area PQBA about the line XX, or $\Sigma(y \,.\, \delta A)$;

and $$I_{XX} = \text{area } P_2Q_2B_2A_2 \times d^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or second moment of area PQBA about XX.

And about a parallel axis through the centre of gravity

$$I_G = I_{XX} - \frac{(\text{area } P_1Q_1B_1A_1)^2}{(\text{area PQBA})} \,.\, d^2 \quad . \quad . \quad . \quad . \quad (2)$$

*Proof.*—Let the areas PQBA, $P_1Q_1B_1A_1$ and $P_2Q_2B_2A_2$ be represented by A, $A_1$, and $A_2$ respectively, and their width at any distance $y$ from XX be denoted by $z$, $z_1$, and $z_2$ respectively. Then elementary strips

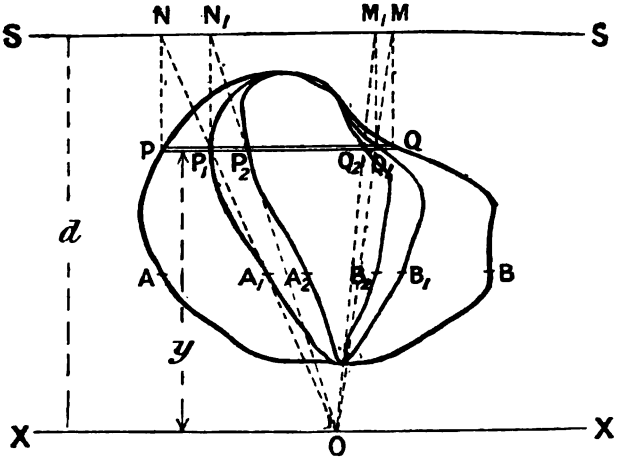



FIG. 64.

PQ, $P_1Q_1$, and $P_2Q_2$, or $\delta A$, $\delta A_1$, and $\delta A_2$ of area are respectively equal to $z \,.\, dy$, $z_1 \,.\, dy$, and $z_2 \,.\, dy$.

In the first derived figure, a strip PQ is reduced to $P_1Q_1$ in the ratio $y$ to $d$, or

$$\delta A_1 = \frac{y}{d} \,.\, \delta A \quad \text{or} \quad z_1 dy = \frac{y}{d} \,.\, z \,.\, dy$$

Taking the sums

$$A_1, \text{ or } \Sigma(\delta A_1), \text{ or } \Sigma\left(\frac{y}{d} \,.\, \delta A\right) = \frac{1}{d}\Sigma(y \,.\, \delta A) = \frac{1}{d}\Sigma(y \,.\, z \,.\, dy)$$

or in integral form

$$\int z_1 dy = \frac{1}{d}\int yz \,.\, dy$$

The area $A_1$ or $\Sigma(\delta A_1)$ is therefore proportional to the moment of the area A about XX, which is equal to $A_1 \,.\, d$.

Then the centroid of the area A is at a distance $\bar{y}$ from XX given by

$$\bar{y} = \frac{\Sigma(y \,.\, \delta A)}{\Sigma(\delta A)} = \frac{A_1}{A} \,.\, d \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, in the second derived figure the strip $P_1Q_1$ is further reduced to $P_2Q_2$ in the ratio $\frac{y}{d}$ and

$$\delta A_2 = \frac{y}{d} \,.\, \delta A_1 = \frac{y^2}{d^2} \,.\, \delta A \text{ or } z_2 dy = \frac{y}{d} \,.\, z_1 \,.\, dy = \frac{y^2}{d^2} \,.\, z \,.\, dy$$

And taking the sums

$$A_2 \text{ or } \Sigma(\delta A_2), \text{ or } \Sigma\left(\frac{y}{d} \,.\, \delta A_1\right) = \frac{1}{d} \Sigma(y \,.\, \delta A_1) = \frac{1}{d^2} \Sigma(y^2 \,.\, \delta A)$$

or

$$\int z_2 dy = \frac{1}{d} \int y \,.\, z_1 \,.\, dy = \frac{1}{d^2} \int y^2 \,.\, z \,.\, dy$$

The area $A_2$ is therefore proportional to the moment of inertia or second moment of the area A about XX, which is equal to $A_2 \times d^2$, or

$$I_X = A_2 \,.\, d^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

And since the distance of the centre of gravity of A from XX is $\frac{A_1}{A} \,.\, d$, by theorem (1) of Art. 66

$$I_G = A_2 \,.\, d^2 - A\bar{y}^2 = A_2 d^2 - A\left(\frac{A_1}{A}\right)^2 d^2 = d^2\left(A_2 - \frac{A_1^2}{A}\right) \quad (5)$$

A slightly modified construction is shown in Fig. 65, where, instead of using a constant pole as at O in Fig. 64, a different one is used for each line, such as PQ or AB, across the area PQBA, viz. the foot of the

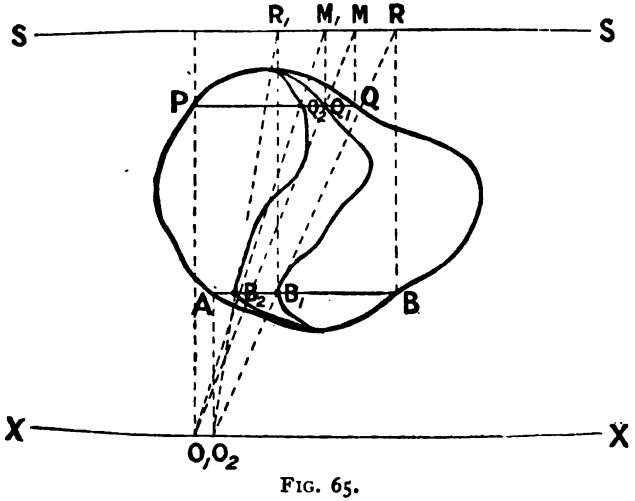



FIG. 65.

perpendicular from the points such as P or A on XX; by this means the left-hand sides of the perimeter of the original and derived areas are the same, the areas A, $A_1$, and $A_2$ being shown by PQBA, $PQ_1B_1A$, and $PQ_2B_2A$ respectively. This construction is rather easier to use in

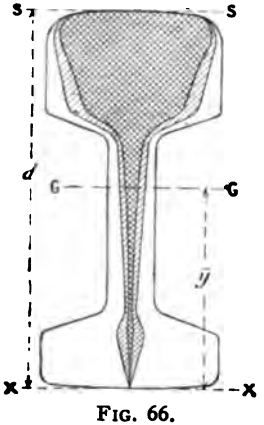



FIG. 66.

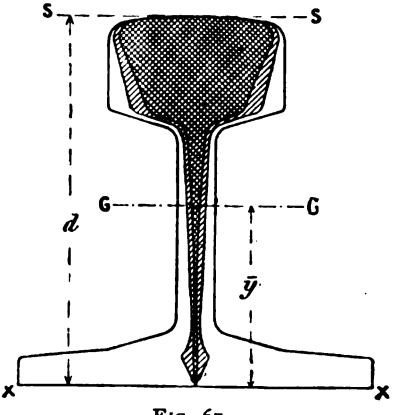



FIG. 67.

many cases, and with the same care should give rather better results than the previous one for areas which are not symmetrical about an axis perpendicular to XX. In a similar manner the $n$th moment of the area about XX is equal to $A_n \cdot d^n$, where $A_n$ is the $n$th derived area.

Illustrations of these graphical methods are shown in Figs. 66 to 72 inclusive. Figs. 66 and 67 represent rail sections, the centroid and moment of inertia being found as in Fig. 64. Figs. 68 and 69*a* represent the modified construction of Fig. 65 applied to the same rails as those in Figs. 66 and 67. Figs. 70 and 71 represent symmetrical I beam sections, the moment of inertia being found as in Fig. 64; but in Fig. 71 the moment of inertia about the central axis GG is found directly for half the section without the use of theorem (1), Art. 52. In this case twice the inner area multiplied by $\left(\frac{d}{2}\right)^2$ gives the moment of inertia of the section about GG.

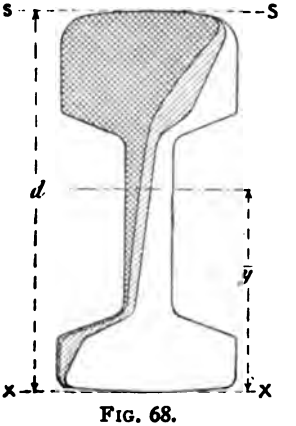



FIG. 68.

This method is preferable to that adopted in Fig. 70, for if the moment of inertia about GG is found by subtraction as in (5) a given percentage error in measuring areas will give rise to a larger percentage error in $I_G$. A similar method although involving more labour would be applicable to unsymmetrical areas such as Figs. 66, 67, 68, and

69 after the centroid has been determined: derived areas on each side of the central axis would be required.

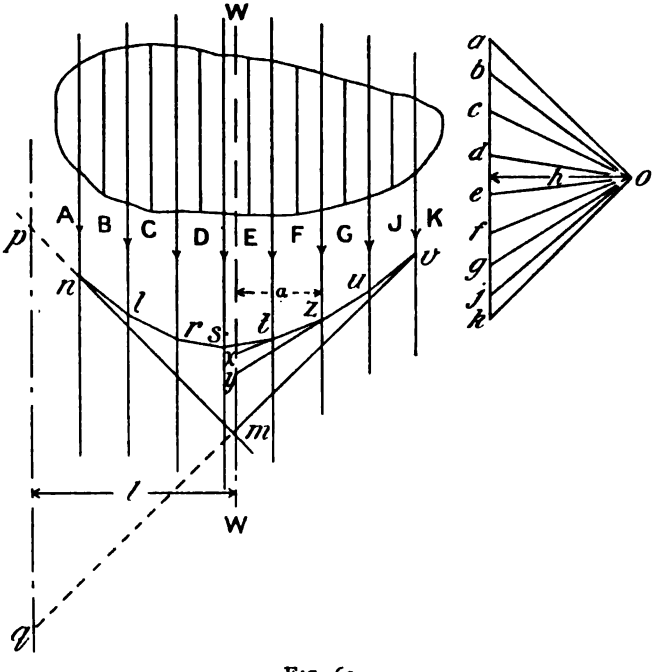



FIG. 69.

Fig. 72 shows the alternative construction of Fig. 65 applied to the same section as that in Fig. 71. The first derived area of a beam

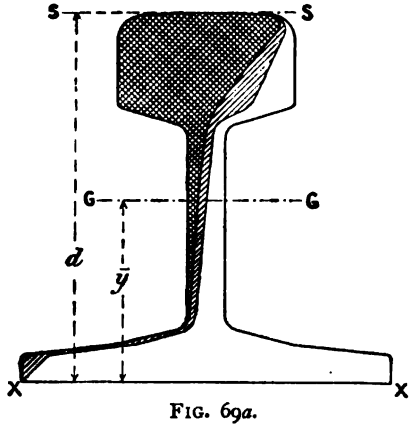



FIG. 69a.

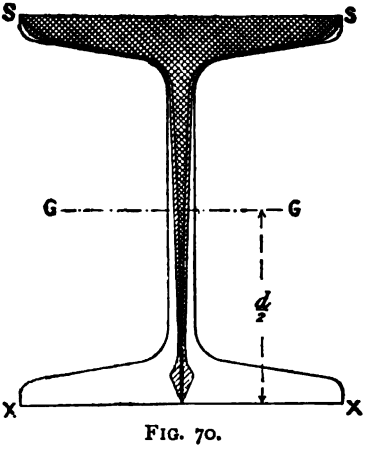



FIG. 70.

section as shown in Figs. 71 and 72 is sometimes called a *modulus figure.*

*Alternative Graphical Methods.*—The centroid of an area may be found by the methods used to find the centre of a system of parallel forces. In the case of an irregular area, if it be divided into parts and each part looked upon as a force acting at the centroid of the partial area an axis through the centroid of the whole area may be found arithmetically from formula (2), Art. 52, or graphically by the link polygon as in Art. 48. This is illustrated in Fig. 69, where an irregular figure is divided into eight strips by parallel lines, the areas of the strips being set off to scale at *ab*, *bc*, *cd*, *de*, etc., and the funicular polygon *nlrstuvm* is drawn. The intersection of the extreme sides AO and KO is at *m*, and the axis WW through *m* passes through the centroid

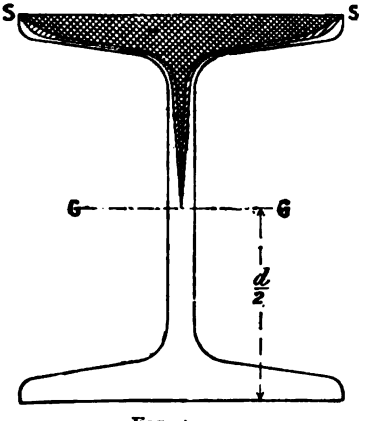



FIG. 71.

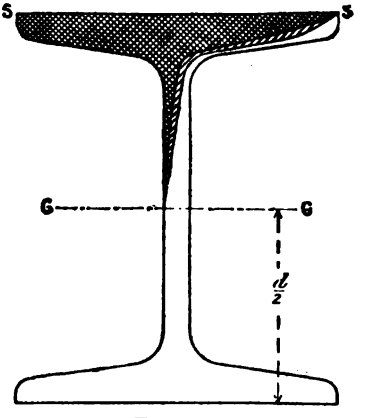



FIG. 72.

of the area. A second axis containing the centroid can, if necessary, be found.

The moment of inertia of the irregular area A, say, may also be found from Fig. 69. Consider the partial area $\delta A$, say, represented by *fg* in the line FG. Let *xy* be the intercept of the sides FO and GO of the link polygon on the central axis WW which is proportional to the moment of $\delta A$ about WW. As in Art. 50 if $a$ is the distance of the line FG from the parallel axis WW,

$$\frac{xy}{a} = \frac{fg}{h} \text{ or } xy = \frac{fg \times a}{h}$$

$$xy \times a = \frac{fg \times a^2}{h}$$

and if the area of the triangle *xyz* is $\delta A'$

$$\delta A' = \tfrac{1}{2}\, xy \times a = \frac{fg}{2h} \times a^2$$

A similar relation holds for each element of the figure, and if A′ be

the area of the whole figure, *nlrstzuvm* made up of triangular areas similar to *xyz*,

$$A' = \frac{\Sigma(fg \cdot a^2)}{2h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$\Sigma(fg \times a^2)$ being proportional to the moment of inertia I, of the irregular figure about WW; hence A′ represents I. Further, if $h$ be taken equal to half the length $ak$, it represents $\frac{1}{2}$A on the same scale that the lengths $fg$, etc., represent the parts $\delta$A, and we may write (6) as

$$A' = \frac{\Sigma(\delta A \cdot a^2)}{2 \times \frac{1}{2}A} \text{ or } A \cdot A' = \Sigma(\delta A \cdot a^2) = I \quad . \quad . \quad . \quad (7)$$

thus the product of the original area A and the area A′ is *equal* to the moment of inertia of A about the central axis WW, or the area A′ is equal to the square of the radius of gyration ($k^2$), about the axis WW, and provided $h = \frac{1}{2} \cdot ak$ the scale of the vector polygon *abc*...*koa* is immaterial. The same construction would hold good for any other axis such as *pq* parallel to WW, and the moment of inertia *pq* being A multiplied by the area *pnl*...*uvq*. This illustrates theorem (1) of Art. 52, the triangle *pmq* evidently being equal to the product A × $l^2$. The value of the result in using this method depends upon the degree of subdivision, and actually the polygon *nlrstzuv* should be a smooth curve touching the sides and giving an increased area A′.

Another graphical method of finding the "second moment" is to find the intercepts on WW from a second link polygon of which the first intercepts *xy* (with due regard to sign) form the vector polygon. But the method given first in this article is probably the best to employ.

**54. Ellipse of Inertia, or Momental Ellipse.**—*Principal Axes of a Section.*—The principal axes OX and OY of a plane area may be defined as the rectangular axes in its plane, and through the centroid such that the sum $\Sigma(xy \cdot \delta A)$, called the *product of inertia* (or product moment), is zero, $x$ and $y$ being the rectangular co-ordinates of an element $\delta$A of the area with reference to OX and OY.

Let
$$\Sigma(y^2 \cdot \delta A) = I_x = k_x^2 \cdot \Sigma(\delta A)$$
$$\Sigma(x^2 \cdot \delta A) = I_y = k_y^2 \Sigma(\delta A)$$

Then the moment of inertia of the area about any perpendicular axes OX′ and OY′ in its plane when OX′ is inclined at an angle $\alpha$ to OX may be found by writing from the right-hand side of Fig. 73 for the co-ordinates ($x'$, $y'$) of any point P,

$$OM = x' = x \cos \alpha + y \sin \alpha$$
$$PM = y' = y \cos \alpha - x \sin \alpha$$

hence $I_{y'} = \Sigma(x'^2 \cdot \delta A) = \cos^2 \alpha \Sigma(x^2 \delta A) + \sin^2 \alpha \Sigma(y^2 \delta A) + 2 \sin \alpha \cos \alpha \Sigma(xy \delta A)$

or
$$\left.\begin{aligned} I_{y'} &= I_y \cos^2 \alpha + I_x \sin^2 \alpha \\ k_{y'}^2 &= k_y^2 \cos^2 \alpha + k_x^2 \sin^2 \alpha \end{aligned}\right\} \text{ since } \Sigma(xy\delta A) = 0 \quad . \quad . \quad (1)$$

Also similarly

$$\left.\begin{aligned} I_{x'} &= I_x \cos^2 \alpha + I_y \sin^2 \alpha \\ k_{x'} &= k_x^2 \cos^2 \alpha + k_y^2 \sin^2 \alpha \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

Adding (1) and (2)

$$\left.\begin{aligned} I_{x'} + I_{y'} &= I_x + I_y \\ k_{x'}^2 + k_{y'}^2 &= k_x^2 + k_y^2 \end{aligned}\right\} = \text{constant} \quad \ldots\ldots \quad (3)$$

A result which follows directly from theorem (2) Art. 52.

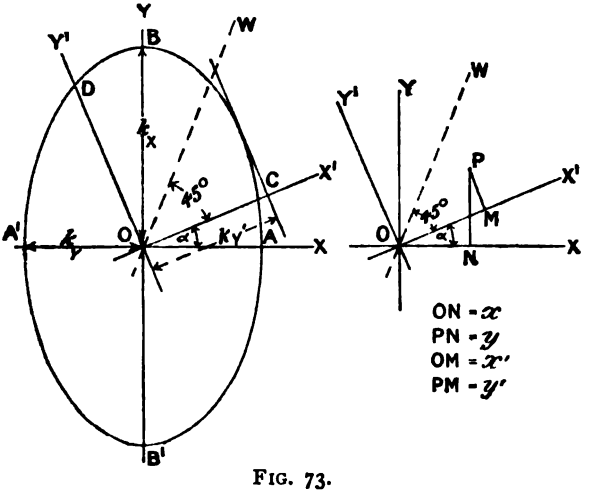



FIG. 73.

If OA = OA', Fig. 73, be set off to represent $k_y$ and OB = OB' to represent $k_x$ and an ellipse ABA'B' be drawn with OA and OB as semi-principal axes, then $k_{y'}$ is represented by OC, the perpendicular distance from the centre O to the tangent parallel to OY' when OX' and OY' are inclined as shown at an angle $\alpha$ to OX and OY respectively. For a property of the ellipse is

$$OC^2 = OA^2 \cos^2 \alpha + OB^2 \sin^2 \alpha$$

which is the relation given by (1). This *momental ellipse* then shows the radius of gyration about any axis, such as OY' by the length of the perpendicular from O on the tangent parallel to OY'. Also since the product OD . OC is constant in an ellipse (viz. equal to OA . OB), the radius of gyration about any axis such as OY' is inversely proportional to the radius vector OD in that direction. Its value is

$$k_{y'} = \frac{k_x \cdot k_y}{OD}$$

If a curve be drawn such that every radius vector measured from O is proportional to the square of $k$, *i.e.* proportional to I about that radius vector, it is called an *inertia curve* for the given section. The radius vector in the direction OX', for example, would be given by equation (2), and others might be found similarly.

It is evident by differentiating (1) with respect to $\alpha$, or by inspection of the ellipse, that $k$ has maximum and minimum values, $k_x$ and $k_y$, the values of $k$ about the two principal axes. It is often important to find the minimum value of $k$ (and I) of a given section, and therefore to find the principal axes. If the section has an axis of symmetry that is evidently one principal axis, for from the symmetry the sum $\Sigma(xy \cdot \delta A)$ must be zero. The other principal axis is then at right angles to the first, and through the centroid of the section; a case in point is an angle section with equal sides.

If a plane figure (such as a circular or square section) has more than two axes of symmetry, its momental ellipse becomes a circle, and its moment of inertia about every axis in its plane and through the centroid is the same. If a section has not an axis of symmetry the principal axis and the principal or maximum and minimum moments of inertia may be found from the moments of inertia about two perpendicular axes OX' and OY', say, and the moment of inertia about a third axis OW, Fig. 73, inclined 45° to each of the other two; these three moments of inertia may be found by the methods described in the preceding articles. Let $I_w$ be the moment of inertia about OW. Then applying (2)

$$I_w = I_x \cos^2(\alpha + 45^\circ) + I_y \sin^2(\alpha + 45^\circ) = \tfrac{1}{2}I_x(1 - \sin 2\alpha) + \tfrac{1}{2}I_y(1 + \sin 2\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$2I_w = I_x + I_y + (I_y - I_x) \sin 2\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence by (3) $$(I_y - I_x) \sin 2\alpha = 2I_w - (I_{x'} + I_{y'}) \quad . \quad . \quad (6)$$

and subtracting (2) from (1)

$$(I_y - I_x) \cos 2\alpha = I_{y'} - I_{x'} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Dividing (6) by (7)

$$\tan 2\alpha = \frac{2I_w - (I_{x'} + I_{y'})}{I_{y'} - I_{x'}} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which determines the directions of the principal axes, $\alpha$ to be measured from OX' in the direction opposite to OW.

Also from (3) and (7)

$$I_x = \tfrac{1}{2}\{I_{x'} + I_{y'} + (I_{x'} - I_{y'}) \sec 2\alpha\} \quad . \quad . \quad . \quad (9)$$
$$I_y = \tfrac{1}{2}\{I_{x'} + I_{y'} - (I_{x'} - I_{y'}) \sec 2\alpha\} \quad . \quad . \quad . \quad (10)$$

which gives the principal moments of inertia in terms of the three known moments of inertia.

## EXAMPLES III.

1. ABCD is a square each side being 20 inches, and E is the middle point of AB. Forces of 7, 8, 12, 5, 9, and 6 lbs. act on a body in the lines and directions AB, EC, BC, BD, CA and DE respectively. Find the magnitude and position of the single force required to keep the body in

equilibrium, state its magnitude, its distance from A, and its inclination to AD.

2. A horizontal beam 15 feet long resting on supports at its ends carries concentrated vertical loads of 7, 9, 5, and 8 tons at distances of 3, 8, 12, and 14 feet respectively from the left-hand support. Find the reactions at the supports.

3. A beam 20 feet long rests on two supports 16 feet apart, and overhangs the left-hand support 3 feet, and the right-hand support by 1 foot. It carries a load of 5 tons at the left-hand end of the beam, and one of 7 tons midway between the supports. The weight of the beam is one ton. Find the reactions at the supports.

4. A horizontal rod AB, 13 feet long, is supported by a horizontal hinge perpendicular to AB at A, and by a vertical upward force at B. Four forces of 8, 5, 12, and 17 lbs. act upon the rod, their lines of action cutting AB at 1, 4, 8, and 12 feet respectively from A, their lines of action making angles of 70°, 90°, 120°, and 135° respectively with the direction AB, each estimated in a clockwise direction. Find the pressure exerted on the hinge, state its magnitude, and its inclination to AB.

5. A horizontal beam AB is 10 feet long. Draw on it as a chord an arc of a circle subtending an angle of 120° at the centre. Divide the arc into 8 equal parts at A, C, D, E, F, G, H, K, B, and the line AB into 8 equal parts at A, L, M, N, O, P, Q, R, B. Forces of 3, 4, 2, 4·5, 3 and 7 cwts. act in the lines CL, DM, EN, FP, HQ and KR respectively. Draw (*a*) the link polygon through A and B, the side through A being inclined 45° to the horizontal; (*b*) the link polygon passing through AB and G.

6. An I-section of a girder is made up of three rectangles, viz. two flanges having their long sides horizontal, and one web connecting them having its long side vertical. The top flange section is 6 inches by 1 inch, and that of the bottom flange is 12 inches by 2 inches. The web section is 8 inches deep and 1 inch broad. Find the height of the c.g. of the area of cross-section from the bottom of the lower flange.

7. Find the c.g. of a T section, the height over all being 8 inches, and the flange width 6 inches, the metal being $\frac{1}{8}$ inch thick in the vertical web, and 1 inch thick in the horizontal flange.

8. A girder of I-shaped cross-section has two horizontal flanges 5 inches broad and 1 inch thick, connected by a vertical web 9 inches high and 1 inch thick. Find the "moment of inertia of the area" of the section about a horizontal axis in the plane of the section and through its c.g. or centroid.

9. Find the moment of inertia and radius of gyration of the area of the section in Problem 7 about an axis through the c.g. of the section and parallel to the flange.

Numerous examples on centroids, and moments of inertia, etc., of plane areas may be found in the tables of standard sections (see tables in the Appendix).

# CHAPTER IV

## *BENDING MOMENTS AND SHEARING FORCES*

**55. Beams and Bending.**—A bar of material acted on by external forces (including loads and reactions) oblique to its longitudinal axis is called a beam, and the components perpendicular to the axis cause the straining called flexure or bending. This and the following four chapters deal only with beams which are straight or nearly straight. As beams are frequently horizontal, and the external forces are weights, it will be convenient to speak always of the beams as being horizontal and the external forces as vertical, although the same conclusions would hold in other cases. Members of structures are often beams as well as struts or ties; that is, there are some transverse forces acting upon them in addition to longitudinal ones.

**56. Straining Actions on Beams. Shearing Force and Bending Moment.**—Before investigating the stresses and strains set up in bending, the straining actions resulting from various systems of loading and supporting beams will be considered.

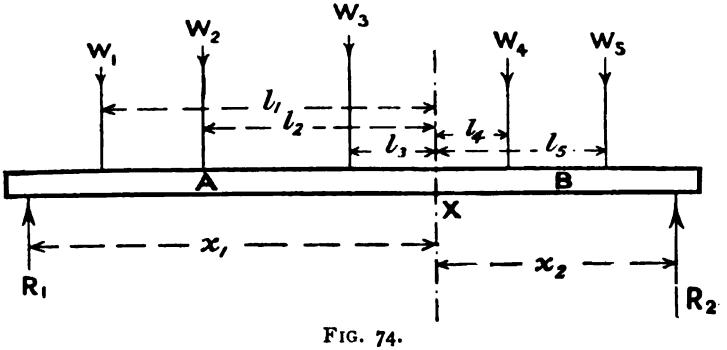



FIG. 74.

If we consider a beam carrying a number of transverse loads, as in Fig. 74, the whole beam is in equilibrium under the action of the loads $W_1$, $W_2$, $W_3$, etc., and the supporting forces or reactions $R_1$ and $R_2$; further, if we divide the beam into two parts A and B by an ideal plane of section X, each part is in equilibrium. The system which keeps A in equilibrium consists of the forces $W_1$, $W_2$, $W_3$, and $R_1$,

together with the forces exerted on A by B across the section X in virtue of the state of stress in the beam. We may conveniently consider these latter forces by estimating their total horizontal and vertical components and their moments. Applying the ordinary conditions of equilibrium, Art. 46, we conclude—

(1) Since there are no horizontal forces acting on the piece A except those across the section X, the algebraic total horizontal component of those forces is zero.

(2) Since the algebraic sum of the vertical *downward* forces on A is

$$W_1 + W_2 + W_3 - R_1$$

the total or resultant *upward* vertical force exerted by B on A is $W_1 + W_2 + W_3 - R_1$, which is also equal to an upward force

$$R_2 - (W_4 + W_5)$$

*Shearing Force.*—The resultant vertical force exerted by B on A is then equal to the algebraic sum of the vertical forces on either side of the plane of section X; the action of A on B is equal and opposite. This total vertical component is the *shearing force* on the section in question.

(3) If the distances of $W_1$, $W_2$, $W_3$, and $R_1$ from X be $l_1$, $l_2$, $l_3$, and $x_1$ respectively the moment of the external forces on A about the section X is

$$M = R_1x_1 - W_1l_1 - W_2l_2 - W_3l_3$$

which is also equal to $Wl_4 + W_5l_5 - R_2x_2$, and is of clockwise sense if the above expressions are positive. The moment exerted by B on A must balance the above sum, and is therefore of equal magnitude.

*Bending Moment.*—The above quantity M is the algebraic sum of the moments of all the forces on either side of the section considered, and is called the *bending moment.* The balancing moment which B exerts on A is called the *moment of resistance* of the beam at that section. The statical conditions of equilibrium show that the moment of resistance and the bending moment are numerically equal.

**57. Diagrams of Shearing Force and Bending Moment.**—Both shearing force and bending moment will generally vary in magnitude from point to point along the length of a loaded beam; their values at any given cross-section can often be calculated arithmetically, or general algebraic expressions may give the bending moment and shearing force for any section along the beam. The variation may also be shown graphically by plotting curves the basis of which represent to scale the length of the beam, and the vertical ordinates the bending moments or shearing forces, as the case may be. Some simple typical examples of bending moment and shearing force curves follow in Figs. 75 to 89, inclusive. In each case M represents bending moment, F shearing force, and R reaction or supporting force, with

appropriate suffixes to denote the position to which the letters refer. Other cases of bending-moment and shearing-force diagrams will be

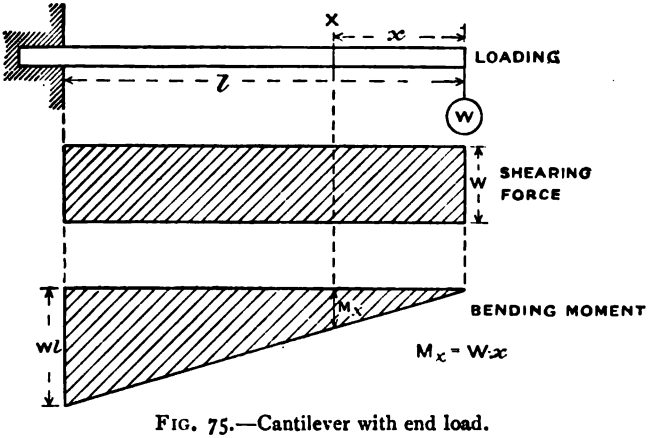



FIG. 75.—Cantilever with end load.

dealt with later (see Arts. 100 to 105 and 121). In the case of moving loads the straining actions change with the position of the load; such cases are dealt with in Chapter VI. When a beam carries several

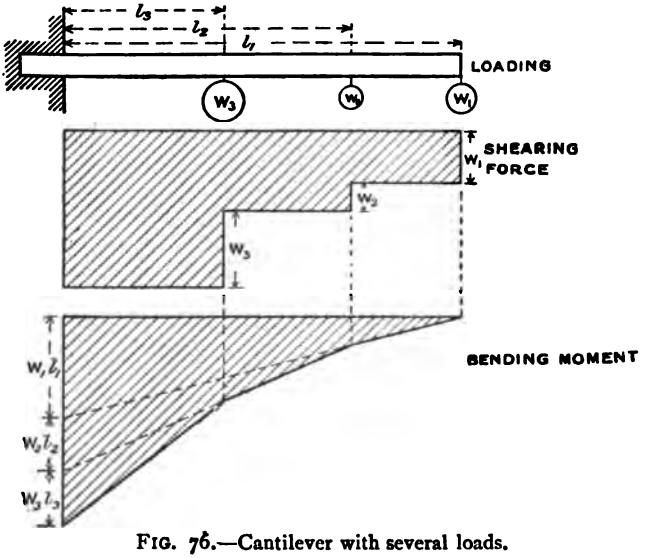



FIG. 76.—Cantilever with several loads.

different concentrated or distributed loads the bending moment at any and every cross-section is the algebraic sum of the bending moments produced by the various loads acting separately. In plotting the

diagrams it is sometimes convenient to add the ordinates of diagrams for two separate loads and plot the algebraic sum, or to plot the two

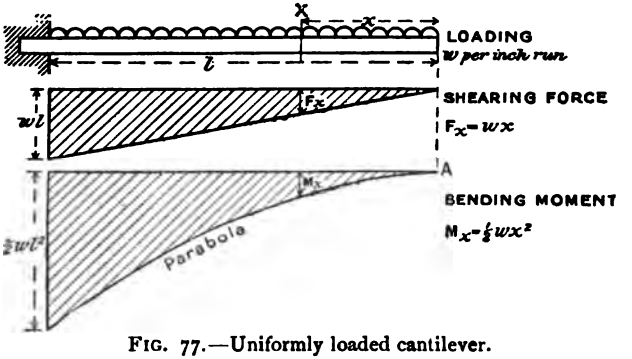



FIG. 77.—Uniformly loaded cantilever.

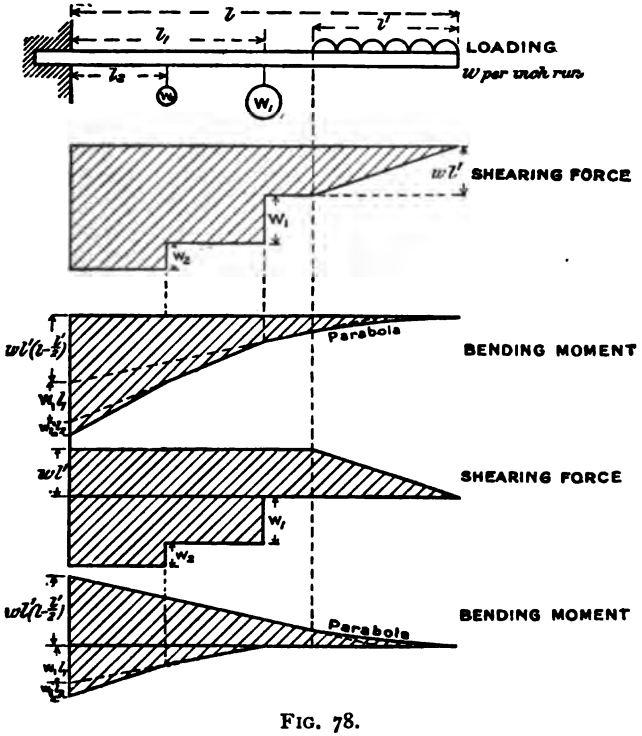



FIG. 78.

curves on opposite sides of the same base-line, and measure resultant values (vertically) directly from the extreme boundaries of the resultant diagram.

The two methods are illustrated in order in Fig. 78. Figs. 75, 76, 77, and 78 represent cantilevers, *i.e.* beams firmly fixed at one end and free at the other. Figs. 79, 80, 81, 82 represent beams resting freely on supports at each end, and carrying various loads as shown. In calculating the shearing force or bending moment at any given point, or obtaining a symbolic expression for either quantity for every point

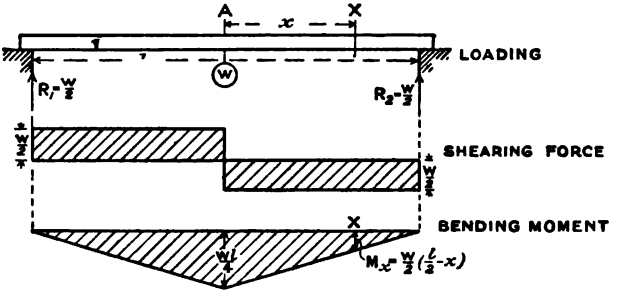



FIG. 79.—Freely supported beam with central load.

over part or all the length of the beam, the first step is usually to find the value of the unknown supporting forces or reactions ($R_1$ and $R_2$). These can conveniently be found by considering the moments of all external forces about either support, and equating the algebraic sum to zero. When all the external forces are known, the shearing force and bending moment are easily obtained for any section, the former being the algebraic sum of the external transverse forces to either side

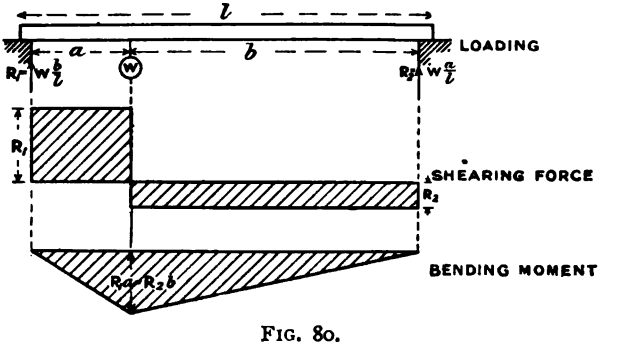



FIG. 80.

of the section, and the latter being the algebraic sum of the moments of the external forces to either side of the section.

The question of positive or negative sign of the resulting sums is arbitrary and not very important; it is dealt with in Art. 59, but in a diagram it is well to show opposite forces and moments on opposite sides of the base-line. Take the case in Fig. 82 fully as an example. The load is uniformly spread at the rate of $w$ per inch run over a

length $c$ of the beam. The distances of the centre of gravity of the load from the left- and right-hand supports of the beam are $a$ and $b$ respectively, so that $a + b = l$, the span of the beam between the supports.

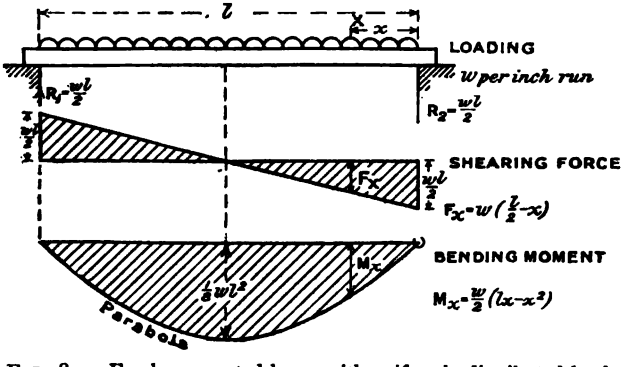



FIG. 81.—Freely supported beam with uniformly distributed load.

Taking moments about the right-hand support

$$R_1 \times l = w \,.\, c \times b$$

$$R_1 = w \,.\, c \,.\, \frac{b}{l} \qquad R_2 = wc \,.\, \frac{a}{l}$$

The shearing force (F) from the left support to the beginning of the load is equal to $R_1$.

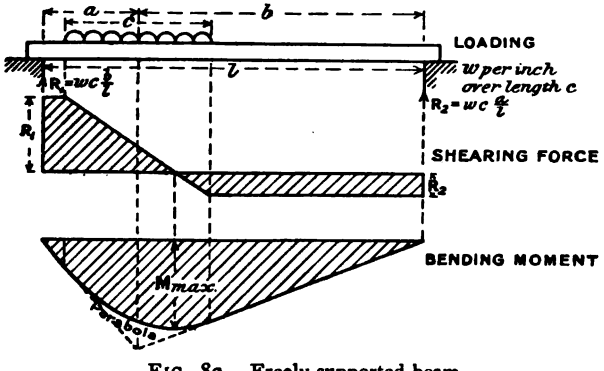



FIG. 82.—Freely supported beam.

Over the loaded portion, at a distance $x$ from the left end, *i.e.* from $x = a - \frac{c}{2}$ to $x = a + \frac{c}{2}$,

$$F_x = R_1 - w\left\{x - \left(a - \frac{c}{2}\right)\right\} = wc\frac{b}{l} - wx + w\left(a - \frac{c}{2}\right)$$

or, $$w\left(\frac{cb}{l} + a - x - \frac{c}{2}\right)$$

which equals zero when $x = \frac{b}{l} + a - \frac{c}{2}$.

For the remainder of the length to the right-hand support the shearing force is numerically equal to $R_2$, or algebraically to $R_1 - wc$, *i.e.*

$$F = w\left(\frac{cb}{l} - c\right) \quad \text{or} \quad wc\left(\frac{b-l}{l}\right) \quad \text{or} \quad -wc\frac{a}{l}$$

The bending moment (M) at a section distant $x$ from the left-hand end to the beginning of the load, *i.e.* if $x$ is less than $a - \frac{c}{2}$, estimating moments on the left of the section, is

$$M_x = R_1 \,.\, x = wc\frac{b}{l} \cdot x \text{ (a straight line)}$$

Over the loaded portion, *i.e.* if $x$ is greater than $a - \frac{c}{2}$ and less than $a + \frac{c}{2}$,

$$M_x = R_1 \,.\, x - \left\{x - \left(a - \frac{c}{2}\right)\right\} \times w \cdot \tfrac{1}{2}\left\{x - \left(a - \frac{c}{2}\right)\right\}$$
$$= wc\frac{b}{l}x - \frac{w}{2}\left(x - a + \frac{c}{2}\right)^2$$

The first term is represented by the left-hand dotted straight line, and the second by the distance between the curve and the straight line, and the value $M_x$ by the vertical ordinate of the shaded diagram.

To the right of the load, *i.e.* when $x$ is greater than $a + \frac{c}{2}$, estimating to the left

$$M_x = R_1 \,.\, x - wc(x - a) = wc \cdot \frac{b}{l} \cdot x - wc(x - a)$$

or, $$M_x = wca - wcx\left(1 - \frac{b}{l}\right) \quad \text{or} \quad wca - wc \cdot \frac{x}{l} \cdot a = wc \cdot \frac{a}{l}(l - x)$$
$$= R_2(l - x) \text{ (a straight line)}$$

which is much more simply found by taking the moments of the sole force $R_2$ to the right of any section in the range considered.

Fig. 83 represents a beam symmetrically placed over supports of shorter span, $l_2$, than the length of the beam, $l_2 + 2l_1$, and carrying equal end loads. Between the supports the shearing force is zero and the bending moment is constant. The magnitudes are unaffected if the positions of the loads and reactions are interchanged.

Fig. 84 shows a beam of length $l_2 + 2_1 l$, with a uniformly spread load placed on supports $l_2$ apart and overhanging them by a length $l_1$ at each end. The bending moment at the supports is

$$M = wl_1 \times \frac{l_1}{2} = \frac{wl_1^2}{2}$$

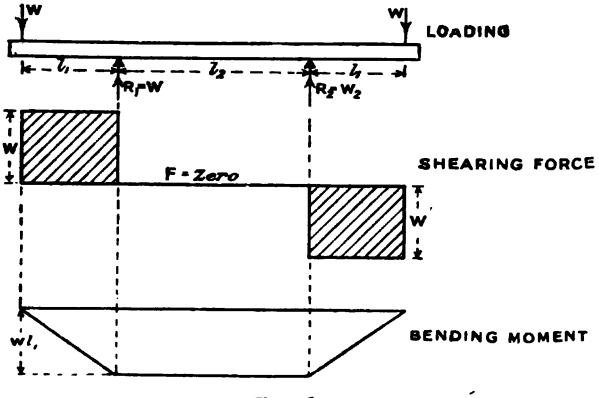



FIG. 83.

Within the span at a distance $x$ from either support the bending moment is

$$M_x = w(l_1 + x) \times \frac{l_1 + x}{2} - R_1 x = \frac{w}{2}(l_1 + x)^2 - wx\left(l_1 + \frac{l_2}{2}\right)$$
$$= \frac{w}{2}l_1^2 - \frac{w}{2}(l_2 x - x^2)$$

the first term of which is the bending moment at the supports, and the second is bending moment for a uniformly loaded span of length $l_2$

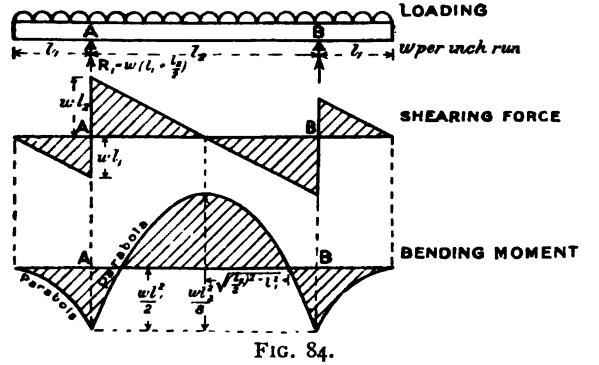



FIG. 84.

(see Fig. 81). The two terms are of opposite sign, and, provided $l_2$ is

long enough, the bending moment will be zero and change sign at two points within the span, viz. when $M_x = 0$, or

$$\frac{w}{2} \cdot l_1^2 - \frac{w}{2} x(l_2 - x) = 0, \quad x^2 - l_2 x + l_1^2 = 0$$

$$x = \frac{l_2}{2} \pm \sqrt{\left(\frac{l_2^2}{2}\right)^2 - l_1^2}$$

*i.e.* at two points distant $\sqrt{\left\{\left(\frac{l_2^2}{2}\right)^2 - l_1^2\right\}}$ on the other side of mid span;

the two points are coincident (at mid span) if $l_2 = 2l_1$, and do not exist if $l_2$ is less than $2l_1$, when the bending moment does not change sign.

*Points of Contraflexure.*—Bending moments of opposite sign evidently tend to produce bending of opposite curvature. In a continuous curve of bending moments change of sign involves passing through a zero

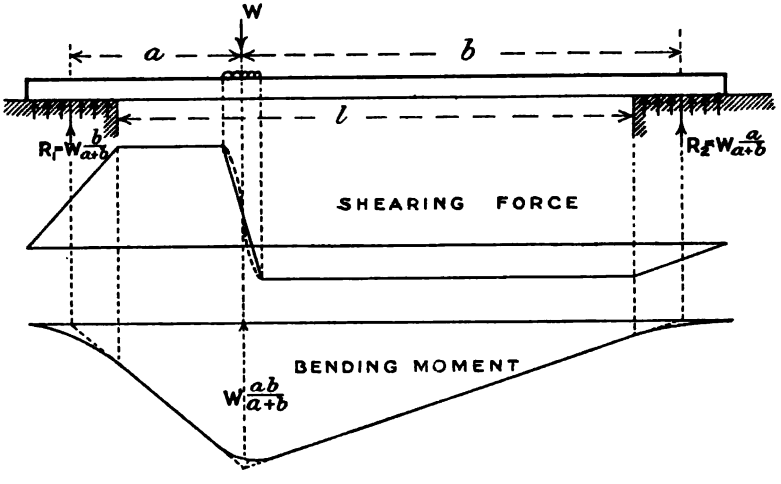



FIG. 85.

value of bending moment, and this point of zero bending moment and change of sign is called a point of inflection or contraflexure, or a virtual hinge. The positions of the points of contraflexure for Fig. 84 have just been determined above from the equation $M_x = 0$.

*Actual Reactions and Effective Span.*—The foregoing diagrams are somewhat conventional as regards the application of the loads and reactions. These will actually be more or less distributed forces and not concentrated in lines (or rather planes perpendicular to the diagrams). The kind of modification which such distribution will produce in the diagrams of shearing force and bending moment is illustrated in Fig. 85, where the load W and both supporting

forces are assumed to be uniformly distributed over short lengths of the beam: a comparison with Fig. 80 shows the effect of such distribution. The three curved portions of the bending moment diagram would be in this case parabolic (as in Fig. 82). The intensity of loading will usually be less at the boundary of the short loaded lengths, and in this case the change of shearing force will be as indicated by the steep dotted curve instead of the uniform rate of change. When the ends of such a beam rest on seatings of finite length, the bending moments everywhere will be greater than if the beam were supported at the ends of the span $l$. The distance $(a + b)$ from centre to centre of the two seatings may be called the *effective span*, which is greater than the *clear span* $l$.

EXAMPLE 1.—A concentrated load of 1 ton is carried 3 feet from the abutment of a beam having a clear span of 9 feet. Calculate the maximum bending moment first if the beam is only 9 feet long and is just supported at its ends; secondly, if it is 11 feet long and rests on seatings 1 foot long at each end and the pressure is uniformly distributed along the seatings.

In the first case the reaction more distant from the load is

$$1 \times \tfrac{3}{9} = \tfrac{1}{3} \text{ ton}$$

And the maximum bending (moment under the load) is

$$\tfrac{1}{3} \times 6 = 2 \text{ tons-feet}$$

In the second case the more distant supporting force found by taking moments about the centre of the near seating is

$$1 \times \frac{3{\cdot}5}{10} = 0{\cdot}35 \text{ ton}$$

And the bending moment under the load is

$$0{\cdot}35 \times 6{\cdot}5 = 2{\cdot}275 \text{ tons-feet}$$

EXAMPLE 2.—A girder of 40 feet effective span supported at its ends has a total load of 56·5 tons uniformly distributed along its length. The load is not carried directly, but is transferred to the girder at its ends and at four consecutive points $a$, $b$, $b'$, $a'$ (cross girders) placed 8, 16, 24, and 32 feet respectively, from the left-hand support. Assuming that each load point $a$, $b$, $b'$, and $a'$ carries the load for 8 feet length, find the bending moment at each point.

Load at each point $= \frac{8}{40} \times 56{\cdot}5 = 11{\cdot}3$ tons
effective reactions at each end $= \frac{1}{2} \times 4 \times 11{\cdot}3 = 22{\cdot}6$ tons
bending moment at $a$ and $a' = 22{\cdot}6 \times 8 = 180{\cdot}8$ tons-feet
bending moment at $b$ and $b' = 22{\cdot}6 \times 16 - 11{\cdot}3 \times 8 = 271{\cdot}2$ tons-ft.

A bending moment diagram for this girder but with greater load is shown in Fig. 279.

EXAMPLE 3.—A girder of 11·25 feet effective span carries a uniformly distributed load of 1 ton total and 2 loads of 30·8 tons each

2½ feet on either side of the centre of the span. Find the maximum bending moment.

This corresponds to the two types of loading shown in Figs. 81 and 83 (reversed) acting together, consequently since the concentrated loads are $\frac{11{\cdot}25}{2} - 2{\cdot}5 = 3{\cdot}125$ feet from the supports, the total bending moment at the centre of the span will be

$$(30{\cdot}8 \times 3{\cdot}125) + \frac{1 \times 11{\cdot}25}{8} = 97{\cdot}7 \text{ tons-feet.}$$

**58. Bending Moments and Shearing Forces from Link and Vector Polygons.**—The vertical breadths of a funicular or link polygon for

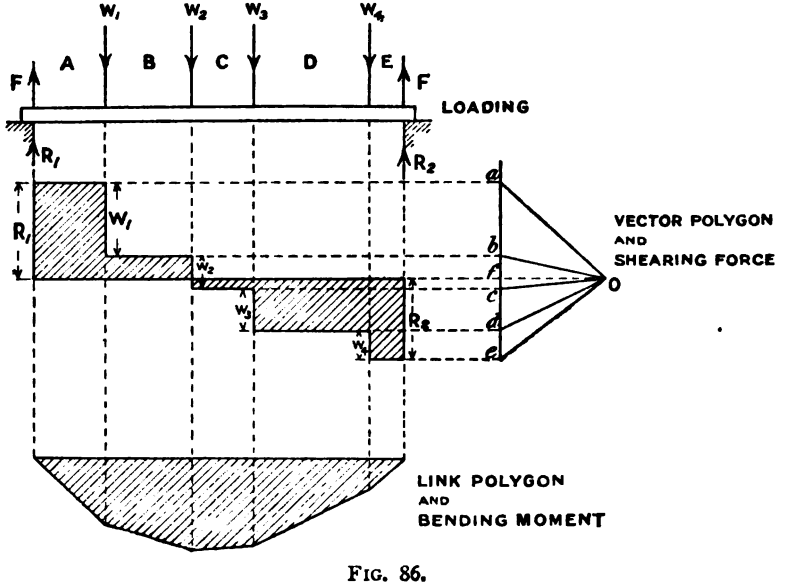



FIG. 86.

a system of vertical forces on a horizontal beam represent to scale the bending moments at the corresponding sections. This has already been proved in Art. 50, and is illustrated in Fig. 86, where the link polygon has been drawn on a horizontal base by making the vector *fo* in the vector polygon horizontal, *i.e.* by choosing a pole *o* in the same horizontal line as the point *f*, which divides the load-line *abcde* in the ratio of the supporting forces. The position of *f* can be calculated or found by means of a trial link polygon with any pole. The scale of bending moment as explained in Arts. 49 and 50 is $p \cdot q \cdot h$ lb.-inches to 1 inch where the scale of force is $p$ lbs. to 1 inch, of distance $q$ inches to 1, and the pole distance *fo* measures $h$ inches. It is not necessary to draw the diagram on a horizontal base, but the distance

$h$ must be estimated horizontally, and the ordinates of bending moment must be measured vertically.

The shearing-force diagram is shown projected from the vertical load-line of the vector polygon.

The same method of drawing the bending-moment diagram to as close approximation as is desired is applicable to loads distributed either uniformly or otherwise by dividing the load into a number of sections along the length of the beam, and treating each part as a load concentrated at its centre of gravity. The resulting funicular polygon will be a figure with straight sides, and the curve of bending moments

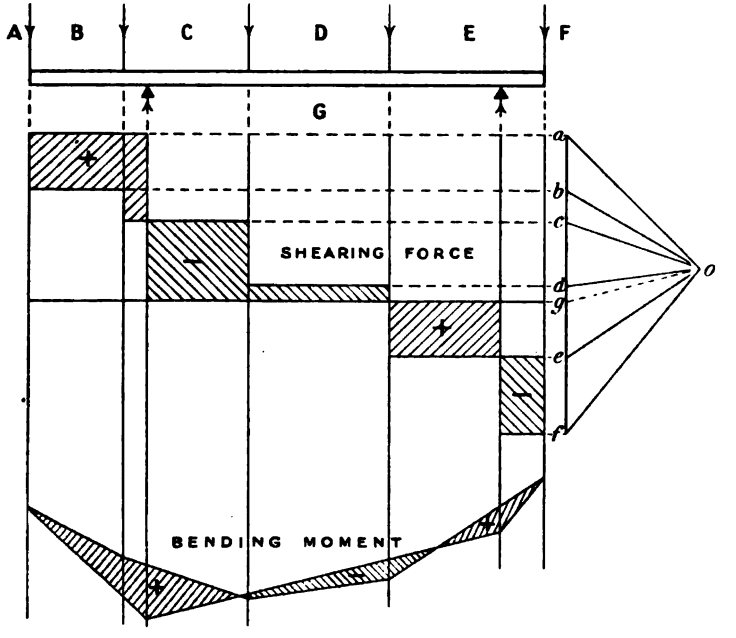
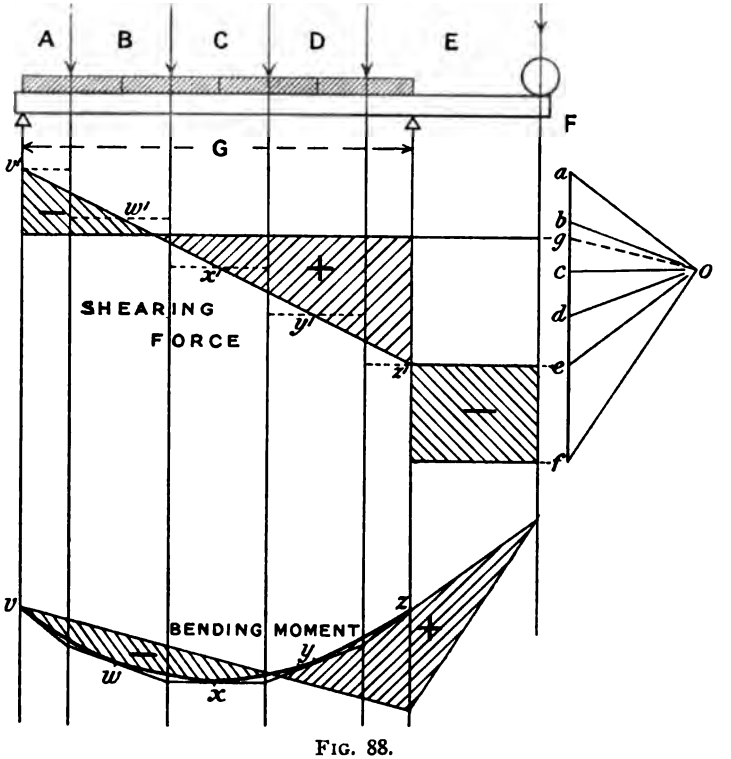



FIG. 87.

is the *inscribed* (not *circumscribed*) curve touching the sides of the polygon, for the polygon evidently gives excessive ordinates at the points of concentration and correct ones at the junctions of the parts into which the loaded lengths are divided. Consideration with a sketch of an extreme case, say a uniform load throughout the span and only two equal divisions, will make this clear. It is also illustrated in Fig. 88.

Fig. 87 shows the bending moment diagram for a beam with overhanging ends. The reactions are found as in Fig. 50, the most convenient order of lettering and setting off the forces on the vector polygon being consecutively round the beam. Care is then required in projecting the shearing-force diagram, as the forces do not follow in


A
B
C
D
E
F
G
v'
w'
x'
y'
z'
SHEARING FORCE
a
b
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c
d
e
f
o
v
w
x
y
z
BENDING MOMENT

FIG. 88.

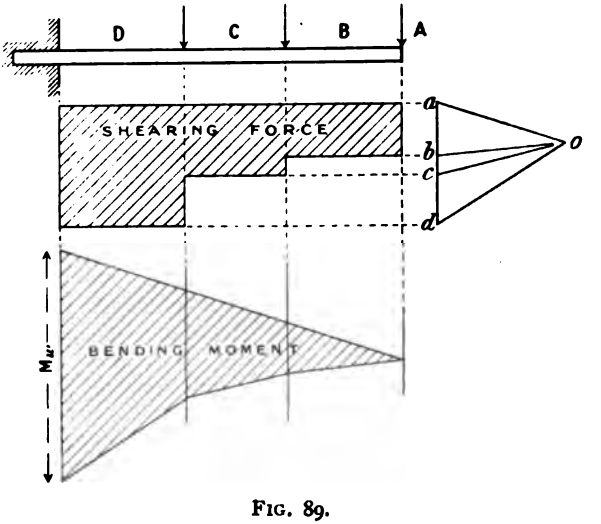

D
C
B
A
SHEARING FORCE
a
b
c
d
o
BENDING MOMENT
$M_d$

FIG. 89.

consecutive order across the paper. It will be found instructive to redraw the vector polygon in such consecutive order and project a shearing-force diagram from it. The choice of signs in Fig. 87 is arbitrary, and those given are in accordance with a convention given in Art. 59. Fig. 88 shows the case of a beam overhanging at one end, and shows how to deal with a uniformly distributed load which here extends over the length between the supports. Only four divisions have been taken, but a curve through $v$, $w$, $x$, $y$, $z$, gives the curve of bending moments, and a straight line through $v'$, $w'$, $x'$, $y'$, $z'$ gives the correct shearing-force diagram in place of the stepped figure. A caution is again required in projecting the shearing-force diagram unless the load line is redrawn.

Fig. 89 represents a cantilever carrying three loads; if $o$ were chosen in the same horizontal line as $a$ the bending moment diagram becomes exactly like that already drawn in Fig. 76. The link polygon for the three given forces is not closed, and for equilibrium an upward force $da$ together with a moment $M_w$ at the wall is required; these are supplied by upward and lesser downward reactions on the clamped end.

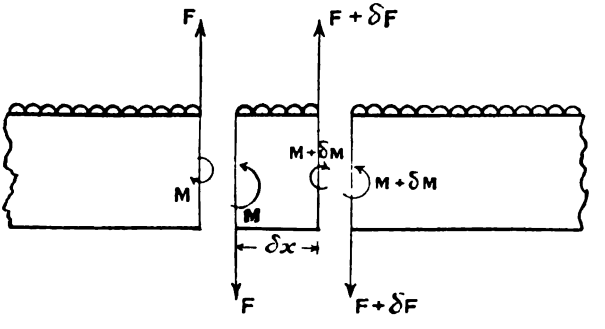



FIG. 90.

**59. Relation between Bending Moment and Shearing Force.**— Consider a small length $\delta x$ of a beam (Fig. 90) carrying a continuous distributed load $w$ per unit of length, where $w$ is not necessarily constant, but $\delta x$ is sufficiently small to take $w$ as constant over that length. Let F and F + δF be the shearing forces, M and M + δM the bending moments at either end of the length $\delta x$ as shown in Fig. 90.

Equating upward and downward vertical forces on length $\delta x$

$$F + \delta F = F + w\delta x$$

$$\delta F = w \,.\, \delta x$$

and

$$\frac{dF}{dx} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

*i.e.* the rate of change of shearing force (represented by the slope of the shearing-force curve) is numerically equal to the intensity of loading.

Or integrating between two sections $x - x_0$ apart—

$$F - F_0 \text{ (the total change in shearing force)} = \int_{x_0}^{x} w \,.\, dx$$

or,
$$F = F_0 + \int_{x_0}^{x} w \,.\, dx$$

taking appropriate signs for each term.

These relations for $w$ = constant, are illustrated in the shearing-force diagrams of Figs. 77, 81, and 84.

Equating moments of opposite kinds, of all external forces on the piece of length $\delta x$, about any point in the left-hand section

$$M + (F + \delta F)\delta x - w \,.\, \delta x \times \frac{\delta x}{2} = M + \delta M$$

$\delta M = F\delta x$, to the first order of small quantities

and
$$\frac{dM}{dx} = F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

*i.e.* the rate of change of bending moment is equal to the shearing force.

Hence, integrating, the total change of bending moment from $x_0$ to $x$ is $\int_{x_0}^{x} F dx$, which is proportional to the area of the shearing-force diagram between the ordinates at $x_0$ and $x$. For example, this area is zero between the ends of the beam in Figs. 79 to 88 inclusive, there being as much area positive as negative.

The relation (2) indicates that the ordinates of the shearing-force diagram are proportional to the slopes or gradients of the bending-moment curve. Where the shearing force passes through a zero value and changes sign, the value of the bending moment is a (mathematical) maximum or minimum, a fact which often forms a convenient method of determining the greatest bending moment to which a beam is subjected, as in Figs. 81, 82, and 84. In Fig. 82, the section at which the shearing force is zero evidently divides the length $c$ in the ratio $\frac{R_1}{R_2}$; or, using the expression given in Art. 57, F is zero at a distance

$$c\frac{b}{l} + a - \frac{c}{2}$$

from the left support. At this point the bending moment is a maximum, and its value is easily calculated.

*Signs.*—It is to be noted that $x$ being taken positive to the right and $w$ positive downwards, F has been chosen as positive in (1) when its action is upwards to the left and downwards to the right of the section considered. Hence, taking account of sign forces being reckoned positive downwards, the shearing force is equal to the downward internal force exerted to the right of any section, or to the algebraic sum of the *upward external forces* to the *right* of the section, or to the

algebraic sum of the *downward external forces* to the *left* of the section. Also M has been chosen as positive in (2) when its action is clockwise on the portion of the beam to the left of the section and contra-clockwise to the right of the section. Hence the bending moment is equal to the *clockwise moment* of the *external forces* to the right of a section or to the contra-clockwise moment of the external forces to the left of the section. It is evident that a positive bending moment will produce convexity upwards and a negative bending moment convexity downwards.

*Concentrated Loads.*—In the case of loads concentrated (more or less) at fixed points along the span, the curve of shearing force (see Figs. 76, 78, 79, 80, 83, 86, 87, 88, and 89) is discontinuous, and so also is the gradient of the bending-moment curve. Between the points of loading, however, the above relations hold, and the section at which the shearing-force curve crosses the base-line is a section having a maximum bending moment (see Figs. 79, 80, 83, 86, 87, and 88). A concentrated load in practice is, as stated in Art. 57, usually a load distributed (but not necessarily uniformly) over a very short distance, and the vertical lines shown in the shear diagrams at the loads should really be slightly inclined to the vertical, there being at any given section only *one* value of the shearing force.

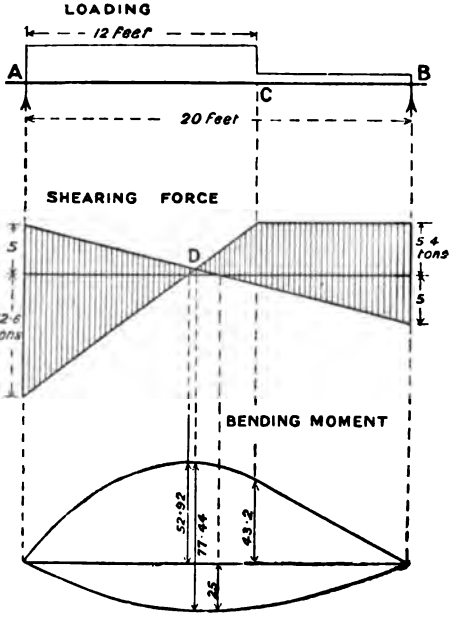



FIG. 91.

EXAMPLE 1.—A beam 20 feet long rests on supports at each end and carries a load of ½ ton per foot run, and an additional load of 1½ ton per foot run for 12 feet from the left-hand end. Find the position and magnitude of the maximum bending moment, and draw the diagrams of shearing force and bending moment.

The loading is indicated at the top of Fig. 91 at ACB.

The reactions due to the ½ ton per foot are 5 tons at A and B. For the 1½ ton per foot load, the centre of gravity of which is 6 feet from A

$$\left.\begin{aligned} (\text{reaction at B}) \times 20 &= 18 \times 6 \\ \text{reaction at B} &= 5{\cdot}4 \text{ tons} \end{aligned}\right\} \text{due to second load}$$

$$\text{hence reaction at A} = 18 - 5{\cdot}4 = 12{\cdot}6 \text{ tons}$$

The shearing-force diagrams for the two loads have been set off separately on opposite sides of a horizontal line, and the resultant diagram is shown shaded.

The bending moment is a maximum where the shear force is zero, as shown at D. The distance from the left support is perhaps most easily found from the fact that the shearing force at the left support is 17·6 tons, and falls off at the rate of 2 tons per foot run, and therefore reaches zero at a distance

$$\frac{17{\cdot}6}{2} \text{ or } 8{\cdot}8 \text{ feet from the left-hand support}$$

The bending moment at 8·8 feet is

$$17{\cdot}6 \times 8{\cdot}8 - 8{\cdot}8 \times 2 \times \frac{8{\cdot}8}{2} = 77{\cdot}44 \text{ tons-feet}$$

The bending-moment diagrams for the two loads have been drawn on opposite sides of the same base-line in Fig. 91, giving a combined diagram for the two, by vertical measurements between the boundaries.

For the ½ ton per foot load alone the maximum bending moment is at the middle of the span, and is

$$5 \times 10 - \tfrac{1}{2} \times 10 \times 5 = 25 \text{ tons-feet}$$

For the 1½ ton per foot-load alone the maximum occurs where the shearing force due to that load would be zero, a distance from A which is given by

$$12{\cdot}6 \div 1{\cdot}5 = 8{\cdot}4 \text{ feet}$$

The maximum ordinate of this curve is then

$$12{\cdot}6 \times 8{\cdot}4 - 8{\cdot}4 \times 1\tfrac{1}{2} \times \frac{8{\cdot}4}{2} = 52{\cdot}92 \text{ tons-feet}$$

At C the ordinate of this curve is—

$$5{\cdot}4 \times 8 = 43{\cdot}2 \text{ tons-feet}$$

and to the right of C it varies directly as the distance from B—the curve being a straight line.

EXAMPLE 2.—A horizontal beam, AB, 24 feet long, is hinged at A, and rests on a support at C, 16 feet from A, and carries a distributed load of 1 ton per foot run, and an additional load of 32 tons at B. Find the reactions, shearing forces, and bending moments. If the load at B is reduced to 8 tons, what difference will it make?

Let $R_C$ be the upward reaction at support C.

Taking moments about A (Fig. 92)

$$16 \cdot R_C = (32 \times 24) + (24 \times 12) = 1056$$
$$R_C = 66 \text{ tons}$$

If the upward reaction at A = $R_A$

$$R_A = 24 + 32 - 66 = -10 \text{ tons}$$

or 10 tons *downward*.

The shearing-force diagram is shown in Fig. 92. From B, where the shearing force is 32 tons, it increases uniformly by 8 to C, where it is reduced by 66 tons to 26 of opposite sign. From C to A the total change at a uniform rate is 16 tons, giving a value 10 at A.

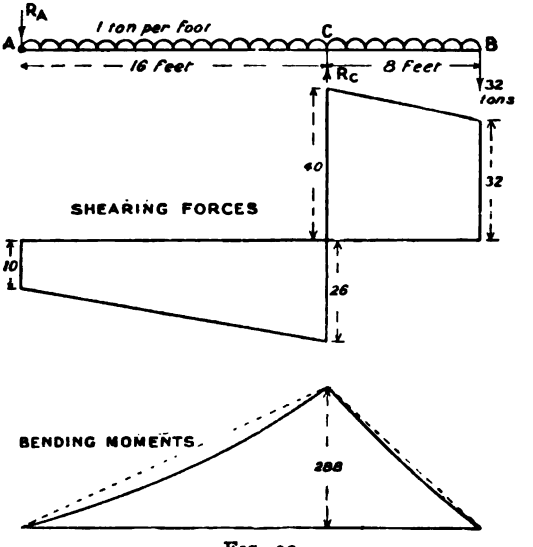



FIG. 92.

The bending moment at C is $(32 \times 8) + (8 \times 4) = 288$ tons-feet,

This falls to zero at A and B, and does not reach a maximum value, in the mathematical sense, in either range. The bending moment 4 feet from B is

$$(32 \times 4) + (4 \times 2) = 136 \text{ tons-feet}$$

Midway between A and C it is $(10 \times 8) + (8 \times 4) = 112$ tons-feet.

The full diagram is shown in Fig. 92.

Treating the problem with only 8 tons load at B

$$16R_C = (24 \times 8) + (12 \times 24) = 192 + 288 = 480$$

$$R_C = 30 \text{ tons}$$

Total load $= 24 + 8 = 32$ tons

$R_A = 2$ tons *upward*

The diagrams of shearing force and bending moment are shown in Fig. 93. The shearing force at B is 8 tons, and increases by a further 8 tons to 16 at C, where it decreases by 30 tons of 14 of opposite sign. From C to A it changes by 16 to 2 tons at A, changing sign and passing through zero between C and A.

The section which has a (mathematical) maximum bending moment between A and C is that for which the shearing force is zero, and since the shear is 2 tons at A and falls off at 1 ton per foot run, the zero value will be, at a section D, 2 feet from A.

The bending moment at C is $(8 \times 8) + (8 \times 4) = 96$ tons-feet
At 4 feet from B it is $(8 \times 4) + (4 \times 2) = 40$ tons-feet
Between A and C, at a distance $x$ from A, it is

$$x \times \frac{x}{2} - 2x \text{ or } x\left(\frac{x}{2} - 2\right)$$

which is zero, for $x = 4$ feet, *i.e.* 4 feet from A, where a point of contraflexure E occurs. This distance might have been inferred otherwise, for it is evidently twice that of the point D from A.

Finally, $M_D = 2 \times 1 - 2 \times 2 = -2$ tons-feet

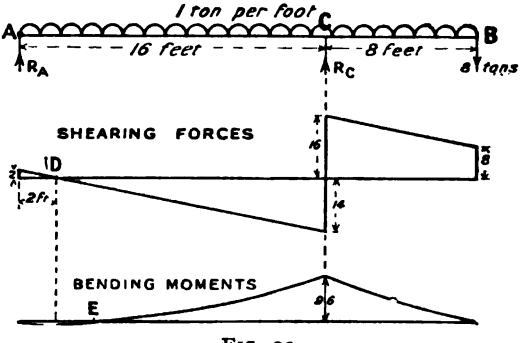



FIG. 93.

EXAMPLE 3.—A beam simply supported at each end has a span of 20 feet. The load is distributed and is at the rate of 1 ton per foot run at the left support, and 4 tons per foot run at the right-hand support, and varies uniformly from one rate to the other along the span. Find the position and amount of the maximum bending moment.

The load may conveniently be divided into a uniformly spread load of 1 ton per foot run, and a second varying from zero at the left to 3 tons per foot run at the right. The first will evidently cause a reaction of 10 tons at each support. The second load has an average intensity of 1·5 ton per foot run, or is 30 tons in all; its centre of gravity will be $\frac{2}{3}$ of the span from the left end, so that the right-hand reaction due to this load will be $\frac{2}{3}$ of 30 tons, or 20 tons, and the left-hand one will be 10 tons.

The total reactions are therefore 20 tons and 30 tons at the left- and right-hand ends respectively.

The load per foot at a distance $x$ feet from the left support is

$$1 + \tfrac{3}{20}x \text{ tons per foot}$$

since it increases $\frac{3}{20}$ ton per foot per foot.

The average over the length $x$ feet is

$$\tfrac{1}{2}(1 + 1 + \tfrac{3}{20}x) \text{ or } 1 + \tfrac{3}{40}x \text{ tons per foot}$$

and the total load on $x$ feet, is $x(1 + \frac{3}{40}x)$

The bending moment is a maximum when the shearing force is zero, *i.e.* at the section where the load carried to the left of it is equal to the left-hand reaction of 20 tons. For this point the shearing force

$$F = 20 - x\left(1 + \frac{3x}{40}\right) = 0$$
$$3x^2 + 40x - 800 = 0$$
$$x = 10{\cdot}96 \text{ feet} = 10 \text{ feet } 11{\cdot}5 \text{ inches}$$

The bending moment at a distance $x$ feet from the support is

$$20x - x \times \frac{x}{2} - \frac{3x^2}{40} \times \tfrac{1}{3}x$$

and when $x = 10{\cdot}96$ feet, $M = 219 - 60 - 33 = 126$ tons-feet

The shearing-force and bending-moment curves may be plotted from the two above expressions for F and M.

## EXAMPLES IV.

1. A cantilever 12 feet long carries loads of 3, 7, 4, and 6 tons at distances 0, 2, 5, and 8 feet respectively from the free end. Find the bending moment and shearing force at the fixed end and at the middle section of the beam.

2. A cantilever 10 feet long weighs 25 lbs. per foot run, and carries a load of 200 lbs. 3 feet from the free end. Find the bending moment at the support, and draw the diagrams of shearing force and bending moment.

3. A beam rests on supports 16 feet apart, and carries, including its own weight, a load of 2 tons (total) uniformly distributed over its whole length and concentrated loads of 1½ ton and ½ ton, 5 feet and 9 feet respectively from the left support. Find the bending moment 4 feet from the left-hand support, and the position and magnitude of the maximum bending moment.

4. Where does the maximum bending moment occur in a beam of 24 feet span carrying a load of 10 tons uniformly spread over its whole length, and a further load of 12 tons uniformly spread over 8 feet to the right from a point 6 feet from the left support? What is the amount of the maximum bending moment, and what is the bending moment at mid-span?

5. A beam of span $l$ feet carries a distributed load, which increases uniformly from zero at the left-hand support to a maximum $w$ tons per foot at the right-hand support. Find the distance from the left-hand support of the section which has a maximum bending moment and the amount of that bending moment. Obtain numerical values when $l = 18$ feet and $w = 2$ tons per foot run.

6. A horizontal beam AB 30 feet long is supported at A and at C 20 feet from A, and carries a load of 7 tons at B and one of 10 tons midway between A and C. Draw the diagrams of bending moment and find the point of contraflexure.

7. Find the point of contraflexure in the previous example if there is an additional distributed load of ¼ ton per foot run from A to C.

8. A girder 40 feet long is supported at 8 feet from each end, and carries a load of 1 ton per foot run throughout its length. Find the bending moment at the supports and at mid-span. Where are the points of contraflexure? Sketch the curve of bending moments.

9. A beam of length $l$ carries an evenly distributed load and rests on two supports. How far from the ends must the supports be placed if the greatest bending moment to which the beam is subjected is to be as small as possible? Where are the points of contraflexure?

10. A beam 18 feet long rests on two supports 10 feet apart, overhanging the left-hand one by 5 feet. It carries a load of 5 tons at the left-hand end 7 tons midway between the supports, and 3 tons at the right-hand end. Find the bending moment at the middle section of the beam and at mid-span, and find the points of contraflexure.

11. If the beam in the previous example carries an additional load of 1 ton per foot run between the supports, find the bending moment at mid-span and the positions of the points of contraflexure.

12. A horizontal beam 24 feet long rests on supports 14 feet apart overhanging the left one by 6 feet. It carries a load of 7 tons at the left-hand end and loads of 5, 4, 12, 9, and 4 tons at 4, 9, 13, 17, and 24 feet respectively from the left-hand end. Draw the diagrams of shearing force and bending moment and measure from the latter the bending moments midway between the supports and at each support. State also the distances of the points of contraflexure from the nearest support.

# CHAPTER V

## *STRESSES IN BEAMS*

**60. Theory of Elastic Bending.** — The relations existing between the straining action, the dimensions, the stresses, strains, elasticity, and curvature of a beam are, under certain simple assumptions, very easily established for the case of *simple bending*, *i.e.* flexure by pure couples applied to a beam without shearing force.

Most of the same simple relations may generally be used as close approximations in cases of flexure which are not "simple," but which are of far more common occurrence, the strains involved from the shearing force being negligible. In such cases, the justification of the "simple theory of bending" must be the agreement of its conclusions with direct bending experiments, and with those of more complex but more exact theory of elastic bending.

**61. Simple Bending.**—A straight bar of homogeneous material subjected only to equal and opposite couples at its ends has a uniform bending moment throughout its length, and if there is no shearing force, is said to suffer simple bending. Such a straining action is illustrated in Fig. 83 for the beam between its two points of support. The beam will be supposed to be of the same cross-section throughout its length, and symmetrical about a central longitudinal plane, in and parallel to which the opposite straining couples act, and parallel to which bending takes place; the intersections of such a plane with transverse section of the beam will be principal axes (see Art. 54) of the sections. In Fig. 94, central longitudinal sections before and after bending and a transverse section are shown, the cross-section being symmetrical about an axis YY.

It will be assumed that transverse plane sections of the beam remain plane and normal to longitudinal fibres after bending, which seems reasonable, since the straining action is the same on every section. The assumption is called Bernoulli's.

Consider any two transverse sections AB and CD very close together. After bending, as shown at A'B' and C'D', they will not be parallel, the layer of material at AC being extended to A'C', and that at BD being pressed to B'D'. The line EF represents the layer of material which is neither stretched nor shortened during bending. This surface EF suffers no longitudinal strain, and is called the *neutral surface*. Its line of intersection ZZ with a transverse section is called the *neutral axis* of that section.

Suppose the section A'B' and C'D' produced to intersect, at an angle $\theta$ (radians), in a line perpendicular to the figure and represented by O,

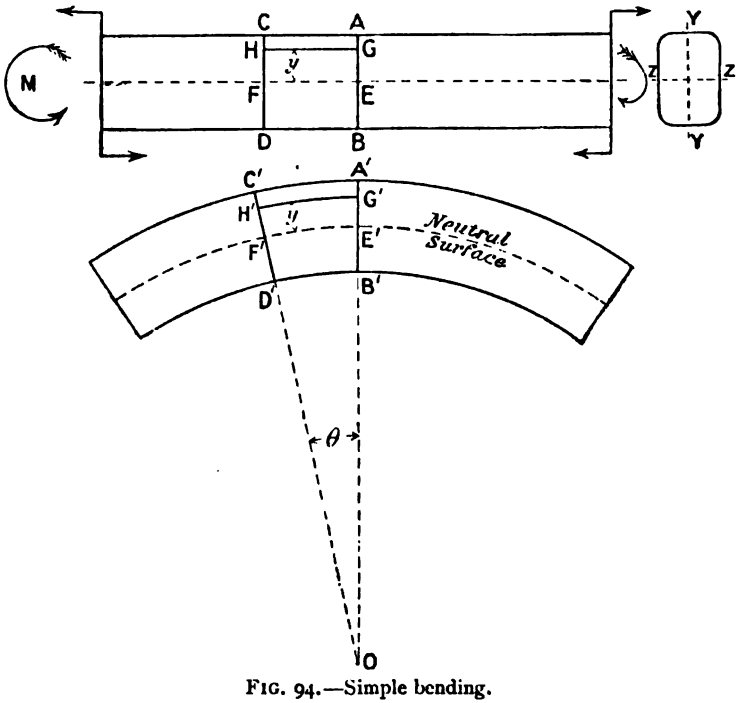



FIG. 94.—Simple bending.

and that the radius of curvature OE' of the neutral surface E'F' about O is R. Let $y$ be the height (E'G') of any layer (H'G') of material originally parallel to the neutral surface FE. Then

$$\frac{H'G'}{E'F'} = \frac{(R+y)\theta}{R\theta} = \frac{R+y}{R}$$

and the strain at the layer H'G' is—

$$e = \frac{H'G' - HG}{HG} = \frac{H'G' - E'F'}{E'F'} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

The longitudinal tensile-stress intensity $p$ at a height $y$ from the neutral surface, provided the limit of elasticity has not been exceeded, is therefore

$$p = E \cdot e = E \cdot \frac{y}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where E is Young's modulus, provided that the layers of material behave under longitudinal stress as if free and are not hindered by the surrounding material, which has not the same intensity of stress.

The intensity of compressive stress will be the same at an equal distance $y$ on the opposite side of the neutral surface, provided E is the same in compression as in tension.

The intensity of direct longitudinal stress $p$ at every point in the cross-section is then proportional to its distance from the neutral axis; its value at unit distance (*i.e.* at $y = 1$) is $\frac{E}{R}$, and it reaches a maximum value at the boundary furthest from the neutral surface. The variation in intensity of longitudinal stress is as shown in Fig. 95, where the arrow-heads denote the direction of the force exerted by the portion R on the portion L at the section AB. Since the stresses on opposite sides of the neutral surface are of opposite sign or kind, they may be represented as at *aeb*.

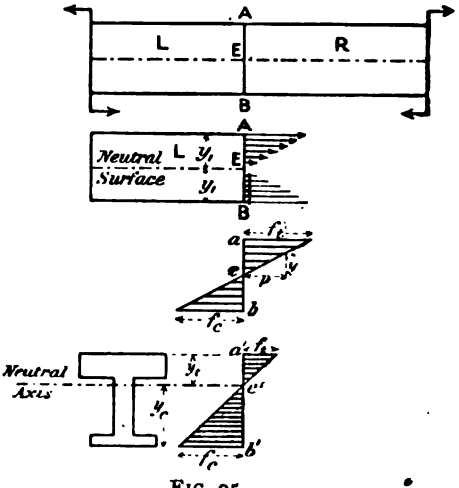



FIG. 95.

**62. Position of the Neutral Axis.**—The beam has been supposed subjected to pure couples only, and therefore the portion, say, to the left of the section AB (Figs. 94 and 95), being in equilibrium under one externally applied couple and the forces acting across AB, these forces must exert a couple balancing the external one in the plane of bending. The (vertical) shearing force being *nil*, the internal forces exerted across AB are wholly horizontal (or longitudinal), and since they form a couple the total tensile forces must balance the compressive ones, *i.e.* the algebraic sum of the horizontal internal forces must, like the external ones, be zero. Putting this statement in symbols, we can find the position of the neutral axis. The cross-section of the beam in Fig. 94 is symmetrical about a horizontal axis, but this is not necessary to the argument. Taking any other forms of cross-sections symmetrical about the plane of bending YY, as in Fig. 96, let $\delta a$ or $z \,.\, \delta y$ be an elementary strip of its area parallel to the neutral axis ZZ, $z$ being the (variable) width of the section. Then, the total horizontal force being zero

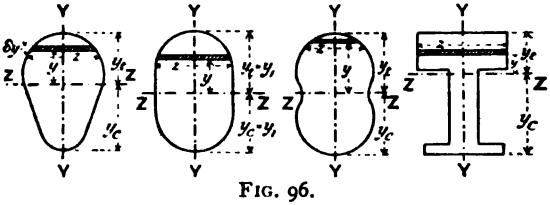



FIG. 96.

$$\Sigma(p \,.\, \delta a) = 0 \quad \text{or} \quad \Sigma(p \,.\, z \,.\, \delta y) = 0$$

and since by (1), Art. 61

$$p = \frac{E}{R} \cdot y$$

$$\frac{E}{R}\Sigma(y \cdot \delta a) = 0 \text{ or } \frac{E}{R}\Sigma(y \cdot z\delta y) = 0. \quad . \quad . \quad . \quad (2)$$

the quantity $\Sigma(y \cdot \delta a)$ or $\Sigma(y \cdot z \cdot \delta y)$ represents the total moment of the area of section about the neutral axis, and this can only be zero if the axis passes through the centre of gravity or centroid of the section.

The use of the value $\frac{E}{R} \cdot y$ for $p$, in all parts of the cross-section involves the assumption that the value of E is the same in compression as in tension, an assumption justified by experiment within the limits of elasticity.

*Assumptions made in the Theory of Simple Bending.*—It may be well to recall the assumptions made in the above theory of "simple bending" under the conditions stated—

(1) That plane transverse sections remain plane and normal after bending.

(2) That the material is homogeneous, isotropic, and obeys Hooke's law, and the limits of elasticity are not exceeded.

(3) That every layer of material is free to expand or contract longitudinally and laterally under stress, as if separate from other layers. Otherwise, E in the relation (1), Art. 61 would not be Young's modulus, but some modified elastic constant; but the relation would otherwise remain unaltered.

(4) That the modulus of direct elasticity has the same value in compression as for tensile strains.

**63. Value of the Moment of Resistance.**—Having found the intensity of longitudinal stress ($p = \frac{E}{R} \cdot y$) at any distance $y$ from the neutral axis, and knowing that these longitudinal internal forces form a couple equal to the bending moment at every section, it remains to express the value of the couple, which is called the moment of resistance (see Art. 56), in terms of the dimensions of the cross-section, and the intensity of stress produced.

Using Fig. 96, as in the previous article, the elementary area of cross-section, at a distance $y$ from the neutral axis, is $\delta a$, or $z \cdot \delta y$, the total stress on the elementary area is $p \cdot \delta a$ or $p \cdot z \cdot \delta y$, and the moment of this stress is $p \cdot y\delta a$ or $p \cdot z \cdot y \cdot \delta y$, and the total moment throughout the section is

$$M = \Sigma(p \cdot y \cdot \delta a) \quad \text{or} \quad M = \Sigma(p \cdot z \cdot y \cdot \delta y)$$

and putting $p = \frac{E}{R} \cdot y$ (Art. 61)

$$M = \frac{E}{R}\Sigma(y^2 \cdot \delta a) \text{ or } \frac{E}{R}\Sigma(zy^2\delta y) \quad . \quad . \quad (3)$$

The sum $\Sigma(y^2\delta a)$, or $\Sigma(zy^2\delta y)$, represents the limiting value of the sum of the products of elements of area, multiplied by the squares of their distances from the axis, when the elements of area are diminished indefinitely, and is usually called the Moment of Inertia of the area of the section about the axis. The values of the moments of inertia for various sections were dealt with in Arts. 52, 53, and 54. If we denote the moment of inertia of the area of the section by I, so that

$$\Sigma(y^2\delta a) = \Sigma(zy^2\delta y) = I$$

the formula (3) becomes

$$M = \frac{E}{R} I \text{ or } \frac{M}{I} = \frac{E}{R} \quad \ldots\ldots (4)$$

and since by (1), Art. 61, $\frac{E}{R} = \frac{p}{y}$ (the stress intensity at unit distance from the neutral axis), we have

$$\frac{p}{y} = \frac{M}{I} = \frac{E}{R} \quad \ldots\ldots (5)$$

These relations are important and should be remembered. If we put this relation in the form—

$$p = \frac{M}{I} \cdot y \quad \text{or} \quad \frac{E}{R} \cdot y$$

we have the intensity of longitudinal stress at a distance $y$ from the neutral axis, in terms of the bending moment and dimensions (I) of cross-section, or in terms of the radius of curvature and an elastic constant for the material. The extreme values of $p$, tensile and compressive, occur at the layers of material most remote from the neutral axis. Thus, in Figs. 95 and 96, if the extreme layers on the tension and compression sides are denoted by $y_t$ and $y_c$ respectively, $f$ and $f_c$ being the extreme intensities of tensile and compressive stress respectively—

$$\frac{p}{y} = \frac{f_t}{y_t} = \frac{f_c}{y_c} = \frac{M}{I} = \frac{E}{R}$$

or,
$$f_t = M \cdot \frac{y_t}{I} \qquad f_c = M \cdot \frac{y_c}{I}$$

or,
$$M = f_t \cdot \frac{I}{y_t} = f_c \cdot \frac{I}{y_c} \quad \ldots\ldots (6)$$

The variation of intensity of stress for an unsymmetrical section is shown in Fig. 95 at $a'c'b'$.

For sections which are symmetrical about the neutral axis, the distances $y_t$ and $y_c$ will be equal, being each half the depth of the section. If we denote the half depth by $y_1$, and the equal intensities of extreme or skin stress by $f_1$, so that—

$$M = f_1 \frac{I}{y_1}$$

the quantity $\frac{I}{y_1}$ is called the modulus of section (see Art. 66), and is usually denoted by the letter Z, so that—

$$M = fZ \quad \text{or} \quad f = \frac{M}{Z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the moment of resistance (M) being proportional to the greatest intensity of stress reached and to the modulus of section.

In the less usual case of unsymmetrical sections, the modulus of section would have the two values—

$$\frac{I}{y_t} \text{ and } \frac{I}{y_c}$$

which may be denoted by $Z_t$ and $Z_c$, so that the relation (6) becomes

$$M = f_t Z_t = f_c Z_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

**64. Ordinary Bending.**—The case of *simple bending*, dealt with in the previous articles, refers only to bending where shearing force is absent, but such instances are not usual, and generally bending action is accompanied by shearing force, which produces a (vertical) shear stress across transverse sections of the beam (see Figs. 75 to 82, etc.). In such cases the forces across any section at which the shearing force is not zero have not only to balance a couple, but also the shearing force at the section, and, therefore, at points in the cross-section there will be tangential as well as normal longitudinal stresses. The approximate distribution of this tangential stress is dealt with in Art. 72, and the deflection due to shearing in Art. 110. When the shearing stresses are not zero, the longitudinal stress at any point in the cross-section is evidently not the principal stress (Arts. 14 and 73) at that point, and the strain is not of the simple character assumed in Art. 61 and Fig. 94, and there is then no reason to assume that plane sections remain plane. St. Venant, a celebrated French elastitician, has investigated the flexure of a beam assuming freedom of every layer or fibre to contract or expand laterally, under longitudinal tension or compression, but without the assumption that plane sections remain plane after bending. His conclusion is that Bernoulli's assumption and equations of the type (5), Art. 63, only hold exactly when the bending moment from point to point follows a straight line law, *i.e.* when the shearing force is constant. For the more exact elastic theory of St. Venant, applicable to other cases, the reader is referred to Todhunter, and Pearson's "History of Theory of Elasticity," vol. ii. pt. 1, pp. 53–69.

For most practical cases the theory of "Simple Bending" (Arts. 61, 62, and 63) is quite sufficient, and gives results which enable the engineer to design beams and structures, and calculate their stresses and strains with a considerable degree of approximation. It may be noticed that in many cases of continuous loading the *greatest* bending moment occurs as a mathematical maximum at the sections for which the shearing force is zero (Art. 59, and Figs. 79 to 86), and for which the conditions correspond with those for *simple* flexure; in numerous

cases where the section of the beam is uniform throughout its length, the maximum longitudinal stress occurs at the section of maximum bending moment; the usefulness of the *simple* theory in such a case is evident. Further, it often happens that where the shearing force is considerable the bending moment is small, and in such cases the intensity of shear stress can be calculated sufficiently nearly by the method of Art. 72.

In this book the usual engineer's practice of using the simple beam theory will be followed, a few modifications in the strains and stresses in certain cases will be mentioned.

**65. Summary of the Simple Theory of Bending.**—At any transverse section of a horizontal beam carrying vertical loads, from the three usual conditions of equilibrium, we have—

(1) The total vertical components of stresses across a vertical section are together equal to the algebraic sum of the external forces to either side of the section, *i.e.* to the shearing force F.

(2) The algebraic total horizontal force is zero.

(3) The total moment of resistance of the horizontal forces across the section is equal to the algebraic sum of the moments of the external forces to either side of the section, *i.e.* to the bending moment M.

If plane sections remain plane, longitudinal strain is proportional to the distance from the neutral axis, $e$ being equal to $\frac{y}{R}$; hence, longitudinal stress intensity at any point in a cross-section is proportional to the same distance, or—

$$p \propto y \quad \text{and} \quad p = \frac{E}{R} \cdot y$$

Summing the moments of longitudinal stress—

$$M = \frac{E}{R} \cdot I = \frac{pI}{y}$$

or,

$$\frac{p}{y} = \frac{M}{I} = \frac{E}{R} = \frac{f_1}{y_1}$$

where $f_1$ and $y_1$ are the intensity of skin stress, and the vertical distance from the neutral axis to the outer boundary of the section respectively.

In applying these relations to numerical examples, it should be remembered that the units must be consistent; as cross-sections are usually stated in inches, and stresses in pounds or tons per square inch, it is well to take the bending moment, or moment of resistance, in lb.-inches or ton-inches.

EXAMPLE 1.—To what radius of curvature may a steel beam of symmetrical section, 12 inches deep, be bent without the skin stress exceeding 5 tons per square inch? (E = 13,500 tons per square inch.)

Since $\frac{E}{R} = \frac{f_1}{y_1}$ $\quad \therefore R = \frac{Ey_1}{f_1}$

$y_1$ being the half depth, which is 6 inches.

Hence $R = \frac{13{,}500 \times 6}{5} = 16{,}200$ inches, or 1350 feet

EXAMPLE 2.—If the elastic limit is not exceeded, find the stress induced in a strip of spring steel, $\frac{1}{20}$ inch thick, by bending it round a drum 2·5 feet diameter? (E = 13,500 tons per square inch.)

$$f_1 = \frac{Ey_1}{R}$$

The greatest value of $y$ is $\frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$ inch. The radius being 15 inches

$$f_1 = \frac{13{,}500 \times \frac{1}{40}}{15} = 22{\cdot}5 \text{ tons per square inch}$$

EXAMPLE 3.—The moment of inertia of a symmetrical section (see B.S.B. 30, Table I. in Appendix) being 2654 inch units, and its depth 24 inches, find the longest span over which, when simply supported, a beam could carry a uniformly distributed load of 1·2 ton per foot run, without the stress exceeding 7·5 tons per square inch.

If $l$ = span in inches, the load per inch run being $\frac{1{\cdot}2}{12}$, or 0·1 ton, the maximum bending moment which occurs at mid-span is

$$M = \tfrac{1}{8} \times 0{\cdot}1 \times l^2 \text{ (see Fig. 65)}$$

And since $M = f_1 \cdot \frac{I}{y_1}$, and $y_1$ the half depth is 12 inches

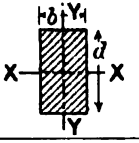
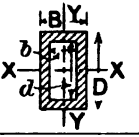
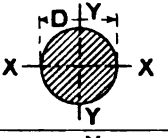
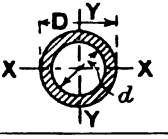
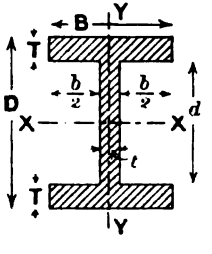
$$\tfrac{1}{8} \times \tfrac{1}{10} \times l = 7{\cdot}5 \times \tfrac{2654}{12}$$

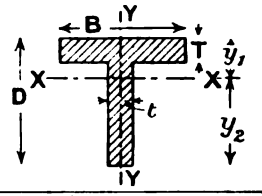
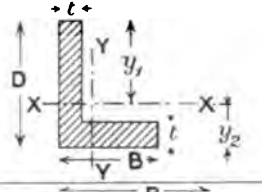
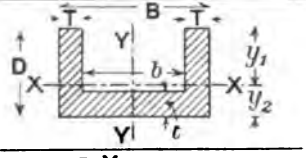
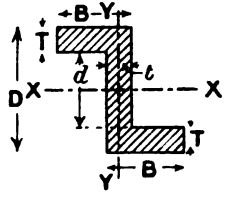
$$l^2 = \frac{80 \times 7{\cdot}5 \times 2654}{12} = 132{,}700$$

$$l = 364 \text{ inches, or } 30 \text{ feet } 4 \text{ inches}$$

**66. Modulus of Section.**—The value of the moment of resistance of a beam is found (Art. 63) by multiplying the extreme value of the intensity of stress by the *modulus of section* (Z) which is the moment of inertia I of the section about the neutral axis divided by the distance to the furthest point in the section from the neutral axis. In the case of sections which are not symmetrical about the neutral axis there will generally be two moduli of section, and two unequal extreme values of stress intensity (tensile and compressive) corresponding to two unequal distances from the neutral axis to the extreme points of the section perimeter.

The following table gives the values of the modulus of section, etc., for sections frequently employed in beams of various kinds :—

| Section. | Distance of centroid from outer edge. | Moment of Inertia I. | Modulus of section Z. |
|---|---|---|---|
|  | $\frac{d}{2}$<br>$\frac{b}{2}$ | $I_{XX} = \frac{1}{12} bd^3$<br>$I_{YY} = \frac{1}{12} db^3$ | $\frac{1}{6}bd^2$<br>$\frac{1}{6}db^2$ |
|  | $\frac{D}{2}$<br>$\frac{B}{2}$ | $I_{XX} = \frac{1}{12}(BD^3 - bd^3)$<br>$I_{YY} = \frac{1}{12}(DB^3 - db^3)$ | $\frac{I}{6D}(BD^3 - bd^3)$<br>$\frac{I}{6B}(DB^3 - db^3)$ |
|  | $\frac{D}{2}$ | $I_{XX} = I_{YY} = \frac{\pi}{64} D^4$ | $\frac{\pi}{32}D^3$ |
|  | $\frac{D}{2}$ | $I_{XX} = I_{YY} = \frac{\pi}{64}(D^4 - d^4)$ | $\frac{\pi}{32} \cdot \frac{D^4 - d^4}{D}$ |
|  | $\frac{D}{2}$<br>$\frac{B}{2}$ | $I_{XX} = \frac{1}{12}(BD^3 - bd^3)$<br>$I_{YY} = \frac{1}{12}(2TB^3 + dt^3)$ | $\frac{I}{6D}(BD^3 - bd^3)$<br>$\frac{I}{6B}(2TB^3 + dt^3)$ |

| Section. | Distance of centroid from outer edge. | Moment of Inertia I. | Modulus of section Z. |
|---|---|---|---|
|  | $y_1 = \frac{tD^2 + (B-t)T^2}{2\{tD + (B-t)T\}}$ | $I_{XX} = \frac{1}{3}\{By_1^3 + ty_2^3 - (B-t)(y_1-T)^3\}$ | $\frac{I_{XX}}{y_1}$ and $\frac{I_{XX}}{y_2}$ |
| | $\frac{B}{2}$ | $I_{YY} = \frac{1}{12}\{TB^3 + (D-T)t^3\}$ | $\frac{2I_{YY}}{B}$ |
|  | $y_2 = \frac{D^2 + (B-t)t}{2(D+B-t)}$ | $I_{XX} = \frac{1}{3}\{ty_1^3 + By_2^3 - (B-t)(y_2-t)^3\}$ | Unsymmetrical bending. See Art. 70. |
| | $y_1 = D - y_2$ | | |
|  | $y_2 = \frac{2D^2T + bt^2}{2(2DT + bt)}$ | $I_{XX} = \frac{1}{3}\{2Ty_1^3 + By_2^3 - b(y_2-t)^3\}$ | $\frac{I_{XX}}{y_1}$ and $\frac{I_{XX}}{y_2}$ |
| | $y_1 = D - y_2$ | | |
| | $\frac{B}{2}$ | $I_{YY} = \frac{1}{12}\{DB^3 - (D-t)b^3\}$ | $\frac{I}{6B}\{DB^3 - (D-t)b^3\}$ |
|  | $\frac{D}{2}$ | $I_{XX} = \frac{1}{12}\{BD^3 - (B-t)d^3\}$ | $\frac{I}{6D}\{BD^3 - (B-t)d^3\}$ |
| | $B - \frac{t}{2}$ | $I_{YY} = \frac{1}{12}\{T(2B-t)^3 + dt^3\}$ | $\frac{2I_{YY}}{2B-t}$ |

The I, T, angle, channel, and Z sections are actually rolled with rounded corners, as shown in Fig. 62 and elsewhere, and the values in the table are those for square-cornered, parallel-limbed sections; they may be applied to give approximate results if mean values of thicknesses are taken. Such sections have been standardized by the Engineering Standards Committee, and tabulated values of their properties with standard dimensions are given in the Appendix. The methods applicable to making calculations of the properties of such sections have been dealt with in Arts. 52 and 53.

A caution is required in applying the tabulated values to such a section as an angle if used *alone;* the principal axes (Art. 54) are not those shown in the table, and XX is not the neutral axis for loading in the plane YY, nor are the distances $y_1$ and $y_2$ shown in the table the extreme distances from the neutral axis. The bending is unsymmetrical, and the subject is treated in Art. 70.

In choosing a section suitable for carrying a given load from such tables as are given in the appendix, it is necessary to select one which shall restrict the bending stress to a safe limit, but it is also often necessary to limit the deflection. This point is dealt with in Chapter VII.

*Modulus Figures.*—The first derived area in Figs. 71 and 72 are sometimes called modulus figures, for the modulus of section is equal to this area multiplied by the extreme distance of the perimeter of the section $\left(\frac{d}{2}\right)$ from the neutral axis GG.

The "centres" of the parallel longitudinal stresses on either side of the neutral axis will evidently be at the centre of area or centroid (or centre of gravity) of the modulus figure. The longitudinal forces across a transverse section are statically equivalent to uniformly distributed stresses of the actual extreme intensity acting on the whole of the modulus figure or to the total of the tensile forces acting at the centroid of the modulus figure on the tension side, together with the (equal) total thrust at the centroid of the modulus figure (which is the centre of pressure) on the compression side.

In comparing algebraic and graphical methods, it is useful to remember that the expression $\frac{1}{y_1}\int yzdy$ represents the area of the modulus figure between the lines corresponding to the limits of integration and parallel to the neutral axis, $y_1$ or $\frac{d}{2}$ being the half depth.

EXAMPLE 1.—A timber beam of rectangular section is to be simply supported at the ends and carry a load of $1\frac{1}{2}$ ton at the middle of a 16-feet span. If the maximum stress is not to exceed $\frac{3}{4}$ ton per square inch and the depth is to be twice the breadth, determine suitable dimensions.

The reactions at the ends are each $\frac{3}{4}$ ton, and the bending moment at the centre is—

$$\tfrac{3}{4} \times 8 \times 12 = 72 \text{ tons-inches}$$

The modulus of section (Z) is given by—

$$\tfrac{3}{4} \times Z = 72 \qquad Z = 96 \text{ (inches)}^3$$

and if

$$b = \tfrac{1}{2}d$$
$$\tfrac{1}{6}bd^2 = \tfrac{1}{12}d^3 = 96$$
$$d = \sqrt[3]{1152} = 10{\cdot}5 \text{ inches nearly}$$
$$b = 5{\cdot}25 \text{ inches}$$

EXAMPLE 2.—Compare the weights of two beams of the same material and of equal strength, one being of circular section and solid and the other being of hollow circular section, the internal diameter being $\frac{3}{4}$ of the external.

The resistance to bending being proportional to the modulus of section, if D is the diameter of the hollow beam and $d$ that of the solid one

$$\frac{\pi}{32}\left\{\frac{D^4 - (\frac{3}{4}D)^4}{D}\right\} = \frac{\pi}{32}d$$
$$(1 - \tfrac{81}{256})D^2 = d^3$$
$$\frac{D}{d} = \sqrt[3]{\tfrac{256}{175}} = 1{\cdot}135$$

The weights are

$$\frac{\text{solid}}{\text{hollow}} = \frac{d^2}{D^2 - (\frac{3}{4}D)^2} = \tfrac{16}{7} \times \left(\frac{d}{D}\right)^2 = \tfrac{16}{7} \times \frac{1}{(1{\cdot}135)^2} = 1{\cdot}77$$

**67. Common Steel Beam Sections.**—Such geometrical figures as rectangles and circles, although they often represent the cross-section of parts of machines and structures subjected to bending action, do not form the sections for the resistance of flexure with the greatest economy of material, for there is a considerable body of material situated about the neutral surface which carries a very small portion of the stress. The most economical section for a constant straining action will evidently be one in which practically the whole of the material reaches the maximum intensity of stress. For example, to resist economically a bending moment which produces a longitudinal direct stress the intensity of which at any point of a cross-section is proportional to the distance from the neutral axis, much of the area of cross-section should be placed at a maximum distance from the neutral axis. This suggests the **I** section, which is the commonest form of steel beams whether rolled in a single piece (see Fig. 62) or built up by riveting together component parts. In such a section most of the area is situated at nearly the full half depth, so that, neglecting the thin vertical web, the moment of inertia $\Sigma(y^2\delta A)$, approximates to—

$$\text{(area of two flanges)} \times \left(\frac{d}{2}\right)^2$$

or the radius of gyration approximates to $\frac{d}{2}$, and the modulus of sec-

tion, Z, which is the moment of inertia divided by $\frac{d}{2}$, approximates to—

$$\text{(area of two flanges)} \times \frac{d}{2}$$

or, $$Z = 2bt \times \frac{d}{2} = b \,.\, t \,.\, d \text{ approximately}$$

where $t$ is the mean thickness of the flange, generally measured in a rolled section at $\frac{1}{4}$ the breadth from either end. These approximations are often very close to the true values, for they exaggerate by taking the flange area wholly at $\frac{d}{2}$ from the neutral axis XX and under-estimate by neglecting the vertical web.

*Plate Girder Sections.*—The plate girder consisting of horizontal plate flanges united to a vertical plate web by angles (see Fig. 97) is of such great importance in structural steel work that it is now considered more fully. Either the depth or the flange area is often varied so that the moment of inertia of every cross-section is roughly proportional to the greatest bending moment to which it is subjected as explained in Chap. XVII., Arts. 186 and 187. Various approximations are in use for estimating the modulus of section and moment of resistance of such a section. For a fairly deep girder perhaps the best approximation is

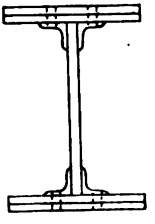


FIG. 97.—Single-web plate girder section.

$$\text{Modulus of section } Z = A \times d$$

where A = net area of one flange, including plates and angles, but no part of web, and $d$ = depth to outside of angles. Sometimes $d$ is taken between the centroids of the flanges and sometimes A includes $\frac{1}{6}$ or $\frac{1}{8}$ of the web. It is usual in calculating A to subtract from the plate and angle sectional areas the area of rivet holes which may lie even approximately in the same plane of cross-section, and a hole $\frac{1}{16}$ in. or $\frac{1}{8}$ in. larger than the nominal rivet diameter is so deducted. It is frequently desirable for purposes of design to work from a simple approximation and then to check, and if necessary adjust the resulting dimensions by a more exact calculation.

*Box Plate Girder.*—This form possesses considerable lateral flexural stiffness, and is in considerable use (see Fig. 98).

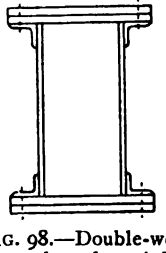


FIG. 98.—Double-web or box plate girder section.

*Compound Girder Section.*—Built-up sections consisting of plate flanges added to rolled I and channel sections are shown in Figs. 99, 100, and 101. The moment of inertia of the (net) plate area about the neutral axis (see theorem I. Art. 52) is added to the known moment of inertia of the rolled sections, and the sum divided by the half depth gives the

modulus of section. An approximate correction for rivet holes in the rolled sections may easily be made (see Example 3 below).

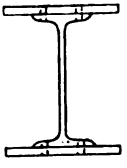


FIG. 99.

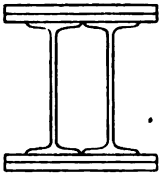


FIG. 100.

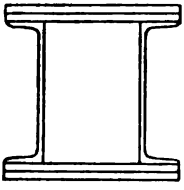


FIG. 101.

EXAMPLE 1.—A box plate girder has a span of 36 feet, and its depth over the angles is 42 ins. It has to carry a load equivalent to 3·5 tons per foot run, with a maximum bending stress of 6 tons per square inch. The two $\frac{3}{8}$ in. webs are connected to the flanges by angles $4 \times 4 \times \frac{1}{2}$ ins. (see B.S.E.A. 11, Table V., Appendix). Calculate the total flange area required at the centre of the span, allowing for two $\frac{7}{8}$-in. rivets in each flange; and if the total thickness is $1\frac{1}{8}$ in., what width of plate will be required?

Central bending moment $= \frac{1}{8}Wl = \frac{3{\cdot}5 \times 36 \times 36 \times 12}{8} = 6804$ ton-ins.

Modulus of section (Z) required $= \frac{6804}{6} = 1134$ (ins.)$^3$

Let B = width required.

Gross area of two angles $= 2 \times 3{\cdot}749 = 7{\cdot}5$ sq. ins.

Net " " " allowing, say, two holes $1 \times \frac{1}{2}$ in each $= 7{\cdot}5 - 2 = 5{\cdot}5$ ins.

Then taking 1 in. holes, net area of flange (A) $= (B - 2)1\frac{1}{8} + 5{\cdot}5 = 1{\cdot}125B + 3{\cdot}25$ sq. ins.

Approximately—

$$Z = (1{\cdot}125B + 3{\cdot}25) \times 42 = 1134$$
$$B = 21{\cdot}1 \text{ ins.}$$

Checking this approximation by the more exact method, we find

$$I = \tfrac{1}{12}\{(B - 2)(44{\cdot}25^3 - 42^3) + 8{\cdot}75(42^3 - 41^3) + 1{\cdot}75(41^3 - 34^3) + \tfrac{3}{4} \times 34^3\} - 1{\cdot}75 \times 2 \times 19^2$$

the last term being an approximation for the horizontal holes through the angles and web. This gives $I = 1046(B - 2) + 9290$ (ins.)$^4$

Now, the required value of I is $22\frac{1}{8} \times Z = 1134 \times 22\frac{1}{8} = 25{,}090$

hence
$$B - 2 = \frac{15800}{1046} = 15{\cdot}1$$
$$B = 17{\cdot}1$$

which shows the approximation to be somewhat far on the safe side in this case. If $\frac{1}{8}$ of the web area were included in the flange area A, we should have had—

$$1{\cdot}125B + 3{\cdot}25 + \tfrac{1}{8} \times 42 \times \tfrac{3}{4} = \tfrac{1134}{42} = 27$$
$$1{\cdot}125B = 19{\cdot}8 \qquad B = 17{\cdot}6 \text{ ins.}$$

which more nearly agrees with the more exact calculation.

EXAMPLE 2.—A single web plate girder has a span of 40 ft., and its depth over the angle is 42 ins. It has to carry a uniformly distributed load of 89 tons with a maximum bending stress of 6 tons per square inch. What thickness of plate 14 ins. wide is required in the flanges if the angles are $6 \times 6 \times \frac{1}{2}$ ins. and the web $\frac{1}{2}$ in. thick? (Allow for 2 rivet holes say 1 in. diameter in each flange and angle.)

$$\text{Central bending moment} = \frac{89 \times 40 \times 12}{8} = 5340$$

$$\text{Modulus of section required} = \tfrac{5340}{6} = 890 \text{ (ins.)}^3$$

$$\text{Approximate flange area required} = \tfrac{890}{42} = 21{\cdot}2 \text{ sq. ins.}$$

and if $t$ = thickness of flats, allowing say two 1-in. holes and four in the angles,

$$12t + 2(11{\cdot}5 - 2)\tfrac{1}{2} = 12t + 9{\cdot}5 = 21{\cdot}2,$$

hence $12t = 11{\cdot}7 \qquad t = 0{\cdot}975$, say 1 in.

Checking by the more exact method, neglecting horizontal holes through angles and web,

$$I = \tfrac{1}{12}\{12(44^3 - 42^3) + 10{\cdot}5(42^3 - 41^3) + 1{\cdot}5(41^3 - 30^3) + \tfrac{1}{2} \times 30^3\}$$
$$= 21{,}982 \text{ (ins.)}^4$$

Allow for neglected rivet holes, say $2 \times 1{\cdot}5 \times 1 \times 17{\cdot}5^2 = 919$.

$$\text{Net value of } I = 21{,}063 \text{ (ins.)}^4$$
$$\text{Value of } Z = \tfrac{21063}{22} = 957 \text{ (ins.)}^3$$
$$\text{Excess} = 957 - 890 = 67 \text{ (ins.)}^3$$

corresponding to an area $\frac{67}{42} = 1{\cdot}5$ sq. in. say, or on flats of 12 ins. net width $\frac{1}{8}$ in. thickness. Hence $\frac{7}{8}$ in. thickness would be sufficient.

This example illustrates the use of the approximate formula, for to have to find $t$ directly by the more exact rule would have involved the unknown quantity in the third power, *i.e.* a cubic equation in $t$.

The limitations of an empirical rule for different proportions may also be noted, for had $\frac{1}{8}$ of the web area been added to the flange area the simple rule would have given too thin a plate to the flange. This would also have been so, but in a smaller degree, if the effective depth had been taken as that between the centres of gravity of the flanges, which in this case is less than the 42-in. depth over the angles. The simpler rules cannot be correct for all cases including large and small angles and varying proportions of depth to flange area, but are nevertheless useful, and may easily be framed so as always to err on the side of safety.

EXAMPLE 3.—A compound girder (as in Fig. 100) is to be made by riveting six $\frac{1}{2}$-in. flats on to the flanges of two $15 \times 6$ ins. I beams (B.S.B. 26, Table I., Appendix). What width of plate will be necessary if the girder has to carry a total uniformly distributed load of 74 tons over a span of 20 feet with a maximum stress of 5 tons per square inch? ($\frac{7}{8}$-in. rivets.)

Referring to column 9, Table I. for the given sections, I = 628·9.

$$\text{Central bending moment} = \frac{74 \times 20 \times 12}{8} = 2220 \text{ ton-ins.}$$

$$\text{Z required} = \tfrac{2220}{5} = 444 \text{ (ins.)}^3$$

$$\text{I required} = 444 \times \tfrac{18}{2} = 3996 \text{ (ins.)}^4$$

$$\text{I for two rolled sections} = 2 \times 628{\cdot}9 = 1258$$

$$\text{difference} = 2738$$

$$\text{Add for holes in I beam flanges say } 4 \times 0{\cdot}8 \times 7{\cdot}5^2 = 180$$

$$\text{I required for flats} = 2918$$

If B = required width,

$$\frac{B-2}{12}\left(18^3 - 15^3\right) = 204{\cdot}75(B-2) = 2918$$

$$B = 2 + \frac{2918}{204{\cdot}75} = 16{\cdot}2 \text{ ins.}$$

EXAMPLE 4.—A girder is made up of two channels (as in Fig. 101) and two flats. The channels are 15 × 4 ins. (see B.S.C. 27, Table II., Appendix). The flats are 14 × ½ ins. What load may the girder carry at its centre over a 14-ft. span (neglecting its own weight) without the extreme bending stress exceeding 5 tons per square inch? Allow for two ⅞-in. rivets in each flange section. Referring to line 1, column 10 of Table II., Appendix, I = 377 per channel = 754 (ins.)⁴ for the two.

$$I = 754 + \tfrac{12}{12}(16^3 - 15^3) - 4 \times 1 \times 0{\cdot}63 \times 7{\cdot}5^2$$

$$= 754 + 721 - 142 = 1333$$

$$Z = \tfrac{1333}{8} = 167 \text{ (inches)}^3$$

Moment of resistance = 167 × 5 = 835 ton-ins.

If W = central load in tons

$$\tfrac{1}{4} \times W \times 14 \times 12 = 835 \qquad \text{or, } W = 19{\cdot}9 \text{ tons.}$$

EXAMPLE 5.—The girder in Example 2, Art. 57, is to carry a live load of 52·5 tons uniformly distributed, and a dead load of 23·17 tons similarly applied at cross girders. If the depth of girder over the angles is 4 ft., width of flanges 21 ins., and angles 4 × 4 × ½ ins., find the necessary flange plates at the central section, using the dynamic method (Art. 41) with a dead-load stress of 6·5 tons per square inch.

Using the result of Example 2, Art. 57, in direct proportion, bending moment due to live load at centre (and at all points between $b$ and $b'$) is

$$271{\cdot}2 \times \frac{52{\cdot}5}{56{\cdot}5} = 252 \text{ ton-ft.}$$

and due to dead load

$$271{\cdot}2 \times \frac{23{\cdot}17}{56{\cdot}5} = 111 \text{ ton-ft.}$$

From Art. 41 (6) we may find the working stress for the total bending moment 252 + 111, or from (7) we may use 6·5 tons per square inch with a bending moment 111 + (2 × 252) = 615 ton-ft. Selecting the latter method, we have,

$$\text{Modulus of section (Z) required} = \frac{615 \times 12}{6{\cdot}5} = 1135 \text{ (ins.)}^3$$

$$\text{Net area of flange required} = \frac{1135}{48} = 23{\cdot}64 \text{ sq. ins.}$$

The two angles (B.S.E.A. 11, Table V., Appendix) less 4 rivet holes $\frac{15}{16}$-in. diameter give,

$$7{\cdot}498 - 1{\cdot}875 = 5{\cdot}623 \text{ sq. ins.}$$

The plates therefore require $23{\cdot}64 - 5{\cdot}62 = 18{\cdot}02$ sq. ins.

Net width of flange allowing 4 rivets $\frac{7}{8}$-in. $= 21 - 4 \times \frac{15}{16} = 17{\cdot}25$ ins.

$$\text{Thickness required} = \frac{18{\cdot}02}{17{\cdot}25} = 1{\cdot}05, \text{ say } 1\tfrac{1}{16} \text{ in.}$$

which may be made up by $\frac{9}{16}$-in. main plates (next to angles) and $\frac{1}{2}$-in. outer plates.

**68. Cast Iron Girders.**—Cast iron is generally five or six times as strong in compression as in tension, but a symmetrical section would in bending get approximately equal extreme intensities of tension and compression so long as the material does not greatly deviate from proportionality between stress and strain (see Art. 63). Cast iron has no considerable plastic yield, so that the distribution of stress beyond the elastic limit will not be greatly different from that within it. Hence a cast-iron beam of symmetrical section would fail by tension due to bending, and it would appear reasonable to so proportion the section that the greatest intensity of compressive stress would be about five times that of the tensile stress. This could be done by making the section of such a form that the distance of its centroid from the extreme compression layers is five times that from the extreme tension layers. This, in a flanged or irregular I section, would involve a large tension flange, and a much smaller compression flange: so great a difference as that indicated above involves serious initial stresses due to the quicker cooling of the small compression flange compared to that of the larger tension flange, and experience shows that distances of the compression and tension edges to the centroid in the ratio of about 2 or 3 to 1 (see Fig. 102) give the most economical results, the tension flange being made wide in order to avoid great thickness, which would involve relatively slow cooling. The moment of inertia of such a section as that shown in Fig. 102 may be estimated by division into rectangles (see Art. 52), or graphically, as in Art. 53.

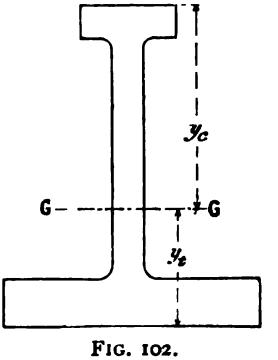


FIG. 102.

**69. Reinforced Concrete Beams.**[1]—Cement and concrete are well adapted to stand high compressive stress, but little or no tension. They

[1] For graphical method, see "The Graphic Statics of Reinforced Concrete Sections," in *Engineering*, December 25, 1908.

can be used to withstand bending by *reinforcement* with metal to take the tension involved, the metal being by various means held fast in the concrete. The usual assumption is that the metal carries the whole of the tension, and the concrete the whole of the compression. In the case of a compound beam of this kind, the neutral axis will not generally pass through the centroid of the area of cross-section because of the unequal values of the direct modulus of elasticity (E) of the two materials (see Art. 62). It may be found approximately by equating the total compressive force or thrust in the cement to the total pull in the metal. As the cross-section of metal usually occupies a very little of the depth, it is usual to take the area of metal as concentrated at the depth of its centre and subject to a uniform intensity of stress equal to that at its centre.

The following simple theory of flexure of ferro-concrete beams must be looked upon as approximate only, since the tension in the concrete is neglected; and further, in a heterogeneous substance like concrete, the proportionality between stress and strain will not hold accurately with usual working loads. More elaborate and less simple empirical rules have been devised and tested by experiment, but the following methods of calculation are the most widely recognised.

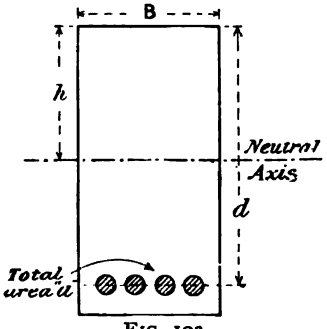


FIG. 103.

Suppose a ferro-concrete beam has the sectional dimensions shown in Fig. 103; assume that, as in Arts. 61 and 65, the strain due to bending is proportional to the distance from the neutral axis and to the direct modulus of elasticity of the material. Let $h$ be the depth of the neutral axis from the compression edge of the section, $f_c$ the (maximum) intensity of compressive stress at that edge, and $f_t$ the intensity of tensile stress in the metal reinforcement, this being practically uniform. Let $E_c$ be the direct modulus of elasticity of the concrete in compression, and $E_t$ that of the steel in tension.

Then $\frac{f_c}{E_c}$ is the proportional strain in the concrete at the compression edge (see Art. 61), and $\frac{f_t}{E_t}$ is the proportional strain in the metal.

The distances from the neutral axis at which these strains occur are $h$ and $(d - h)$ respectively, and since the strains are to be assumed proportional to the distance from the neutral axis (Arts. 61 and 65)

$$\frac{f_c}{E_c} \div \frac{f_t}{E_t} = \frac{h}{d - h}$$

or,

$$\frac{f_c}{f_t} = \frac{h}{d - h} \cdot \frac{E_c}{E_t} \quad \ldots\ldots \quad (1)$$

The ratios of $E_c$ to $E_t$ for given materials are known; for concrete and steel the ratio is usually from $\frac{1}{10}$ to $\frac{1}{15}$.

The total thrust is

(mean intensity of compressive stress) × (compression area) $= \frac{f_c}{2} . h . B$

The total tensile stress, neglecting any in the concrete, is

$$f_t \times (\text{area of section of reinforcement}) = f_t . a$$

And since the total thrust equals the total pull, the two together forming the couple which is the moment of resistance

$$\frac{f_c}{2} \times h . B = f_t . a \text{ or } \frac{f_c}{f_t} = \frac{2a}{h . B} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and therefore from (1)

$$\frac{2a}{h . B} = \frac{h}{d - h} \cdot \frac{E_c}{E_t}$$

which gives a quadratic equation in $h$ in terms of the quantities B, $a$, $d$, and $\frac{E_c}{E_t}$, all of which are supposed to be known.

Ferro-concrete beam sections are generally rectangular, but in case of the compression part of the section having any other shape, we should proceed as follows to state the total thrust in terms of the maximum intensity $f_c$ at the extreme edge at the (unknown) distance $h$ from the neutral axis.

Let $z$ be the width of section parallel to the neutral axis at a height $y$ from it, varying in a known manner with, say, the distance $(h - y)$ from the compression edge, and let $p$ be the intensity of stress at any height $y$ from the neutral axis; then

$$\frac{p}{y} = \frac{f_c}{h}, \quad p = \frac{f_c}{h} . y$$

$$\text{Total thrust} = \int_0^h p . z . dy = \frac{f_c}{h} \int_0^h y . z . dy$$

which can be found when the width $z$ is expressed in terms of, say, $h - y$. This might also be written

$$\text{Total thrust} = f_c \times (\text{area of compression modulus figure})$$

(see end of Art. 66). In the rectangular section of Fig. 103, $z = B =$ constant, this being the simplest possible case.

Frequently the compression area of ferro-concrete is **T**-shaped, consisting partly of a concrete slab or flooring and partly of the upper part of the rectangular supporting beam, the lower part of which is reinforced for tension, the floor and beam being in one piece, or "monolithic" (see Ex. 3 below, and note following it). The breadth is then constant over two ranges, into which the above integrations can conveniently be divided. The thrust in the vertical leg of the **T** (or upper part of the beam) is often negligible compared to that in the cross-piece or slab.

The resisting moment, about the neutral axis, of the total thrust would be

$$\frac{f_c}{h}\int_0^h y^2 . z dy \quad \text{or} \quad f_c . \frac{I}{h}$$

where I is the moment of inertia of the compression area about the neutral axis. The graphical equivalent of this would be

$f_c \times$ (area of compression modulus figure) $\times$ (distance of its centroid from the neutral axis)

the centroid of the modulus figure being with the centre of pressure or thrust, or, using the second derived area as in Art. 53

resisting moment of the thrust $= f_c \times h \times$ second derived area of compression section

The resisting moment, about the neutral axis, of the total tension is evidently $f_t \times a \times (d - h)$, and the total moment of resistance is

total thrust (or pull) $\times$ distance of centre of thrust from reinforcement

EXAMPLE 1.—A reinforced concrete beam 20 inches deep and 10 inches wide has four bars of steel 1 inch diameter placed with their axes 2 inches from the lower face of the beam. Find the position of the neutral axis and the moment of resistance exerted by the section when the greatest intensity of compressive stress is 100 lbs. per square inch. What is then the intensity of tensile stress in the steel? Take the value of E for steel 12 times that for concrete.

Using the symbols of Fig. 103 and those above

$$d = 20 - 2 = 18 \text{ inches}$$

$$\frac{f_c}{E_c} \div \frac{f_t}{E_t} = \frac{\text{maximum compressive strain}}{\text{tensile strain in metal}} = \frac{h}{18 - h}$$

$$\frac{f_c}{f_t} = \frac{E_c}{E_t} \cdot \frac{h}{18 - h} = \frac{h}{12(18 - h)}$$

and equating the total pull in the steel to the thrust in the concrete

$$f_t \cdot 4 \cdot \frac{\pi}{4} = \tfrac{1}{2} f_c \cdot h \cdot 10$$

Therefore

$$\frac{f_c}{f_t} = \frac{4 \times \frac{\pi}{4}}{h \cdot \frac{1}{2} \cdot 10} = \frac{\pi}{5h} = \frac{h}{(18 - h)12}$$

hence $5h^2 + 12\pi h - 216\pi = 0$

and solving this, $h = 8{\cdot}5$ inches

The distance from the neutral axis to the centre of the steel rods $= 18 - 8{\cdot}5 = 9{\cdot}5$ inches. The total thrust is

$$\frac{100}{2} \times 10 \times 8{\cdot}5 = 4250 \text{ lbs.}$$

and the total tension in the metal is equal to this.

The distance of the centre of pressure from the neutral axis is $\frac{2}{3}$ of 8·5 inches, and that of the tension is 9·5 inches.

The moment of resistance is therefore

$$(4250)(9{\cdot}5 + \tfrac{2}{3} \text{ of } 8{\cdot}5) = 64{,}460 \text{ lb.-inches}$$

The intensity of tensile stress in the steel of area $\pi$ square inches is

$$f_t = \frac{4250}{\pi} = 1350 \text{ lbs. per square inch}$$

or thus,

$$\frac{f_t}{100} = 12 \times \frac{9{\cdot}5}{8{\cdot}5} = 1342$$

which checks the above approximate result.

EXAMPLE 2.—A reinforced concrete floor is to carry a uniformly spread load, the span being 12 feet and the floor 10 inches thick. Determine what reinforcement is necessary and what load per square foot may be carried, the centres of the steel bars being placed $1\frac{1}{2}$ inch from the lower side of the floor, the allowable stress in the concrete being 600 lbs. per square inch, and in the steel 12,000 lbs. per square inch, and the modulus of direct elasticity for steel being 10 times that for concrete. If the load per square foot of floor is 300 lbs., estimate the extreme stresses in the materials, assuming bending in one direction only.

Let $h$ = distance of the neutral axis from the compression edge.

Then the distance from the centres of the steel rods is $10 - 1{\cdot}5 - h = 8{\cdot}5 - h$ inches.

The ratio of stress intensities is

$$\frac{\text{intensity of tensile stress}}{\text{maximum intensity of pressure}} = \frac{12{,}000}{600} = \frac{8{\cdot}5 - h}{h} \times 10$$

hence

$$8{\cdot}5 - h = 2h$$
$$h = 2{\cdot}8\dot{3} \text{ inches}$$

Taking a strip of floor 1 inch wide

$$\text{thrust of concrete} = \frac{600}{2} \times 2{\cdot}8\dot{3} \times 1 = 850 \text{ lbs.}$$

The total tension in the steel must also be 850 lbs., and the area of section required is therefore

$$\frac{850}{12{,}000} = 0{\cdot}07083 \text{ square inch}$$

per inch width of floor. If round bars 1 inch diameter are used, they might be spaced at a distance

$$\frac{0{\cdot}7854}{0{\cdot}0708} = 11{\cdot}1 \text{ inches apart}$$

The total moment of resistance is

$$850\{(\tfrac{2}{3} \times 2{\cdot}8\dot{3}) + (8{\cdot}5 - 2{\cdot}8\dot{3})\} = 6422 \text{ lb.-inches}$$

which is the product of the total thrust (or tension), and the distance between the centre of pressure and the centres of the rods.

If $w$ = load per inch run, which is also the load per square inch of the floor, equating the moment of resistance to the bending moment—

$$\tfrac{1}{8}w \times 144 \times 144 = 6422$$

$$144w = \frac{8 \times 6422}{144} = 357 \text{ lbs.}$$

which is the load per square foot.

If the load were only 300 lbs. per square foot, the stresses would be proportionally reduced, and

$$\text{maximum intensity of pressure} = 600 \times \tfrac{300}{357} = 505 \text{ lbs. per square inch}$$

$$\text{intensity of tensile stress} = 12{,}000 \times \tfrac{300}{357} = 10{,}090 \text{ lbs. per square inch.}$$

EXAMPLE 3.—A reinforced beam is of T section, the cross-piece or compression flange being 20 inches wide and 4 inches deep, and the vertical leg 14 inches deep by 8 inches wide. The reinforcement consists of two round bars of steel $1\frac{1}{2}$ inch diameter placed with their axes 2 inches from the lower face. Making the usual assumptions, calculate the intensity of stress in the steel, and the total amount of resistance exerted by a section of the beam when the compressive-stress in the concrete reaches 500 lbs. per square inch. Take the modulus of direct elasticity in steel 12 times that for concrete in compression.

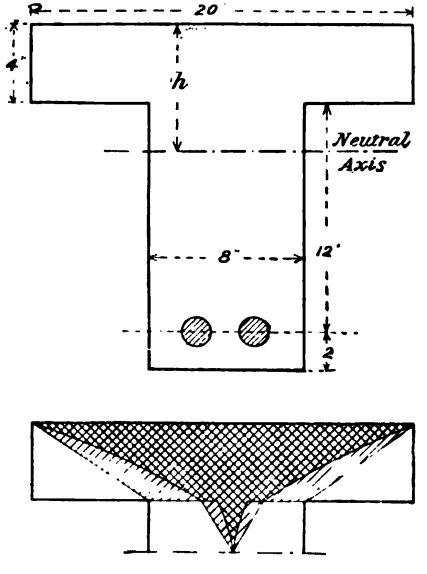



FIG. 104.

Let $f$ = intensity of stress in the steel

$h$ = distance of the neutral axis from the compression edge (see Fig. 104).

The ratio of the stress intensities is then—

$$\frac{f_t}{500} = \frac{16 - h}{h} \times 12$$

whence

$$f_t = \frac{16 - h}{h} \times 6000 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{The total thrust} = \frac{500}{h}\int_{h-4}^{h} 20y\,dy + \frac{500}{h}\int_{0}^{h-4} 8y\,dy$$

$$= \frac{10{,}000}{h \times 2}\{h^2 - (h-4)^2\} + \frac{4000}{2h}(h-4)^2$$

$$= \frac{40{,}000}{h}(h-2) + \frac{2000}{h}(h-4)^2$$

the first term representing the thrust in the cross-piece, and the second that in the vertical leg above the neutral axis. The total tension is—

$$2 \cdot \frac{\pi}{4} \cdot \tfrac{9}{4} f_t = \frac{9\pi}{8} f_t$$

and substituting for $f_t$ from (1) and equating to the total thrust—

$$\frac{9\pi}{8} \cdot \frac{16 - h}{h} \cdot 6000 = \frac{40{,}000}{h}(h-2) + \frac{2000}{h}(h-4)^2$$

from which $h = 6{\cdot}6$ inches and—

$$f_t = 6000 \cdot \frac{16 - 6{\cdot}6}{6{\cdot}6} = 8550 \text{ lbs. per square inch.}$$

The moment of the thrust about the neutral axis is—

$$\frac{500}{6{\cdot}6}\int_{2{\cdot}6}^{6{\cdot}6} 20 \cdot y^2 \cdot dy + \frac{500}{6{\cdot}6}\int_{0}^{2{\cdot}6} 8 \cdot y^2 \cdot dy = \frac{500 \times 20}{6{\cdot}6 \times 3}\{(6{\cdot}6)^3 - (2{\cdot}6)^3\}$$

$$+ \frac{500 \times 8}{6{\cdot}6 \times 3}(2{\cdot}6)^3 = 139{,}000 \text{ lb.-inches}$$

The moment of the tension is—

$$8550 \times \frac{9\pi}{8} \times 9{\cdot}4 = 284{,}000 \text{ lb.-inches}$$

and the total moment of resistance is—

$$139{,}000 + 284{,}000 = 420{,}000 \text{ lb.-inches.}$$

The values found for total thrust and the moment of resistance would not be greatly altered by the omission of the second term in the respective integrals, *i.e.* by neglecting the small thrust in the vertical leg of the section above the neutral axis. The moment of resistance might be estimated graphically by drawing the modulus figure for the compression area with a pole on the neutral axis (see Fig. 104); the moment of resistance for compression would then be—

500 × (area of compression modulus figure) × (distance of its centroid from the axis)

or if a second derived figure be drawn, the moment would be—

500 × 6·6 × (area second derived figure).

The total tension moment would be—

500 × (area of first compression modulus figure) × 9·4

*Note.*—A very common example of a T section occurs in ferro-concrete floors with monolithic cross-beams, the floor forming the cross-piece of the T. The cross-piece is then often very wide in proportion to the remainder of the T section, and with a moderately high intensity of stress in the reinforcement the neutral axis would fall within the cross-piece instead of below it. This would involve tension in the lower side of the floor slab, which is not reinforced for tension in that direction, and might start cracks. This undesirable result can be avoided by employing more reinforcement at a consequently lower intensity of stress in the cross-beam or vertical leg of the T section.

**70. Unsymmetrical Bending.**—In considering simple bending (Art. 61) it was assumed that the beam had a cross section symmetrical about the axis through its centroid and in the plane of bending. The planes of bending and that of the external bending couple will be parallel if the axis of cross-section in the plane of the external moment is a principal axis (Art. 54). If this condition is not fulfilled, let OY', Fig. 105, be the plane of the external bending moment (shown by its trace on the section which is in the plane of the figure) inclined at an angle $\alpha$ to the principal axis OY, or let the bending couple M be in a plane perpendicular to OX'.

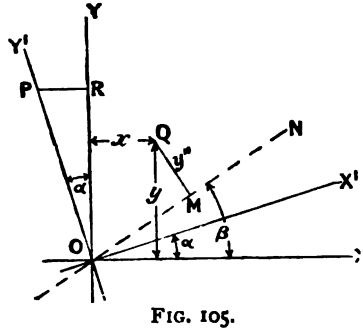


FIG. 105.

If the couple M, represented by OP, say, be resolved into components represented by OR and RP about the principal axes OX and OY, these components will be—

$$M \cos \alpha \text{ and } -M \sin \alpha \text{ respectively.}$$

The intensity of bending stress and the strain everywhere on the section can then be found by taking the algebraic sum of the effects produced by the component bending moments about the two principal axes. Thus, the unit stress at any point Q the co-ordinates of which referred to the principal axes OX and OY are $x$, $y$ will be from (5) Art. 63—

$$p = \frac{y \cdot M \cos \alpha}{I_x} - \frac{xM \sin \alpha}{I_y} \quad \ldots\ldots \quad (1)$$

where $I_x$ and $I_y$ are the principal moments of inertia of the beam section about OX and OY respectively. For a point the co-ordinates of which are $-x$, $y$:

$$p = \frac{y \cdot M \cos \alpha}{I_x} + \frac{xM \sin \alpha}{I_y} \quad \ldots\ldots \quad (2)$$

For points on the neutral axis, putting $p = 0$ in (1)—

$$y = x \frac{I_x}{I_y} \tan \alpha \quad \ldots\ldots\ldots \quad (3)$$

which is a straight line ON through the centroid of the section inclined to OX at an angle $\beta$, so that—

$$y = x \tan \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\tan \beta = \frac{I_x}{I_y} \tan \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It may be noted that the relation (5), which may be written

$$\tan \beta = \frac{k_x^2}{k_y^2} \tan \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is that between the slopes of conjugate axes of the momental ellipse (Art. 54), the principal semi-axes of which are the radii of gyration $k_y$ about OY in the direction OX and $k_x$ about OX in the direction OY. Consequently, if the momental ellipse is drawn the direction of the neutral axis ON (Fig. 105) may be found by drawing the diameter conjugate to OY', which is easily accomplished by joining O to the point of bisection of a chord parallel to OY'.

To find the maximum stress in a given section resulting from a given bending moment in any given plane we first calculate the direction of the principal axes and values of the principal moments of inertia as described in Art. 54. Then calculate the direction of the neutral axis from (5) and draw it on the given section and find by inspection the point in the section furthest from the neutral axis and apply equation (1). The intensity of stress might also be stated in terms of $y''$, the distance from the neutral axis (Fig. 105) for

$$QM = y'' = y \cos \beta - x \sin \beta \quad . \quad . \quad . \quad . \quad (7)$$

and from (5)—

$$y\frac{\cos \alpha}{I_x} - x\frac{\sin \alpha}{I_y} = \frac{y \cos \beta}{x \sin \beta} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

hence

$$\left(\frac{y \cos \alpha}{I_x} - \frac{x \sin \alpha}{I_y}\right) \div y'' = \frac{x \sin \alpha}{I_y} \div x \sin \beta \quad . \quad . \quad (9)$$

and substituting this in (1) and then for sin $\alpha$ from (8)—

$$p = \frac{M \cdot y''}{I_y} \cdot \frac{\sin \alpha}{\sin \beta} = \frac{M \cdot y''}{\sqrt{I_x^2 \cos^2 \beta + I_y^2 \sin^2 \beta}} \quad . \quad . \quad (10)$$

The maximum value $f_1$, tensile or compressive of $p$, can be found by writing the maximum value of $y''$ on the tensile or compressive side of the neutral axis.

*Another form of the result.*—The value of $p$ might also be stated directly in terms of the moment of inertia of the section about the neutral axis ON from the general formula (5) Art. 63, for the component bending moment about ON resulting from the bending moment M about OX' is M cos $(\beta - \alpha)$, hence—

$$p = \frac{y'' \cdot M \cos (\beta - \alpha)}{I_N} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $I_N$ is the moment of inertia about the neutral axes ON, which

may be found graphically as described in Art. 54 from the momental ellipse or from (2) Art. 54, writing $\beta$ for $\alpha$, which gives from (11) above

$$p = \frac{y'' M \cos(\beta - \alpha)}{I_x \cos^2 \beta + I_y \sin^2 \beta} \quad \ldots\ldots \quad (12)$$

a formula easily reduced to the form (10) by the relation (5) between $\beta$ and $\alpha$.

The choice of one or other method of dealing with a case of unsymmetrical bending will depend partly on the type of section. Thus in rectangular sections a corner will always be a point of maximum stress, and formula (2) may be applied directly. In other sections it may be more convenient to draw the neutral axis to determine for which point in the section the unit stress is a maximum.

EXAMPLE.—Calculate the allowable bending moment on a British Standard unequal angle $6'' \times 3\frac{1}{2}'' \times \frac{3}{8}''$, carrying a load on the short edge with the long edge vertically downwards, if the stress is limited to 6 tons per square inch and the area, principal moments of inertia and position of the centroid of the section are given.

The particulars from the standard tables are given in Fig. 106, and as follows. Tan XOX′ = tan $\alpha$ = 0·344, hence $\alpha = 19°$; $I_x = 13{\cdot}908$ (inches)$^4$; $I_y = 1{\cdot}963$ (inches)$^4$; area = 3·424 square inches, hence $k_x = 2{\cdot}015$ inches, $k_y = 0{\cdot}757$ inches.

These may be obtained approximately from Table IV. (B.S.U.A. 20) in the Appendix. $I_x$ and $I_y$ are obtained by substituting the values given in columns 9 and 10 in equations (9) and (10) of Art. 54.

The position of the neutral axis may be found by (5)

$$\tan \beta = \frac{13{\cdot}908}{1{\cdot}963} \times 0{\cdot}344 = 2{\cdot}437 = \tan 67{\cdot}7°$$

The neutral axis ON is set off on the left of Fig. 106, and by inspection it is evident that P is the furthest point in the section from ON; its distance from OX is $3{\cdot}84'' = -y$, and its distance from OY is $0{\cdot}83'' = +x$, hence from (1) putting $p = 6$ tons per square inch

$$6 = -\frac{3{\cdot}84M \cos 19°}{13{\cdot}908} - \frac{0{\cdot}83M \sin 19°}{1{\cdot}963} = -M(0{\cdot}261 + 0{\cdot}1375)$$

hence $M = -15{\cdot}05$ ton-inches, the negative sign merely indicating the kind of bending moment, P being, say, on the tension side of the neutral axis ON. The compressive stress at the point Q can readily be found from (1).

*Graphical Solution.*—Set out the momental ellipse on the right of Fig. 106, such that tan XOX′ = 0·344 or angle XOX = 19°, O′A = $k_x$ = 2·015″, O′B = 0·757″ (on any scale). Draw any chord RS parallel to OY′, and bisect it in V; draw NO′N′ the neutral axis through O′ and V. Set out this neutral axis ON on the section, as shown to the left of the figure, and look out the distance from it of the most remote point P which measures 2·22″. Through C draw the tangent to the ellipse parallel to ON, and measure its perpendicular distance from

NO'N' which is 1·04". Then the moment of inertia of the section about ON is

$$(1\cdot04)^2 \times 3\cdot424 = 3\cdot70 \text{ (inches)}^4$$

Then measuring the angle NOX' as 48·7° and applying (11)

$$6 = 2\cdot22 \times M \times \cos 48\cdot7^\circ = 0\cdot396M$$

and M = 15·15 ton-inches, confirming approximately the previous result.

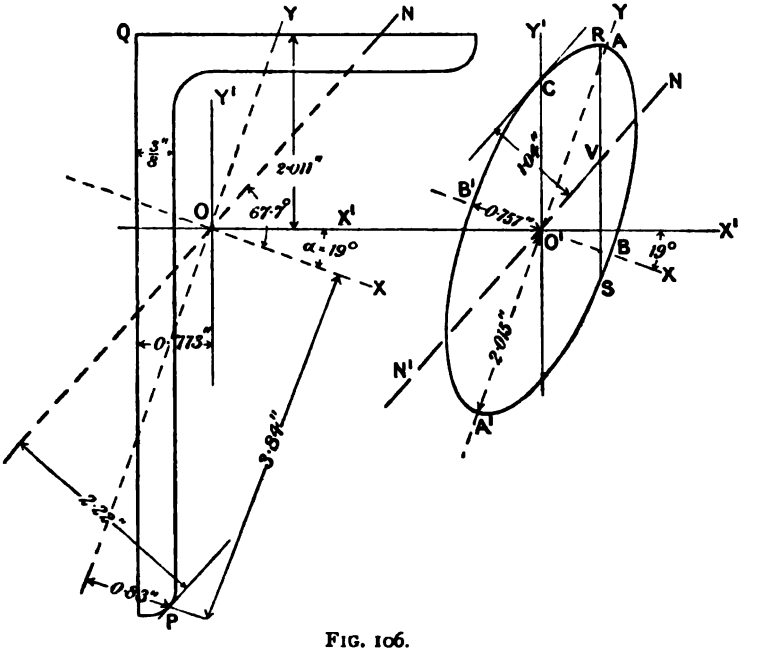



FIG. 106.

EXAMPLE 2.—A British Standard equal angle section measures $4\frac{1}{2}'' \times 4\frac{1}{2}'' \times \frac{3}{8}''$ and is rounded to a radius of 0·275 inch at its outer ends or toes. Its area of section is 3·236 square inches, and the distance of its centroid from either outside edge is 1·244 inch. Its principal moments of inertia are 9·768 (inches)$^4$ and 2·514 (inches)$^4$, the former being about an axis through the intersection of the outer edges. A beam of this section, and simply supported at its ends, has one side of the angle horizontal and carries on it a vertical load of $\frac{1}{2}$ ton midway between the supports, which are 5 feet 4 inches apart. Find the greatest tensile and compressive stresses in the material.

In this case from the symmetry $\alpha = 45^\circ$, and the given values may be obtained from Table V. (BSEA 12) in the Appendix. One principal moment of inertia is found from columns numbered 3 and 10, and the other then follows from equation (1), Art. 54.

If $\beta$ is the angle which the neutral axis makes with the principal axis passing through the intersection of the edges, from (5)

$$\tan \beta = \frac{9{\cdot}768}{2{\cdot}514} = 3{\cdot}885$$

Hence from tables $\beta = 75{\cdot}6^\circ$

The neutral axis is inclined to the loaded edge at an angle

$$75{\cdot}6^\circ - 45^\circ = 30{\cdot}6^\circ$$

The most distant point in tension may be measured from a drawing to scale or calculated; it occurs on the curved toe, as in Fig. 106. The co-ordinates of the centre of the curve referred to axes parallel to the angle edges are known, and hence the distance from the neutral axis is easily calculated about an oblique neutral axis; the distance to the curved toe exceeds the distance to the centre by the radius 0·275". Either method gives $y'' = 2{\cdot}26''$.

About the neutral axis

$$I_N = 9{\cdot}768 \cos^2 75{\cdot}6^\circ + 2{\cdot}514 \sin^2 75{\cdot}6 = 2{\cdot}96 \text{ (inches)}^4$$

which may be checked by drawing the momental ellipse. The bending moment M midway between the supports is

$$\tfrac{1}{4} \times \tfrac{1}{2} \times 64 = 8 \text{ ton-inches}$$

Hence from (11)—

$$\text{Maximum tensile stress} = \frac{2{\cdot}26 \times 8 \times \cos 30{\cdot}6^\circ}{2{\cdot}96} = 5{\cdot}26 \text{ tons per sq. inch}$$

Also from the neutral axis to the intersection of the outer edges where the compressive stress is greatest measures 1·70" (viz. $1{\cdot}244 \times \sqrt{2} \times \sin 75{\cdot}6^\circ$). Hence, similarly, the maximum compressive stress is

$$\frac{1{\cdot}70 \times 8 \times \sin 75{\cdot}6^\circ}{2{\cdot}96} = 3{\cdot}97 \text{ tons per sq. inch.}$$

**71. Beams of Uniform Strength.**—The bending moment generally varies from point to point along a beam in some way dependent on the manner of loading; if the cross-section does not vary throughout the length of the beam, it must be sufficient to carry the maximum bending to which the beam is subjected anywhere, and will therefore be larger than necessary elsewhere. Evidently less material might be used by proportioning the section everywhere to the straining action which it has to bear. This, with practical limitations, is attempted in compound girder sections of various types (see Art. 67). In other cases there is seldom any practical advantage in adopting an exactly proportioned variable cross-section, although variable sections are common, *e.g.* ship masts, carriage springs, and many cantilevers.

A brief indication of the type of variation of section for uniformity of strength will be given. Considering only direct stresses resulting from bending, in order to reach the same maximum stress intensity $f$ at

every cross-section of a beam under a variable bending moment M, the condition

$$M = fZ \quad \text{or} \quad Z = \frac{M}{f} \quad \text{or} \quad f = \frac{M}{Z}$$

must be fulfilled, where Z is the variable modulus of section of the beam. In other words, since $f$ is to be constant, the modulus Z must be proportional to the bending moment. Taking rectangular beams in which $Z = \frac{1}{6}bd^2$ (Art. 66), either $b$ or $d$ (or both) may be varied so that $bd^2$ is proportional to M. If the beam is a cantilever with an end load W (see Fig. 75), in which the bending moment at a distance $x$ from the free end is $W \cdot x$, uniform strength for direct stresses may be attained by varying the breadth $b$ proportionately to $x$, *i.e.* by making the beam of constant depth $d$ and triangular in plan, thus

$$\tfrac{1}{6}bd^2 = \frac{Wx}{f} \quad \text{or} \quad b = \frac{6W}{fd^2} \cdot x$$

In general, for rectangular beams of constant depth the condition of uniform strength would be that the width should vary in the same way as the height of the bending-moment diagram, for

$$b = \frac{6}{fd^2} \cdot M \ (f \text{ and } d \text{ being constant})$$

If the breadth is made constant the square of the depth should be proportional to the bending moment, *i.e.* the depth should be everywhere proportional to the square root of the bending moment, or

$$d^2 = \frac{6}{f \cdot b} \cdot M \ (f \text{ and } b \text{ being constant})$$

For solid circular sections in which the diameter varies

$$Z = \frac{\pi}{32} d^3 = \frac{M}{f} \quad \text{or} \quad d^3 = \frac{32M}{\pi f}$$

and the diameter varies as the cube root of the bending moment.

**72. Distribution of Shear Stress in Beams.**—In considering the equilibrium of a portion of a horizontal beam in Art. 56 it was found convenient to resolve the forces across a vertical plane of section into horizontal and vertical components. The variation in intensity of the horizontal or longitudinal components of stress has been investigated in Arts. 61, 62, and 63, and we now proceed to examine the distribution of the tangential or shearing stress over the vertical cross-section. The vertical shear stress at any point in the cross-section is accompanied by a horizontal shear stress of equal intensity (see Art. 8), the tendency of the former being to produce a vertical relative sliding on either side of the section, and the tendency of the latter being to produce relative horizontal sliding on either side of a horizontal or longitudinal section. The intensity of shear stress at a height $y$ from the neutral axis for a beam may be found approximately as follows :—

In Fig. 107 let AD and BC be two cross-sections of the beam distant EK or $\delta x$ apart measured along the neutral surface GH; let the variable breadth at any height $y$ from GH be denoted by $z$; let the

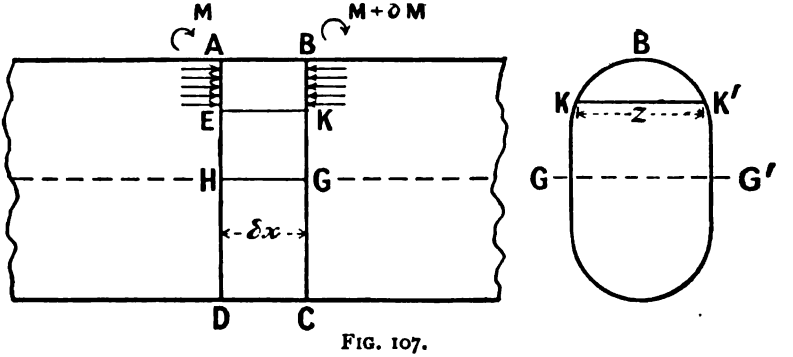



FIG. 107.

bending moment at the section AD be M, and at BC be $M + \delta M$. Then, at any height $y$ from the neutral surface, the longitudinal or horizontal direct stress intensity on the section AD is

$$p = \frac{My}{I} \quad \text{(Art. 61)}$$

where I is the moment of inertia of the cross-section. Consider the equilibrium of a portion ABKE between the two sections. On any element of cross-section, of area $z dy$, the longitudinal thrust at AE is—

$$p \,.\, z \,.\, dy \text{ or } \frac{My}{I} \,.\, z \,.\, dy$$

But at BK, on the element at the same height, the thrust is—

$$\frac{(M + \delta M)y}{I} \,.\, z \,.\, dy$$

The thrusts on any element at BK being in excess of those at AE by the difference in the above quantities, viz.—

$$\frac{\delta M}{I} \,.\, y \,.\, z \,.\, dy$$

the total excess thrust on the area BK over that at AE will be—

$$\int_y^{y_1} \frac{\delta M}{I} \,.\, y \,.\, z \,.\, dy \text{ or } \frac{\delta M}{I} \int_y^{y_1} y \,.\, z \,.\, dy$$

where $y_1$ is the extreme value of $y$, *i.e.* HA, and $z$ represents the variable breadth of section between EK and AB. Since the net horizontal force on the portion ABKE is zero, the excess thrust at BK must be balanced by the horizontal shearing force on the surface EK; hence, if $q$ represents the mean intensity of shear stress at a height $y$ (neglecting any

change in $q$ in the length $\delta x$), the shearing stress on EK is $q \,.\, z \,.\, \delta x$, and

$$q \,.\, z \,.\, \delta x = \frac{\delta M}{I} \int_{y}^{y_1} y \,.\, z \,.\, dy$$

hence
$$q = \frac{\delta M}{\delta x} \cdot \frac{1}{I \,.\, z} \int_{y}^{y_1} y \,.\, z \,.\, dy = \frac{F}{I \,.\, z} \int_{y}^{y_1} y \,.\, z \,.\, dy \quad . \quad . \quad (1)$$

where $F = \frac{dM}{dx}$ (Art. 59 (2)) = total shearing force on the cross-section of the beam. Actually the intensity of shear stress at a height $y$ varies somewhat laterally, being greatest at the inside.

In the expression $\frac{F}{Iz} \int_{y}^{y_1} y \,.\, z \,.\, dy$, the symbol $z$ outside the sign of integration, and the symbol $y$, which is the lower limit of integration, refer to a particular pair of values corresponding to the height above HG for which $q$ is stated, while in the product $y \,.\, z$ within the sign of integration each letter refers to a variable over the range $y_1$ to $y$, or A to E (Fig. 107). It may be noted that the quantity—

$$\int_{y}^{y_1} y \,.\, z \,.\, dy$$

is the moment of the area KBK′ about the neutral axis GG′, which is equal to the area multiplied by the distance of its centre of gravity or centroid from GG′, or the area of so much of a modulus figure (see Art. 68) as lies above KK′, multiplied by the height HA or $y_1$ so that—

$$q = \frac{F}{I \times KK} \times (\text{area KBK}') \times (\text{distance of its centroid from GG}') \quad (2)$$

or—

$$q = \frac{F \times y_1}{I \times KK} \times (\text{area of modulus figure between B and KK}') \quad . \quad (3)$$

which give graphical methods of calculating the intensity of shear stress at any part of the cross-section.

It is obvious from the above expressions (1) or (3) that $q$ is a maximum when the lower range of integration is zero (*i.e.* at the neutral surface), and that it is zero at either edge ($y = y_1$ or $y = -y_1$). If the graphical method with modulus figures be used, the areas on opposite sides of the neutral axis should be reckoned of opposite signs.

*Rectangular Section* (Fig. 108).—Width $b$, depth $d$. At any height $y$ from the neutral axis, since $z$ is constant and equal to $b$—

$$q = \frac{F}{Iz} \int_{y}^{\frac{d}{2}} y \,.\, z \,.\, dy = \frac{F}{I} \int_{y}^{\frac{d}{2}} y \,.\, dy = \frac{12F}{bd^3} \left( \tfrac{1}{2} y^2 \right)_{y}^{\frac{d}{2}} = \frac{6F}{bd^3} \left\{ \left( \frac{d}{2} \right)^2 - y^2 \right\} \quad (4)$$

L

If the various values of $q$ are shown by ordinates on $d$ as a base-line, as in Fig. 108, the curve is a parabola, and when $y = 0$—

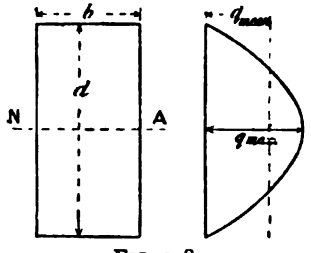


FIG. 108.

$$q = \tfrac{3}{2}\frac{F}{bd}$$

The mean intensity of shear stress is $F \div bd$; the greatest intensity is thus 50 per cent. greater than the mean.

*Rectangular* I *Section with Sharp Corners* (Fig. 109).—In the flange, at a height $y$ from the neutral axis—

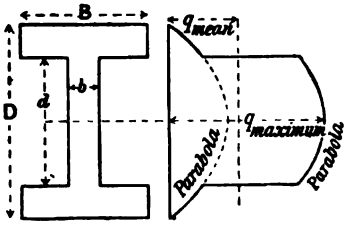


FIG. 109.

$$q = \frac{F}{IB}\int_{y}^{\frac{D}{2}} y \,.\, B \,.\, dy = \frac{F}{2I}\left(\frac{D^2}{4} - y^2\right)$$

and when $y = \frac{d}{2}$ at the inner edge of the flange—

$$q = \frac{F}{I}\cdot\frac{D^2 - d^2}{8}$$

In the web $q = \frac{F}{Ib}\int_{y}^{\frac{D}{2}} y \,.\, z \,.\, dy$

where $z = B$ over part of the range and $z = b$ over the remainder (the web). The integral may conveniently be split up thus—

$$q = \frac{F}{Ib}\left(B\int_{\frac{d}{2}}^{\frac{D}{2}} y\,dy + b\int_{y}^{\frac{d}{2}} y\,dy\right) = \frac{F}{I}\left(\frac{B}{b}\cdot\frac{D^2 - d^2}{8} + \frac{d^2}{8} - \frac{y^2}{2}\right)$$

When $y = \frac{d}{2}$, just inside the web—

$$q = \frac{F}{I}\cdot\frac{D^2 - d^2}{8} \times \frac{B}{b} \quad \text{or} \quad \frac{B}{b} \text{ times that just inside the flange.}$$

And when $y = 0$—

$$q = \frac{F}{I}\left\{\left(\frac{B}{b}\cdot\frac{D^2 - d^2}{8}\right) + \frac{d^2}{8}\right\}$$

The curves in Fig. 109 show the variation in intensity at different heights, both parts being parabolic.

The shear stress intensity anywhere might conveniently be stated from (*a*) above; thus, in the web at level $y$—

$q = \frac{F}{Ib} \times$ (moment of section area above level $y$ about neutral axis)

*e.g.* the maximum stress, when $y = 0$ is (taking moments of parts)—

$$q = \frac{F}{Ib}\left\{B\left(\frac{D}{2} - \frac{d}{2}\right)\left(\frac{D}{2} + \frac{d}{2}\right)\tfrac{1}{2} + b\cdot\frac{d}{2}\cdot\frac{d}{4}\right\}$$

which agrees with the previous result.

*Rolled* **I** *Section.*—This may best be treated graphically by the method of the modulus figure given above. An example is shown

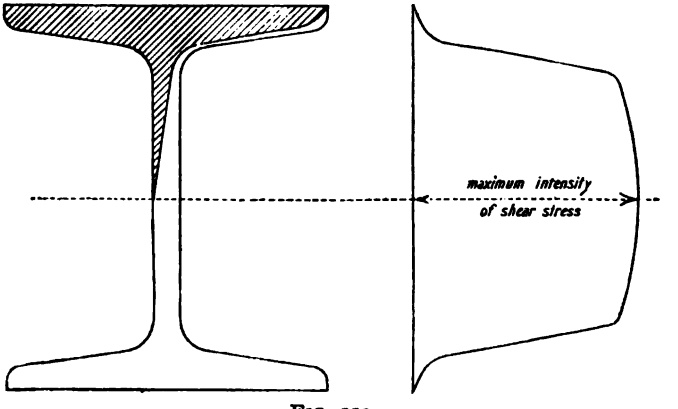



FIG. 110.

in Fig. 110. Every ordinate is proportional to the area of modulus figure above it, divided by the corresponding breadth of the cross-section.

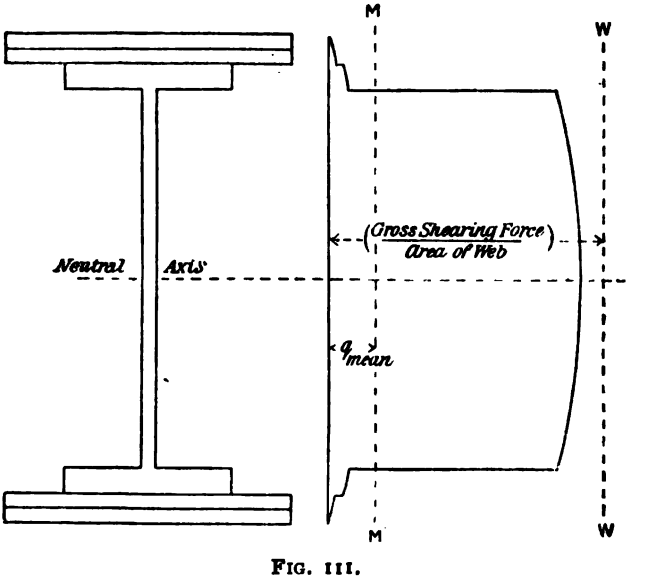



FIG. 111.

*Built-up Girder Section.*—Fig. 111 shows the intensity of shear stress at different parts of the section of a built-up girder. The stress intensities have been calculated, as in Fig. 110, for the **I** section, but

the integration requires splitting into three parts, as there are three different widths of section.

*Approximation.*—The usual approximation in calculating the intensity of shear stress in the web is to assume that the web carries the whole vertical shearing force with uniform distribution. Fig. 111 shows that the intensity in the web does not change greatly. The intensity of shear stress according to the above approximation is shown by the dotted line WW, which represents the quotient when the whole shearing force on the section is divided by the area of the section of the web. Judging by Fig. 111, this simple approximation to the mean shear stress in the web for such a section is a good one. The line MM shows the mean intensity of shearing stress, *i.e.* the whole shearing force divided by the whole area of section; this is evidently no guide to the intensity of shear stress in the web, which everywhere greatly exceeds it.

EXAMPLE 1.—A beam of I section 20 inches deep and $7\frac{1}{2}$ inches wide has flanges 1 inch thick and web 0·6 inch thick, and carries a shearing force of 40 tons. Find what proportion of the total shearing force is carried by the web and the maximum intensity of stress in it, given I = 1647 inch units.

At any height $y$ from the neutral axis of the section the intensity of shearing stress in the web section is—

$$q = \frac{40}{1647 \times 0{\cdot}6}\left(7{\cdot}5\int_9^{10} y\,dy + 0{\cdot}6\int_y^9 y\,dy\right)$$

$$= \frac{40}{1647 \times 0{\cdot}6 \times 2}\{(7{\cdot}5 \times 19) + 0{\cdot}6(81 - y^2)\}$$

$$= 3{\cdot}87 - 0{\cdot}01213y^2$$

The stress on a strip of web of depth $dy$ situated at a height $y$ from the neutral axis is—

$$q \times 0{\cdot}6 \times dy$$

and the whole shearing force carried by the web section is—

$$0{\cdot}6\int_{-9}^{9} q\,dy = 0{\cdot}6\int_{-9}^{9}(3{\cdot}87 - 0{\cdot}01213y^2)\,dy$$

$$= 1{\cdot}2(34{\cdot}83 - 0{\cdot}00404 \times 729) = 38{\cdot}26 \text{ tons}$$

or 95·6 per cent. of the whole.

The maximum value of $q$ (when $y = 0$) is evidently 3·87 tons per square inch.

Testing the usual approximation of taking all the shearing stress as spread uniformly over the web section—

$$\frac{40}{0{\cdot}6 \times 18} = 3{\cdot}70 \text{ tons per square inch}$$

which is intermediate between the mean value of $q$ in the web, viz.—

$$\frac{38{\cdot}26}{0{\cdot}6 \times 18} \text{ or } 3{\cdot}54 \text{ tons per square inch}$$

and the maximum intensity 3·87 tons per square inch.

**73. Principal Stresses in Beams.**—The intensity of direct stress due to bending, as found in Arts. 61 to 65, and the intensity of horizontal and vertical shear stress, as found in Art. 72, are only, as indicated in Arts. 56, 64, and 65, component stresses in conveniently chosen directions. Within the limitations for which the simple theory of bending is approximately correct (Art. 64), the methods of Arts. 18 and 19 may be applied to find the direction and magnitude of the principal stresses, the greater of which, at any point, has the same sign as the longitudinal direct component there, and makes the smaller (acute) angle with it. Fig. 112 shows the directions of the principal stresses at numerous points in a simply supported beam of rectangular cross-section carrying a uniformly distributed load, as well as the intensities of the component horizonal direct and vertical shear stresses on certain vertical sections, and the intensities of the two opposite

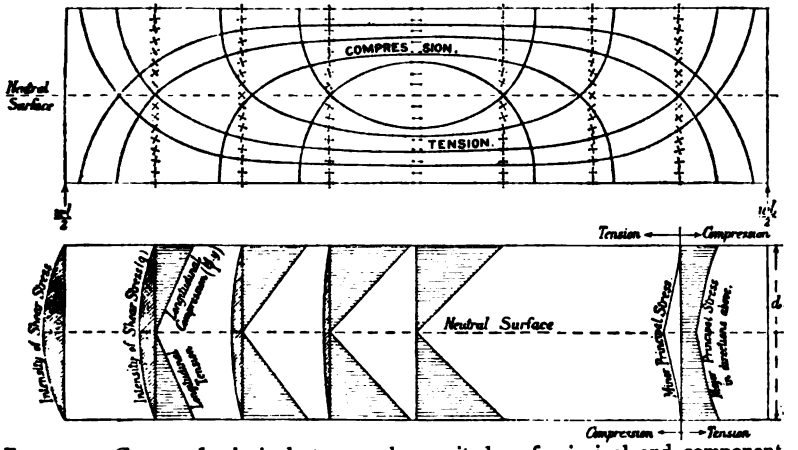



FIG. 112.—Curves of principal stress and magnitudes of principal and component stresses.

principal stresses on one section. The distribution of horizontal direct component stress over a given section is as shown in Fig. 95, and the values of its intensity for a given height vary along the length of the beam, as shown in the bending-moment diagram, Fig. 81. The distribution of tangential or shear stress across vertical sections is as in Fig. 108, and the intensities at a given height vary along the length of the beam, as in the shearing-force diagram in Fig. 81. For the purpose of illustration, the intensity of vertical shearing stress has been made excessive for a rectangular section by taking a span, $l$, only four times the depth of the beam. The maximum intensity of vertical (and horizontal) shear stress, which occurs at the middle of the end section, is, by Fig. 81 and Art. 72—

$$q = \tfrac{3}{2} \cdot \frac{\frac{1}{2}wl}{bd} = \tfrac{3}{4}\frac{wl}{bd}$$

where $w$ is the load per inch run on the span $l$.

The maximum intensity of horizontal direct stress, which occurs at the top and bottom of the middle section, is, by Fig. 81 and Art. 63 (7)—

$$f = \tfrac{1}{8}wl^2 \div \tfrac{1}{6}bd^2 = \tfrac{3}{4}\frac{wl^2}{bd^2}$$

hence $$\frac{\text{maximum } q}{\text{maximum } f} = \frac{d}{l} = \tfrac{1}{4}$$

The *magnitudes* of the principal stresses for all points in the one cross-section $\frac{1}{8}l$ from the right-hand support have been calculated from the formula (3) in Art. 19 and are shown in Fig. 112. The two principal stresses are of opposite sign, and the larger one has the same sign as the direct horizontal stress, *i.e.* it is compressive above the neutral axis and tensile below it. The diagram does not represent the direction of the principal stresses at every point in this section.

For such a large ratio of depth to span as $\frac{1}{4}$, the simple theory of bending could not be expected to give very exact results, but with larger spans the shearing stresses would evidently become more insignificant for a rectangular section. The magnitudes shown in Fig. 105 must be looked upon as giving an idea of the variation in intensity rather than an exact measure of it.

*Curves of Principal Stress.*—Lines of principal stress are shown in Fig. 112 on a longitudinal section of the beam. They are such that the tangent and normal at any point give the direction of the two principal stresses at that point. There are two systems of curves which cut one another at right angles: both cross the centre line at 45° (see Arts. 8 and 15). The intensity of stress along each curve is greatest when it is parallel to the length of the beam and diminishes along the curve to zero, where it cuts a face of the beam at right angles. For larger and more usual ratios of length to depth, for rectangular beams the curves would be much flatter, the vertical shearing stress being smaller in proportion about mid-span.

*Maximum Shearing Stress.*—At any point in the beam the intensity of shear stress is a maximum on two planes at right angles, inclined at 45° to the principal planes, and of the amount shown in Art. 19 (4), viz. half the algebraic difference of the principal stress intensities, which is, in the case shown in Fig. 112, half the arithmetic sum of the magnitudes of the principal stress intensities taken with like sign.

*Principal Stress in* I *Sections.*—In I sections, whether rolled in one piece or built up of plates and angles, it has been shown (Art. 67) that the web area is of little importance in resisting the longitudinal direct stresses due to bending, or, in other words, it contributes little to the modulus of section; and in Art. 72 (Fig. 111) it was shown that the flanges carry little of the shear stress. It should be noticed, however, that in the web near the flange the *intensity* of longitudinal direct stress is not far below the maximum on the section at the outer layers, while the intensity of vertical shear stress is not much lower than the maximum, which occurs at the neutral plane. The principal stress in such a position may consequently be of higher intensity than either of the maximum component stresses (see example below). Only low shear-

stress intensities are allowed in cross-sections of the webs of I section girders; it should be remembered that the shear stresses involve tensile and compressive principal stresses, which may place the thin web in somewhat the condition of a long strut. See also remarks in Art. 24 on the strength of material acted on by principal stresses of opposite kinds, which is always the case in the webs of I sections, where, in the notation of Art. 19—

$$p = \frac{p_1}{2} \pm \sqrt{\left(\frac{p_1}{2}\right)^2 + q^2}$$

The stresses in and design of plate girder webs is further dealt with in Art. 188.

EXAMPLE.—A beam of I section, 20 inches deep and $7\frac{1}{2}$ inches wide, has flanges 1 inch thick, and web 0·6 inch thick. It is exposed at a particular section to a shearing force of 40 tons, and a bending moment of 800 ton-inches. Find the principal stresses (*a*) at the outside edges, (*b*) at the middle of the cross-section, (*c*) $1\frac{1}{2}$ inch from the outer edges.

The moment of inertia about the neutral axis is—

$$\tfrac{1}{12}(7{\cdot}5 \times 20^3 - 6{\cdot}9 \times 18^3) = 1647 \text{ (inches)}^4$$

(*a*) At the outside edges $f = \dfrac{800 \times 10}{1647} = 4{\cdot}86$ tons per square inch pure tension or compression, the other principal stress being zero.

(*b*) At the middle of the cross-section the intensity of vertical and horizontal sheer stress is—

$$q = \frac{40}{1647 \times 0{\cdot}6}\left(7{\cdot}5\int_9^{10} y\,dy + 0{\cdot}6\int_0^9 y\,dy\right) = 3{\cdot}87 \text{ tons per square inch}$$

as in example at end of Art. 72.

This being a pure shear, the equal principal stresses of tension and compression are each inclined 45° to the section, and are of intensity 3·87 tons per square inch.

(*c*) Intensity of direct stress perpendicular to the section is—

$$p_1 = \frac{800 \times 8{\cdot}5}{1647} = 4{\cdot}13 \text{ tons per square inch.}$$

The intensity of vertical shear stress on the section is—

$$q = \frac{40}{1647 \times 0{\cdot}6}\left(7{\cdot}5\int_9^{10} y\,dy + 0{\cdot}6\int_{8{\cdot}5}^{9} y\,dy\right)$$

$$= \frac{40}{1647 \times 2 \times 0{\cdot}6}\{(7{\cdot}5 \times 19) + 0{\cdot}6(81 - 72{\cdot}25)\}$$

$$q = 2{\cdot}99 \text{ tons per square inch}$$

Hence, the principal stresses are, by Art. 19—

$$p = \frac{p_1}{2} \pm \sqrt{\left\{\left(\frac{p_1}{2}\right)^2 + q^2\right\}} = 2{\cdot}065 \pm 3{\cdot}63$$

which are 5·695 and −1·565 tons per square inch, and the major principal stress is inclined at an angle—

$$\tan^{-1}\frac{2\cdot99}{5\cdot695} \quad \text{or} \quad 27^\circ\ 40'\ \text{(see Art. 19 (2))}$$

to the corresponding direct stress along the flange, or 62° 20′ to the cross-section.

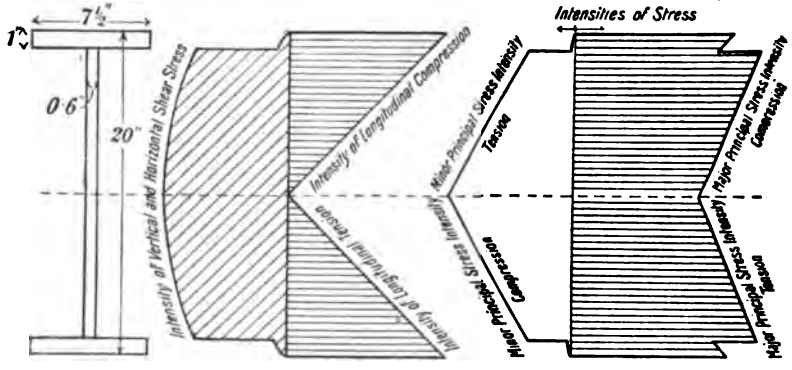



FIG. 113.—Magnitudes of component and principal stress intensities in I-section beam.

This illustrates the fact that just within the flange of an I section, carrying a considerable bending moment and shearing force, the intensity of the principal stress (5·695) may exceed that at the extreme outside layers of the section.

The intensities of principal stress in the web, calculated as above, are shown in Fig. 113, which shows that the material bears principal stresses the greater of which is nowhere greatly less than the maximum. In accepting such conclusions as to principal stresses, the limitations of the simple theory of bending should be borne in mind: these results can only be looked upon as approximations giving a useful idea of the nature of the stresses.

**74. Bending beyond the Elastic Limit. Modulus of Rupture.**—If bending is continued after the extreme fibres of a beam reach the limit of elasticity, the intensity of longitudinal stress will no longer be proportional to the longitudinal strains, and the distribution of stress will not be as shown in Fig. 95. For moderate degrees of bending beyond the elastic limit, the assumption that plane sections remain plane is often nearly true. In this case the *strains* will be proportional to the distances from the neutral axis (Art. 61), and the longitudinal stress intensities will vary from the neutral axis to the extreme layers, practically as in stress-strain diagrams for direct stress. Different types of distribution will occur according as the elastic limit is reached first in tension or compression, or simultaneously. The true elastic limit for cast iron is very low in tension or compression, but at, say, 8 tons per square inch the strain in tension is much greater, and

deviates much more from proportionality to stress than in compression. The distribution of stress on a symmetrical section will therefore be somewhat as in Fig. 114; the neutral surface will no longer pass through the centroid of the area of cross-section, but will be nearer the compression edge, which, yielding less than the tension edge, will have a greater intensity of stress. If the beam is of constant breadth, *i.e.* of rectangular cross-section, the neutral surface will move from half-depth in such a way that the areas OPQ and ORS remain equal, for the total tension and total thrust are of equal magnitude, and form a couple.

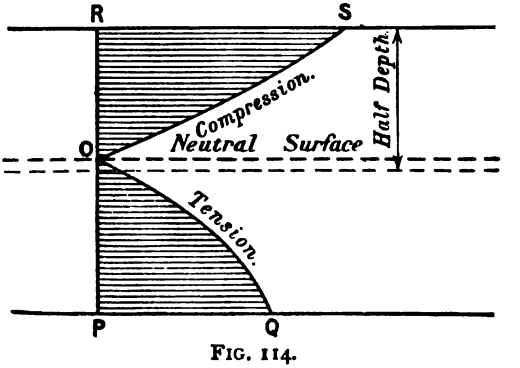


FIG. 114.

If the material of a beam has the same stress-strain diagram in tension and in compression, the neutral surface will continue to pass through the centroid of the area of cross-section, the distribution of tension and compression being symmetrical, but the intensity of stress will not in either case be proportional to the distance from the neutral surface (see Fig. 115) after the elastic limit is exceeded; the material nearer the neutral surface will carry a *higher* intensity of stress than if the stress were proportional to the distance from the neutral surface, the intensities being intermediate between a proportional and a uniform distribution.[1]

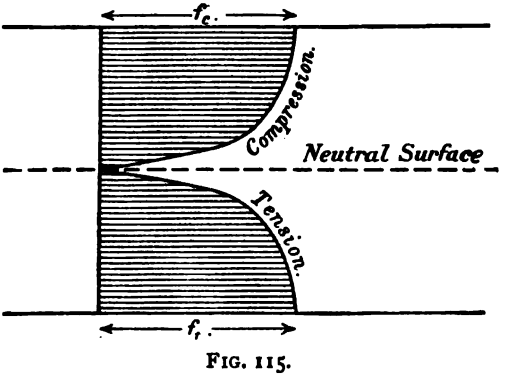


FIG. 115.

*Modulus of Rupture.*—When a bar of metal is tested by bending until rupture takes place, the intensities of stress at the outer layers at rupture are not those given by the formula (6) in Art. 63, viz.—

$$f_t = M\frac{y_t}{I} \quad \text{and} \quad f_c = M\frac{y_c}{I}$$

[1] Some experiments on the distribution of strain on cross-sections of beams will be found in a paper by Dr. J. Morrow, *Proc. Roy. Soc.*, vol. 73, p. 13.

since the condition of elasticity there assumed has ceased to hold good. Nevertheless, the quantity—

$$M\frac{y_1}{I} \quad \text{or} \quad \frac{M}{Z}$$

where M is the bending moment at rupture, is very often used as a guide to the quality of cast iron, the bending test with a central load being easily arranged. It is evidently not a true intensity of stress, and is called the transverse *modulus of rupture*. The term is practically confined to the tests of a rectangular section, and in cast iron the modulus is much higher than the ultimate tenacity in a tensile test, for two reasons. Firstly, because the tensile strain at comparatively low stress at one edge allows a distribution of stress similar to that sketched in Fig. 114, thereby using the high compressive strength of cast iron to advantage. And secondly, because the inner layers of material under the distribution of stress previous to rupture carry a higher intensity of stress than is contemplated by the formula—

$$f_1\frac{y_1}{I} \quad \text{or} \quad f_1 \div \tfrac{1}{6}bd^2 \text{ (for a rectangular beam)}$$

for the moment of resistance, thereby increasing the resistance. This second reason would not apply in any considerable degree to a thin I section, in which the direct stress is borne almost entirely by the flanges, and with comparatively uniform distribution in them, both before and after the elastic limit is passed (see Fig. 115), near outside edges. Practically, however, the term "modulus of rupture" and the transverse test to rupture are confined to cast iron and timber and to the rectangular section.

## EXAMPLES V.

1. Find the greatest intensity of direct stress arising from a bending moment of 90 ton-inches on a symmetrical section 8 inches deep, the moment of inertia being 75 inch units.

2. Calculate the moment of resistance of a beam section 10 inches deep, the moment of inertia of which is 145 inch units when the skin stress reaches 7·5 tons per square inch.

3. What total distributed load may be carried by a simply supported beam over a span of 20 feet, the depth of section being 12 inches, the moment of inertia being 375 inch units, and the allowable intensity of stress 7·5 tons per square inch? What load at the centre might be carried with the same maximum stress?

4. To what radius may a beam of symmetrical section 10 inches deep be bent without producing a skin stress greater than 6 tons per square inch, if E = 13,500 tons per square inch? What would be the moment of resistance, if the moment of inertia of the section is 211 inch units?

5. A wooden beam of rectangular section 12 inches deep and 8 inches wide has a span of 14 feet, and carries a load of 3 tons at the middle of the span. Find the greatest stress in the material and the radius of curvature at mid span. E = 800 tons per square inch.

6. What should be the width of a joist 9 inches deep if it has to carry a

uniformly spread load of 250 lbs. per foot run over a span of 12 feet, with a stress not exceeding 1200 lbs. per square inch.

7. A floor has to carry a load of 3 cwt. per square foot. The joists are 12 inches deep by 4½ inches wide, and have a span of 14 feet. How far apart may the centre lines be placed if the bending stress is not to exceed 1000 lbs. per square inch?

8. Compare the moments of resistance for a given maximum intensity of bending stress of a beam of square section placed (*a*) with two sides vertical, (*b*) with a diagonal vertical, the bending being in each case parallel to a vertical plane.

9. Over what length of span may a rectangular beam 9 inches deep and 4 inches wide support a load of 250 lbs. per foot run without the intensity of bending stress exceeding 1000 lbs. per square inch?

10. A beam of I section 12 inches deep has flanges 6 inches wide and 1 inch thick, and web ⅝ inch thick. Compare its flexural strength with that of a beam of rectangular section of the same weight, the depth being twice the breadth.

11. A rolled steel joist 10 inches deep has flanges 6 inches wide by ¾ inch thick. Find approximately the stress produced in it by a load of 15 tons uniformly spread over a span of 14 feet.

12. Find the bending moment which may be resisted by a cast-iron pipe 6 inches external and 4½ inches internal diameter when the greatest intensity of stress due to bending is 1500 lbs. per square inch.

13. Find in inch units the moment of inertia of a T section, about an axis through the centroid or centre of gravity of the section and parallel to the cross-piece. The height over all is 4 inches, and the width of cross-piece 5 inches, the thickness of each piece being ½ inch.

14. The compression flange of a cast-iron girder is 4 inches wide and 1½ inch deep; the tension flange 12 inches wide by 2 inches deep, and the web 10 inches by 1½ inch. Find (1) the distance of the centroid from the tension edge; (2) the moment of inertia about the neutral axis; (3) the load per foot run which may be carried over a 10-foot span by a beam simply supported at its ends without the skin tension exceeding 1 ton per square inch. What is then the maximum intensity of compressive stress?

15. A compound girder consists of two rolled I sections 18 × 7 inches (BSB 28, Table I. in Appendix) and four ⅝ inch flats, 18 inches wide forming the flanges (2 on each). For what maximum span may this girder be used to support a load of 3 tons per foot run including its own weight if the maximum bending stress is not to exceed 7·5 tons per square inch, neglecting the weight of the girder? Allow two ⅞-inch rivet holes in each flange.

16. Find the maximum stress in a compound girder consisting of three I beams 14 × 16 inches (BSB 23) having four ½ inch flats 20 inches wide on the flanges (2 on each), when carrying a load of 50 tons at the centre of a span of 18 feet in addition to its own weight, which is 280 lbs. per foot. Allow for three ⅞-inch rivets in each flange section.

17. A box plate girder is to be 30 inches deep over the angles for a span of 30 feet and is to carry a load of 43 tons at its centre with a working stress of 5 tons per square inch. The two webs are each ⅜ inch thick and the 4 angles are each 4 × 4 × ½ inches (BSEA 11, Table V. Appendix). The flanges are each to be made of 3 plates, the outer one ⅜ inch, the next ⅜ inch, and the inner one ½ inch. Find the necessary width of flanges.

18. A single web plate girder has a span of 34 feet and depth over angles of 36 inches, the web being ½ inch thick. It is required to carry a load of 72 tons evenly distributed along its length. If the angles are 6 × 6 × ½ inch what thickness of flats 14 inches wide will be required in order that the working stress shall be about 5 tons per square inch?

In Examples Nos. 19 to 24 inclusive the tension in the concrete is to be neglected, and the modulus of direct elasticity of steel in tension taken as 15 times that of concrete in compression. The concrete is to be taken as perfectly elastic within the working stresses.

19. A reinforced concrete beam 10 inches wide and 22 inches deep has four $1\frac{1}{4}$-inch bars of round steel placed 2 inches from the lower edge. If simply supported at the ends, what load per foot run would this beam support over a 16-feet span if the compressive stress in the beam reaches 600 lbs. per square inch? What would be the intensity of tensile stress in the reinforcement?

20. A reinforced concrete floor is 9 inches thick, and the reinforcement is placed 2 inches from the lower face. What area of section of steel reinforcement is necessary per foot width if the stress in the concrete is to reach 600 lbs. per square inch, when that in the steel is 15,000 lbs. per square inch, and what load per square foot could be borne with these stresses over a span of 10 feet?

21. A concrete beam is 18 inches deep and 9 inches wide, and has to support a uniformly distributed load of 1000 lbs. per foot run over a span of 15 feet. What area of section of steel reinforcement is necessary, the bars being placed with their centres 2 inches above the lower face of the beam, if the intensity of pressure in the concrete is not to exceed 600 lbs. per square inch?

22. A ferro-concrete floor is 8 inches thick, and carries a load of 200 lbs. per square foot over a span of 12 feet. What sectional area of steel reinforcement 2 inches from the lower surface is necessary per foot width of floor if the pressure in the concrete is to be limited to 600 lbs. per square inch? What would then be the working stress in the steel?

23. Part of a concrete floor forms with a supporting beam a T section, of which the cross-piece is 30 inches wide by 6 inches deep, and the vertical leg is 8 inches wide, and is to be reinforced by bars placed with their centres 12 inches below the under side of the floor. What area of cross-section of steel will bring the neutral axis of the section in the plane of the under side of the floor? What would then be the intensity of tension in the steel when the maximum compression reaches 600 lbs. per square inch?

24. A reinforced concrete beam of T section has a cross-piece 24 inches wide and 5 inches deep, the remainder being 10 inches wide by 18 inches deep. The reinforcement consists of two 2-inch round bars placed with their centres 3 inches from the lower face of the beam. Find the intensity of tension in the steel and moment of resistance of the section when the extreme compressive stress in the concrete reaches 600 lbs. per square inch.

25. A (reinforced) flitched timber beam consists of two timber joists each 4 inches wide and 12 inches deep, with a $\frac{1}{2}$-inch steel plate 9 inches deep placed symmetrically between and firmly attached to them. What is the total moment of resistance of a section when the bending stress in the timber reaches 1200 lbs. per square inch, and what is the greatest intensity of stress in the steel? (E for steel may be taken 20 times that for the timber.)

26. Find the greatest intensity of vertical shear stress on an I section 10 inches deep and 8 inches wide, flanges 0·97 inch thick, and web 0·6 inch thick, when the total vertical sheer stress on the section is 30 tons. What is the ratio of the maximum to the mean intensity of vertical shear stress?

27. The section of a plate girder has flanges 16 inches wide by 2 inches thick; the web, which is 36 inches deep and $\frac{3}{4}$ inch thick, is attached to the

flanges by angles 4 × 4 × $\frac{5}{8}$ inch, and the section carries a vertical shearing force of 100 tons. Find approximately the intensity of vertical shear stress over all parts of the section and plot a curve showing its variation. (Neglect the rivet holes and rounded corners of the angle plate.)

28. If the above section in No. 27 is also subjected to a bending moment of 5000 ton-inches, find the principal stresses in the web 7 inches from the outer edge of the tension flange.

# CHAPTER VI

## *MOVING LOADS*

**75. Maximum Straining Actions.**—The bending moment and shearing force diagrams found in Chapter IV. give the straining actions at all sections of beams subjected only to a stationary load. In designing a bridge girder, it is necessary to know the greatest bending moment and shearing force which every section has to resist for all possible dispositions of the movable load, and in this chapter various cases of moving loads will be examined to find these maximum straining actions at every section of a beam simply supported at its ends.

*Signs.*—It may be well to recall the convention of signs adopted in Art. 59, viz. Positive shearing force at any section of a horizontal beam is numerically equal to the upward external force to the right of the section, and positive bending moment is that which tends to produce upward convexity, and is equal to the clockwise moment of the external forces to the right of the section, or to the contra-clockwise moment of the external forces to the left of the section.

**76. Uniformly Distributed Load longer than the Span.**—*Shearing Force.*—Suppose the load $w$ per foot approaches a section X of a span AB of length $l$, from the left support A (Fig. 116). When the load covers a length AC $= y$ from A, the positive shearing force at X, to the right of C will be

$$\text{positive } F_X = R_B = \frac{wy^2}{2l} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the moment of the load about A divided by $l$. As the load advances this value increases until when the load reaches X, $y = x$, and

$$\text{positive } F_X = R_B = \frac{wx^2}{2l} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As soon as the load passes the section X the shearing force at X decreases, for the increase in upward force at B is obviously less than the downward force to the right of X. Hence the positive shear is a maximum at the section X when the load extends from A to X, and its amount is

$$\text{maximum positive } F_X = \frac{wx^2}{2l} \quad . \quad . \quad . \quad . \quad (3)$$

The curve of maximum positive shear is thus a parabola with vertex at A and reaching an ordinate $\frac{wl}{2}$ at B. Similarly the maximum negative shear at X occurs when in approaching from the right or receding towards the right the load covers the portion BX of the span, when

$$\text{maximum negative } F_X = -R_A = -\frac{w(l-x)^2}{2l} \quad . \quad . \quad (4)$$

The curve of maximum negative shear being a parabola with vertex at B and reaching an ordinate $\frac{wl}{2}$ at A.

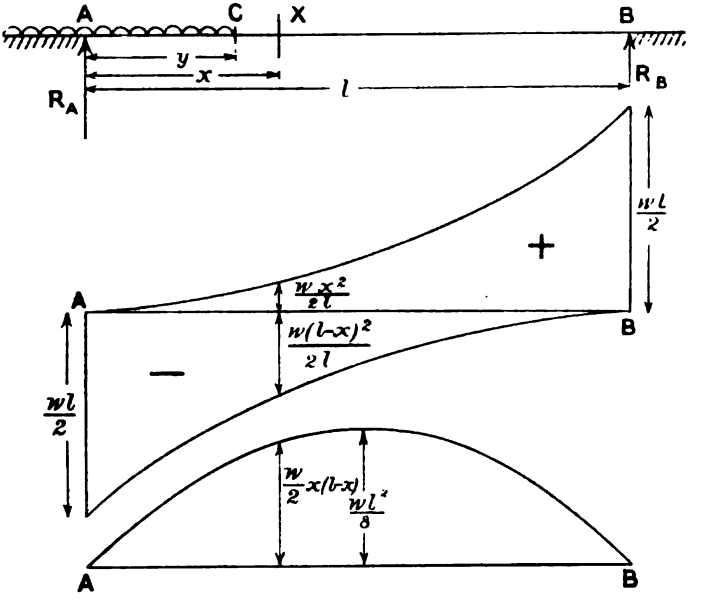



FIG. 116.—Uniformly distributed load longer than the span.

*Bending Moment.*—As the load approaches X from A the bending moment at X (which is always negative), taking contra-clockwise moments of extreme forces to the right of X, is

$$M_X = -R_B(l-x) = -\frac{wy^2}{2l}(l-x) \quad . \quad . \quad . \quad (5)$$

and increases in magnitude as C approaches X. After C passes X the bending moment at X, taking clockwise moments to the left, is

$$M_X = -R_A x + \frac{wx^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

of which only $R_A$ varies with the position of C, and the greatest magni-

tude is reached when $R_A$ is greatest, viz. when the load covers the whole span, and then

$$\text{maximum } M_X = -\frac{wl}{2}x + \frac{wx^2}{2} = -\frac{w}{2}x(l-x) \quad . \quad . \quad (7)$$

The curve of maximum bending moments (Fig. 116) is a parabola having a maximum ordinate $-\frac{wl^2}{8}$ at $x = \frac{l}{2}$. It is evidently the same curve as for a fixed load $w$ per foot covering the whole span (Art. 57 and Fig. 81).

**77. Single Concentrated Load.**—*Shearing Force.*—As the load distant $y$ from A approaches the section X (Fig. 117) from A

$$\text{positive } F_X = R_B = \frac{y}{l}W \quad . \quad . \quad . \quad . \quad . \quad (1)$$

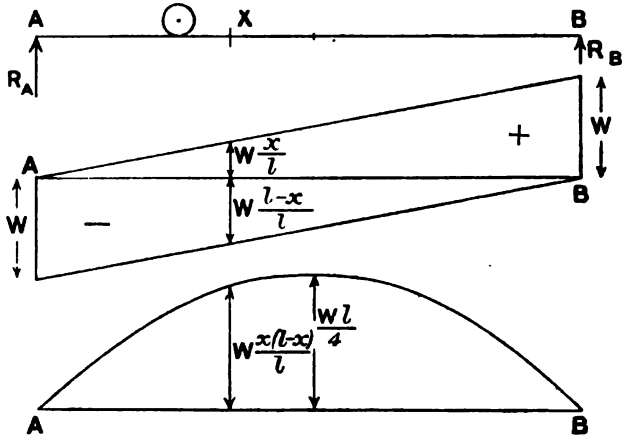



FIG. 117.—Single concentrated load.

which increases as W approaches X and reaches the value

$$\text{maximum positive } F_X = \frac{x}{l}W \quad . \quad . \quad . \quad . \quad . \quad (2)$$

when W reaches X ($y = x$). When W passes to the right of X the shearing force at X is evidently negative. Taking the upward force to the left of X

$$\text{negative } F_X = -R_A = -\frac{l-y}{l}W \quad . \quad . \quad . \quad . \quad (3)$$

which has its greatest magnitude when $y = x$, when

$$\text{maximum negative } F_X = -\frac{l-x}{l}W \quad . \quad . \quad . \quad . \quad (4)$$

The curves of positive and negative shearing force are straight lines shown in Fig. 117.

*Bending Moment.*—As W approaches X from A the bending moment (negative) at X from contra-clockwise moments to the right of X is

$$M_X = -R_B(l-x) = -\frac{y}{l}W(l-x) \quad . \quad . \quad . \quad . \quad (5)$$

which increases in magnitude until $y = x$. As soon as W has passed X the bending moment from clockwise moments to the left of X is

$$M_X = -R_A x = -\frac{l-y}{l}.W.x \quad . \quad . \quad . \quad . \quad (6)$$

which is greatest when $y = x$. Thus both (5) and (6) give the same maximum bending moment—

$$\text{maximum } M_X = -\frac{W}{l}x(l-x) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The curve of maximum bending moments is a parabola having a maximum ordinate $-\frac{1}{4}Wl$ at $x = \frac{l}{2}$.

**78. Uniformly distributed Load shorter than the Span.**—*Shearing Force.*—Let $c$ be the length covered by a uniformly distributed load $w$

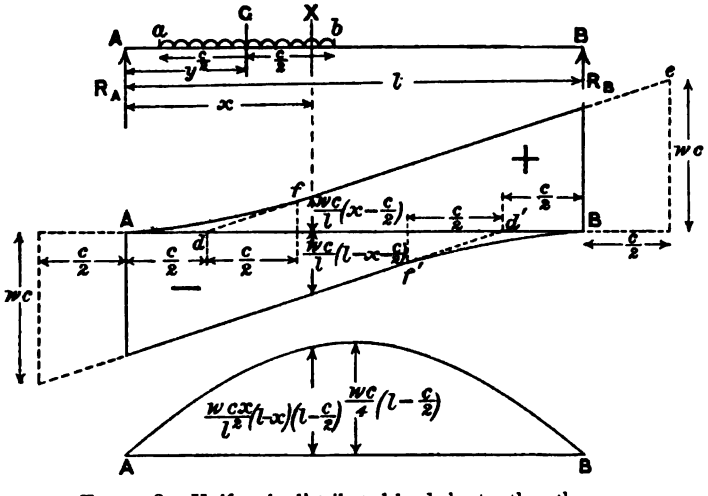



FIG. 118.—Uniformly distributed load shorter than the span.

per foot approaching from the left-hand support A (Fig. 118). Before the leading point $b$ reaches the section X, provided the whole load is on the span, the positive shear at X is

$$\text{positive } F_X = R_B = wc.\frac{y}{l} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $y$ is the distance of the centre of gravity G of the load from A. This evidently increases with $y$ until $b$ reaches X. After this the value of $F_X$ diminishes, for the downward load to the right of X evidently

more than balances any increase in $R_B$ upward; hence the maximum shearing force occurs when $b$ reaches X, and then $y = x - \frac{c}{2}$ and

$$\text{maximum positive } F_X = wc\frac{x - \frac{c}{2}}{l} \quad . \quad . \quad . \quad (2)$$

The curve of maximum positive shearing force for this part of the span (where the shearing force is equal to the reaction due to the whole load $wc$) is from (2), a straight line reaching $\frac{2l - c}{2l} \times wc$ at B($x = l$), and which would reach o at $x = \frac{c}{2}$ and $wc$ at $x = l + \frac{c}{2}$ if it applied to these points (see Fig. 118). But the whole load $wc$ only gets on a length $c$, and the straight line (2) only applies from $x = c$ to $x = l$; it is easily drawn by joining the points $de$. For points between $x = o$ and $x = c$ the maximum positive shearing force is evidently as for a load longer than the span, viz. when $b$ reaches the section considered as in (2), Art. 76.

$$\text{maximum positive } F_X = R_B = \frac{wx^2}{2l} \quad . \quad . \quad . \quad . \quad (3)$$

the curve $Af$ (Fig. 118) being a parabola as in Fig. 116, and the ordinate being the same as for the straight line (2) when $x = c$, viz. $\frac{wc^2}{2l}$.

The maximum negative shearing force is evidently found in a similar manner; writing $l - x$ for $x$ in (2), from $x = o$ to $x = l - c$

$$\text{maximum negative } F_X = -wc \cdot \frac{l - x - \frac{c}{2}}{l} \quad . \quad . \quad (4)$$

and corresponding to (3) from $x = l - c$ to $x = l$

$$\text{maximum negative } F_X = -\frac{w}{2l}(l - x)^2 \quad . \quad . \quad . \quad (5)$$

*Bending Moment.*—As $b$ approaches the section X the (negative) bending moment

$$M_X = -R_B(l - x)$$

increases. After $b$ has passed the section X as in Fig. 118

$$M_X = -R_B(l - x) + \frac{w}{2}\left(y + \frac{c}{2} - x\right)^2$$

$$= -wc\frac{y}{l}(l - x) + \frac{w}{2}\left(y + \frac{c}{2} - x\right)^2 \quad . \quad . \quad . \quad . \quad (6)$$

Differentiating to find the value of $y$ for a maximum (negative) bending moment

$$\frac{dM_X}{dy} = -wc\frac{l - x}{l} + \frac{w}{2}(2y + c - 2x)$$

which is zero when $$y + \frac{c}{2} - x = \frac{l-x}{l} \cdot c \quad \ldots \ldots \quad (7)$$

or the length $$bX = \frac{BX}{AB} \cdot c \quad \ldots \ldots \quad (8)$$

that is, the section X divides the loaded length $c$ in the same ratio $\left(\frac{BX}{AX}\right)$ as it divides the span AB.

Inserting the value of $y$ from (7) in (6)

$$\text{maximum (negative) } M_X = -wc \cdot \frac{l-x}{l}\left(c\frac{l-x}{l} - \frac{c}{2} + x\right) + \frac{w}{2}\left(c \cdot \frac{l-x}{l}\right)^2$$

$$= -\frac{wcx(l-x)}{l}\left(1 - \frac{c}{2l}\right) \quad \ldots \ldots \quad (9)$$

The curve of maximum $M_X$ is a parabola, Fig. 118 having a maximum ordinate found by putting $x = \frac{l}{2}$ in (9) to be

$$-\frac{wc}{4}\left(l - \frac{c}{2}\right) \quad \ldots \ldots \quad (10)$$

**79. Two Concentrated Loads.**—Let $W_1$ and $W_2$ (Fig. 119) be the

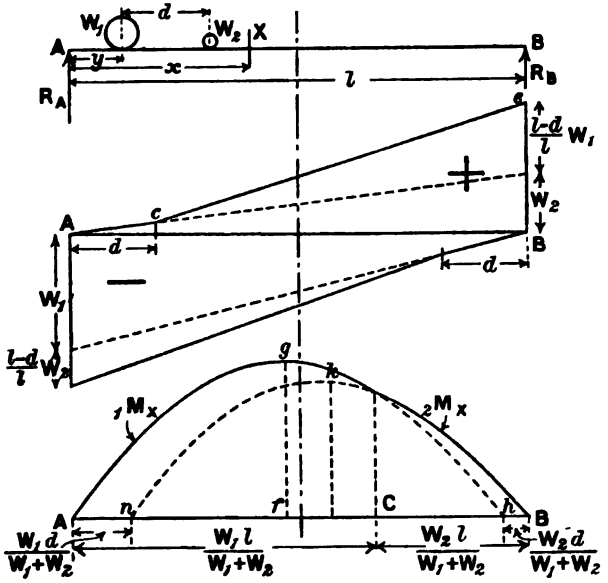



FIG. 119.—Two concentrated loads.

loads ($W_1$ being the greater) at a fixed distance $d$ apart, $d$ being less than $\frac{W_2}{W_1 + W_2}$ times the span $l$.

*Shearing Force.*—As the loads approach any section X from the left support A

$$\text{positive } F_X = R_B = \frac{1}{l}\{W_1 y + W_2(y + d)\}$$

which increases until $W_2$ reaches X, after which it suddenly decreases.[1]

If $x$ is greater than $d$ both loads are on the span when $W_2$ reaches X, ($y = x - d$), and the maximum shearing force which then occurs is (from moments about A)

$$\text{maximum positive } F_X = \frac{1}{l}\{W_2 x + W_1(x - d)\} \quad . \quad . \quad (1)$$

the curve being a straight line $ce$ rising from $\frac{d}{l}W_2$ at $c$ (when $x = d$) to $W_2 + \frac{l-d}{l}W_1$ at B. From A to $c$ the curve is a straight line

$$\text{maximum positive } F_X = \frac{x}{l}W_2 \quad . \quad . \quad . \quad . \quad (2)$$

Similarly, when the load $W_1$ is over any section X,

$$\text{maximum negative } F_X = -\frac{1}{l}\{W_1(l - x) + W_2(l - x - d)\} \quad (3)$$

if $x$ is less than $l - d$, and for values of $x$ greater than $l - d$, the maximum negative shear occurs when the load $W_2$ is off the span, and

$$\text{maximum negative } F_X = -\frac{l-x}{l}W_1 \quad . \quad . \quad . \quad (4)$$

*Bending Moment.*—As $W_2$ approaches X, the bending moment is

$$M_X = -R_B(l - x) = -\frac{1}{l}\{W_1 y + W_2(y + d)\}(l - x) \quad (5)$$

which evidently increases in negative magnitude as $W_2$ approaches X, reaching the value

$${}_2M_X = -\frac{1}{l}\{W_1(x - d) + W_2 x\}(l - x)$$

or

$$= -\frac{1}{l}\{(W_1 + W_2)x - W_1 d\}(l - x) \quad . \quad . \quad . \quad (6)$$

when $W_2$ is at X (or $y + d = x$), provided $W_1$ is then on the span. The curve is a parabola which has evidently zero values for $x = l$ and $x = \frac{W_1}{W_1 + W_2} \cdot d$; it is shown at $nk$B (Fig. 119).

[1] $F_X$ then increases again until $W_1$ passes X. As $W_1$ reaches X, $F_X$ has risen to $R_B - W_2 = \frac{1}{l}\{W_2(x + d) + W_1 x\} - W_2$ or $\frac{1}{l}\{W_2(x + d - l) + W_1 x\}$. This would exceed the value (1) if $W_2(d - l)$ exceeds $-W_1 d$, *i.e.* if $d$ exceeds $\frac{W_2}{W_1 + W_2}l$. It has been assumed that $d$ is less than $\frac{W_2}{W_1 + W_2}l$. The other case presents no special difficulty.

When $W_2$ has passed the section X and $W_1$ has not reached it, the bending moment at X

$$ {}_0M_X = -R_B(l-x) + W_2(y+d-x) $$
$$ = -\frac{1}{l}\{W_1 y + W_2(y+d)\}(l-x) + W_2(y+d-x) \quad . \quad (7) $$

and differentiating with respect to $y$

$$ \frac{d_0M_X}{dy} = -\frac{1}{l}(W_1+W_2)(l-x) + W_2 \quad . \quad . \quad . \quad (8) $$

which does not vary or vanish for any value of $y$ (except for the section $x = \frac{W_1}{W_1+W_2}l$ when it is zero for all possible values of $y$), hence the greatest and least values of ${}_0M_X$ occur at the limits of the range of equation (7), viz. $x = y$ and $x = y + d$. This will be clear from Fig. 120, which represents the bending moments at a given

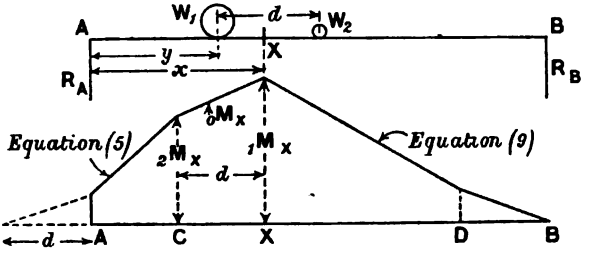



FIG. 120.

section X for various values of $y$ (the distance of $W_1$ from A). The values of $y$ are shown horizontally from A along the base AB. If the section X is at a distance $\frac{W_1}{W_1+W_2}l$ from A, the curve above CX would be horizontal. If such curves as Fig. 120 were drawn for every section of the beam, the maximum ordinates of each would be ordinates of the curve of maximum bending moment shown in Fig. 119.

When $W_1$ has passed the section X, the bending moment

$$ M_X = -R_A \cdot x = -\frac{1}{l}\{W_1(l-y) + W_2(l-y-d)\}x \quad . \quad (9) $$

which decreases in negative magnitude as $W_1$ recedes to the right from X, its maximum value being

$$ {}_1M_X = -\frac{x}{l}\{W_1(l-x)+W_2(l-x-d)\} \text{ or } -\frac{x}{l}\{W_1+W_2)(l-x)-W_2d\} \quad (10) $$

when $W_1$ is at X (and $y = x$), provided $W_2$ is still on the span, *i.e.* not to the right of B. The curve is a parabola which has evidently zero values for $x = 0$ and $x = l - \frac{W_2}{W_1+W_2}d$; it is shown by $Agh$ (Fig. 119).

The value ${}_1M_X$ from (10) is of greater (negative) magnitude than ${}_2M_X$ from (6), if

$$-W_2xd \text{ exceeds } -W_1(l-x)d$$

*i.e.* if $l-x$ exceeds $\frac{W_2}{W_1}x$, or $x$ is less than $\frac{W_1}{W_2}(l-x)$

Hence if a point C (Fig. 119) divides the span AB so that

$$\frac{AC}{CB}=\frac{W_1}{W_2} \text{ or } AC=\frac{W_1}{W_1+W_2}l$$

then in the range AC, ${}_1M_X$ gives the maximum bending moment; and in the range CB, ${}_2M_X$ gives the maximum bending moment. If $W_1$ is greater than $W_2$, the greatest bending moment anywhere in the span evidently occurs where ${}_1M_X$ is a maximum. This occurs at a value of $x$ midway between the values which give ${}_1M_X=0$, or, differentiating (10) with respect to $x$.

$$\frac{d_1M_X}{dx}=-\frac{1}{l}\{W_1(l-x)+W_2(l-x-d)-W_1x-W_2x\} \quad (11)$$

which is zero for

$$x=\frac{l}{2}-\frac{W_2}{W_1+W_2}\cdot\frac{d}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

*i.e.* the greatest bending moment on the beam occurs at a section under $W_1$ when the centre of the span AB is midway between $W_1$ and the centre of gravity of the two loads $W_1$ and $W_2$, the centre of gravity being a distance $\frac{W_2}{W_1+W_2}\cdot\frac{d}{2}$ to the right of midspan, and $W_1$ an equal distance to the left. This is a particular case of the general theorem given in the next article.

The greatest bending moment anywhere is found by substituting the value (12) of $x$ in (10) which gives—

$$-\frac{W_1+W_2}{4l}\left(l-\frac{W_2}{W_1+W_2}d\right)^2 \quad . \quad . \quad . \quad . \quad (13)$$

which is the ordinate $gf$ (Fig. 119); the distance $\frac{W_2}{W_1+W_2}d$ is the distance of the centre of gravity of $W_1$ and $W_2$ from the greater load $W_1$. The value (10) for ${}_1M_X$ only holds good so long as $W_2$ is on the span, *i.e.* to the left of B. This condition is complied with from A to C, where ${}_1M_X$ gives the maximum bending moment, for we have supposed that $d$ is less than the length CB, or $\frac{W_2}{W_1+W_2}\cdot l$. Similarly the value (6) for ${}_2M_X$ holds good over the portion CB, for since $d$ is less than $\frac{W_1}{W_1+W_2}\cdot l$, the load $W_1$ is always on the span when $W_2$ is between C and B. If $d$ should exceed $\frac{W_2}{W_1+W_2}\cdot l$, the maximum bending moment curve is made up of parts of *three* parabolas, viz. the curves from (10) and (6), and the curve of maximum bending moment for $W_1$ alone as in Fig. 177, Art. 77.

The intersection of this curve with the other two may easily be found by equating $-\frac{W_1}{l}x(l-x)$ from (7) (Art. 77) to (10) and (6) of the present article. If the distance $d$ between the two loads is sufficiently great compared to $l$, the greatest maximum bending moment anywhere may occur when $W_1$ only is on the span, viz. when $d$ exceeds the value obtained by equating $-\frac{1}{4}W_1 l$ (see Art. 77) to the value (13).

*Reversed Order.*—If the pair of loads $W_1$ and $W_2$ may cross the span AB with either $W_1$ or $W_2$ to the left (as in the case of a traction engine crossing a bridge in either direction), the diagrams of maximum shearing force may be found from Fig. 119 by taking the greatest of the two ordinates, positive or negative, at a given distance from the centre

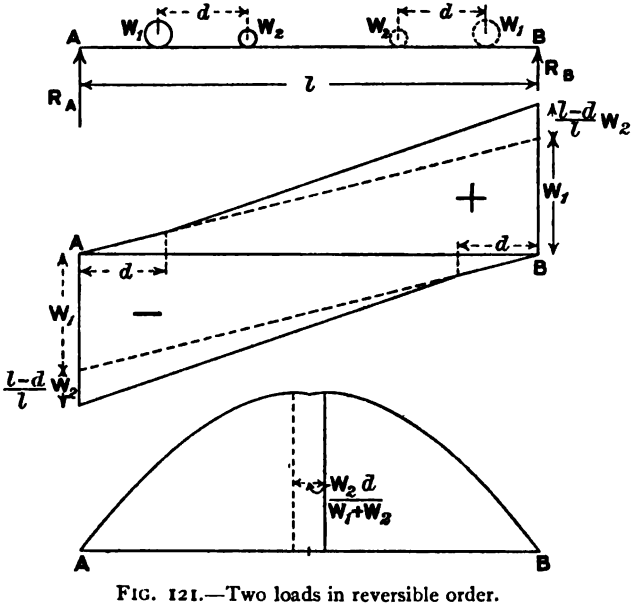



FIG. 121.—Two loads in reversible order.

or ends, and using this in both positive and negative shearing force diagrams as shown in Fig. 121. Similarly the diagram of maximum bending moment may be drawn by setting up on either side of the centre of the span ordinates equal to the greatest of the two ordinates at the same distance from the centre in Fig. 119; this is also shown in Fig. 121.

**80. Several Loads.**—The methods of the previous article become complex with more than two loads, and a graphical method which will give results as nearly correct as may be desired is usually adopted.[1]

[1] An interesting exact graphical construction of maximum bending moment diagrams for several loads is given in the *Proc. Inst. C.E.*, vol. cxli. (1900), p. 93.

*Theorem.*—When a series of wheel loads pass over a beam simply supported at its ends, the maximum bending moment under any given wheel occurs when its axis and the centre of gravity of the whole load on the span are equidistant from the centre of the span (or from opposite ends of the span).

Let AB (Fig. 122) be the span, and let W be the total load. Let the

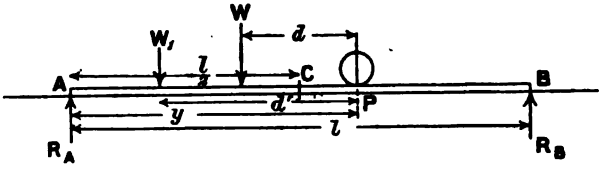



FIG. 122.

given wheel have reached a position P distant $y$ from A. Let $W_1$ be the load on the length AP, and let $d$ and $d'$ be the distances of W and $W_1$ respectively from P. Taking moments about B

$$R_A = \frac{W}{l}(l - y + d)$$

The bending moment under the wheel is

$$M_P = W_1 d' - R_A \cdot y = W_1 d' - \frac{W}{l}(l - y + d)y$$

differentiating with respect to $y$

$$\frac{dM_P}{dy} = -\frac{W}{l}(l - 2y + d)$$

The maximum value of $M_P$ occurs when $\frac{dM_P}{dy} = 0$, *i.e.* for $y = \frac{l}{2} + \frac{d}{2}$, *i.e.* when the wheel P and the centre of gravity of W are each $\frac{d}{2}$ on opposite sides of C the centre of the span AB.

*General Method.*[1]

*Bending Moment.*—Let AB, BC, CD, and DE (Fig. 123) be four loads which cross a span equal in length to $X_1Y_1$. Set off the force line *abcde* to represent the magnitudes of the four loads; choose a pole *o*, and draw the rays *ao*, *bo*, *co*, *do*, and *eo*, and the open funicular polygon *ut*, *uv*, $vx_1$, $x_1w$, and $wy_1$; the extreme sides meet at *t* in the vertical line through the centre of gravity of the whole load. This polygon will serve as bending moment diagram for various positions of the loads on the span if we consider the span as moving to the left instead of the loads moving to the right. Divide the span into say five equal parts (ten would give greater accuracy, but five are used to avoid complication in the figure and explanation). Take the first position of the span between verticals through $X_1Y_1$, so that the large load CB nearest

[1] This method, with examples, will be found explained in articles on "Moving Loads on Railway Under-bridges," by Mr. H. Bamford, in *Engineering*, Sept. 7, 1906; also published separately.

the c.g. of the whole load is over the left abutment ($X_1$). Draw vertical lines through the abutments to meet the funicular polygon in $x_1$ and $y_1$. Joining $x_1y_1$ then $x_1y_1w$ is the bending moment diagram, and the bending moment anywhere may be scaled off from the vertical distance between $x_1y_1$ and the lines $x_1w$ and $wy_1$, *e.g.* the bending moment $\frac{2}{5}$ of the span from the left abutment is $z_1y_4$.

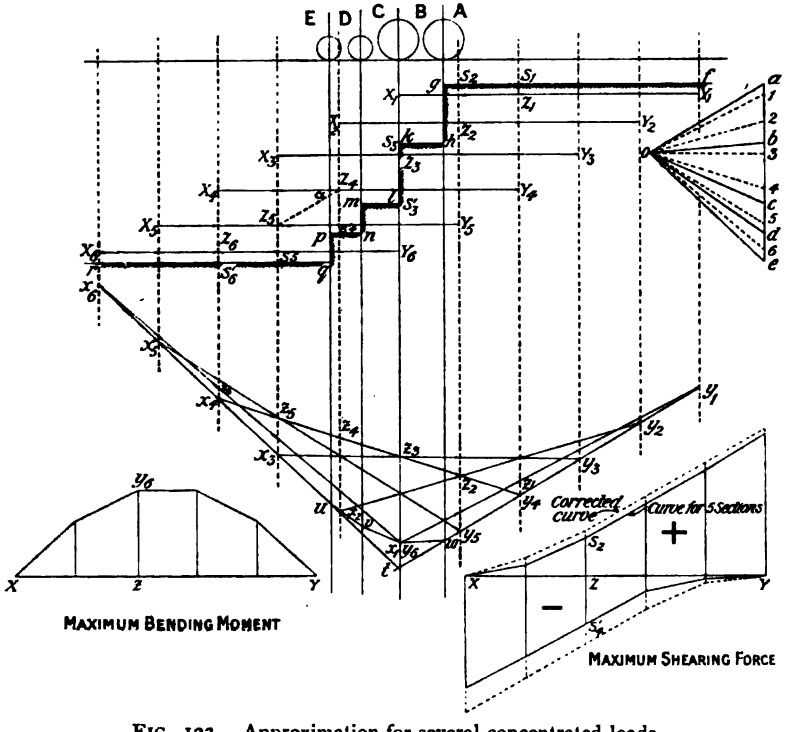



FIG. 123.—Approximation for several concentrated loads.

Now if the span moves $\frac{1}{5}$ of its length to the left, the base line of the bending moment diagram $x_2v x_1 w y_2$ is the closing side $x_2y_2$ vertically under the span $X_2Y_2$; *e.g.* the bending moment $\frac{2}{5}$ of the span from the left support ($X_2$) is $z_2y_5$. Similarly moving the span successive fifths of its length to the left gives the base lines $x_3y_3$, $x_4y_4$, $x_5y_5$, and $x_6y_6$, under the span positions $X_3Y_3$, $X_4Y_4$, $X_5Y_5$, and $X_6Y_6$ respectively. The approximate greatest bending moment at each $\frac{1}{5}$ of the span may now be measured and set off on a base line XY equal to the span, *e.g.* for the six positions taken the bending moments $\frac{2}{5}$ of the span from the left support are $z_1y_4$, $z_2y_5$, $z_3y_6$ (or $z_3x_1$), $z_4x_2$, $z_5x_3$, and $z_6x_4$, and the greatest of these is $z_3y_6$, which is then set off at $zy_6$ in the diagram of maximum bending moments on the base XY. The maximum bending moment under the load CB occurs when the centre of the span is

midway between the vertical through $t$ and the line CB; the base line of bending moments for this position is not shown, but would join a point nearly midway between $x_3$ and $x_4$ to a point nearly midway between $y_3$ and $y_4$, and could easily be drawn. The greatest bending moment anywhere under load CB would occur at a short distance (half the distance of $t$ from the vertical line CB) from the centre of the span XY. It is evident that this too would be greater than any occurring under any other of the loads, which have their maxima values further from midspan. A curve through the maximum bending moments at each $\frac{1}{6}$ of the span gives an approximate diagram of maximum bending moments at all points of the span, which, like Fig. 119, would be really made up of a number of parabolic arcs; further subdivision of the span would give a result more closely approximating to the true curve.

*Maximum Shearing Force.*—Through the pole $o$ draw lines $o1$, $o2$, $o3$, $o4$, $o5$, and $o6$, parallel respectively to $x_1y_1$, $x_2y_2$, $x_3y_3$, $x_4y_4$, $x_5y_5$, and $x_6y_6$, meeting *abcde*, in 1, 2, 3, 4, 5, and 6 respectively. Draw horizontal lines through *a*, *b*, *c*, *d*, and *e*, crossing the spaces A, B, C, D, and E respectively; these lines joined by the verticals *gh*, *kl*, *mn*, and *pq*, are shown cross-hatched and form the shearing force diagrams. The bases are the lines $X_1Y_1$, $X_2Y_2$, $X_3Y_3$, etc., corresponding to the position of the span. Taking, for example, the position $X_3Y_3$, the line $o3$ parallel to $x_3y_3$, divides the load line *ae* into reactions $e3$, and $3a$ at the left and right supports respectively, and the shear diagram *qpnmlkhg* on the base $X_3Y_3$ follows as in Art. 58. The maximum positive and negative shearing forces at each $\frac{1}{6}$ of the span are scaled from the various base lines, and plotted on the line XY of the maximum shearing force diagram, *e.g.* at $\frac{2}{6}$ of the span from the left support the positive shearing forces in the first three positions are $Z_1S_1$, $ZS_2$, and $Z_3S_3$, after which there is no positive shear at this section. The maximum positive ordinate for this section is $Z_2S_2$, which is set up at $ZS_2$. The negative shearing forces for this section in the last four positions of the span are $Z_3S'_3$, $Z_4S'_4$, $Z_5S'_5$, and $Z_6S'_6$, and the greatest is $Z_4S'_4$, which is set downwards at $ZS'_4$.

Some inaccuracy of the maximum shear diagram results from the fewness of the parts (five) into which the span has been divided, *e.g.* the maximum negative shearing force $\frac{2}{6}$ of the span from the left-hand support is measured from the line $X_3Y_3$ under the load BC. If a base line intermediate to $X_5Y_5$ and $X_4Y_4$ were drawn, the value of negative shear obtained for this section occurs just as the load ED passes over it, and the amount is readily found by joining the points $Z_4$, $Z_5$, meeting the vertical line ED in $a$; then $aq$ is the maximum negative shearing force for this section. The same method may be applied sometimes as an exact, and sometimes as an approximate one to other points, for over a moderate range the ends X and Y and consecutive positions of other selected points on the span are collinear; for as long as the whole load is on the span the changes in the end reactions (and therefore shears between axles) are proportional to the travel. Corrected in this way the method yields a much closer approximation, and for a

short load on a long span, exact values in the parts of the diagram which are important, *i.e.* where the ordinates are greatest. It will be noticed in drawing out the figure (which the reader should actually do to appreciate the method) that the greatest ordinates are determined by the outer loads, *i.e.* the positive and negative maximum shears at most sections occur when AB and ED respectively are passing over them. But the five divisions of the span were given positions with respect to the load CB which dominates the greatest ordinates of the maximum bending moment diagram; consequently this diagram gave a closer approximation to the true values. Increased accuracy in the bending moment diagram may be obtained by marking vertical reaction lines through X and Y, and a vertical line midway between them on a piece of tracing paper, and picking out maximum bending moments under dominating loads such as BC for other points on the span. The result of Art. 81 may be employed in conjunction with the graphical method to obtain increased accuracy if desired.

Sometimes a diagram such as Fig. 123 is set off from a horizontal base line either by taking the pole *o* on the level of *a* say, or by calculating the moments of the loads about some point, and setting these off in succession as intercepts on a vertical line through the chosen point, and thus drawing in a polygonal bending moment diagram without the use of the force diagram *oabcde*, the inverse of the process in Art. 50 and Fig. 53.

These methods are particularly convenient for a series of spans, for the diagrams of bending moment and shearing force for say a 300-feet span will serve with movable base lines for picking out maximum values for all smaller spans. A sheet of tracing-paper with parallel lines ruled on it forms a convenient movable base line.

*Reversed Order.*—If, as is usual, the order of loads is reversible, the maximum bending moment diagram will reach equal ordinates at equal distances from the centre of the span, and the shearing forces of opposite signs will also be of equal magnitude at equal distances on opposite sides from the centre of the span; the larger values in Fig. 123 will give those required on either side of the centre.

**81. Position of Load for Maximum Bending Moment at any Section of a Beam.**—For a series of concentrated loads the position to give a maximum bending moment to any given section C (Fig. 124) in passing over a beam AB may be found as follows. Let $W_1$ be the load to the left of C, W the total load on AB, and $x$ the distance of its centre of gravity from A; let the (constant) distance of the centre of gravity of $W_1$ to the left of W be $d$. Then the bending moment at C,

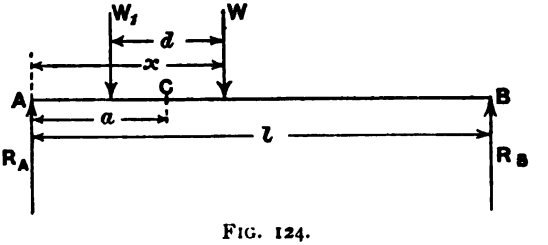



FIG. 124.

$$M_C = -R_A a + W_1(d + a - x) = -\frac{W(l-x)}{l}a + W_1(d + a - x) \quad (1)$$

For a small movement to the right—

$$\frac{dM_C}{dx} = W\frac{a}{l} - W_1, \text{ which is also equal to } \frac{a}{l}\left(W - W_1 - W_1\frac{l-a}{a}\right) \quad (2)$$

For a maximum value of (negative) $M_C$ this must change from negative to positive. (For a distributed load it will attain a zero value, but for concentrated loads it will pass discontinuously through zero as a load passes over C.) This can only take place as a load passes C moving to the right. (A load passing B decreases W, and cannot change (2) from negative to positive; a load passing A increases $W_1$ more than it does $W\frac{a}{l}$, and therefore cannot change (2) from negative to positive.) Regarding the load just passing C as partly on either side, and equating (2) to zero,

$$\frac{dM_C}{dx} = 0 = \frac{Wa}{l} - W_1 \quad \text{or} \quad \frac{Wa}{l} = W_1 \quad \text{or} \quad \frac{W}{l} = \frac{W_1}{a} \quad . \quad (3)$$

*i.e.* the average load per foot between A and C is equal to the average load per foot for the whole span; hence also the average load $\frac{W - W_1}{l - a}$ between B and C has also the same value. It may perhaps most conveniently be remembered that the passage of the load over the section C changes

$$(W - W_1) - \frac{l-a}{a}W_1$$

from negative to positive, *i.e.* changes the quantity

$$\text{load to right of C} - \frac{l-a}{a} \times \text{load to left of C}$$

from negative to positive. This defines the position of the load to give a maximum bending moment at any section C. In the passage of a given set of wheel loads two or more maximum values may occur each satisfying condition (3). The value $M_C$ for each position must be calculated, and the greatest of these values is the maximum required. At the centre of a span $a = \frac{l}{2}$, and (3) becomes

$$W = 2W_1 \text{ or } W - W_1 = W_1 \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

*i.e.* the load on either side of the centre is equal, which condition is satisfied when a load is passing over the centre of the span changing $(W - W_1) - W_1$ from negative to positive; that is, changing

$$\text{load on right} - \text{load on left of centre}$$

from negative to positive.

*Case for a Braced Girder.*—If the beam is a jointed frame or truss carrying the loads on say the bottom joints, the foregoing investigation will only hold good for maximum moments about those joints. For

any other, such as C in Fig. 125, let $W_1$ be the load on panels from A to E, $W_2$ the load on the panel ED in which C lies, and W the total load on the span AB, the positions of the centres of gravity being as shown in Fig. 125, the horizontal distance of C from E being $h$ and the length of the panel ED being $k$. The portion of the load $W_2$ carried at E will be—

$$W_2 \times \frac{FD}{ED} = W_2 . \frac{AD - x - d'}{k} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the bending moment at C is

$$M_C = -R_A a + W_1(a - x) + W_2\left(\frac{AD - x - d'}{k}\right)h$$

$$= -\frac{Wa}{l}(l - x - d) + W_1(a - x) + W_2\frac{h}{k}(AD - x - d') \quad . \quad (5)$$

$$\frac{dM_C}{dx} = W\frac{a}{l} - W_1 - W_2\frac{h}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For a maximum (negative) value of $M_C$ this changes from negative to positive;

$$\frac{dM_C}{dx} = 0 = W\frac{a}{l} - W_1 - W_2\frac{h}{k}$$

or

$$\frac{W}{l} = \frac{W_1 + \frac{h}{k}W_2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This only differs from the result (3) for a solid beam or over one of the loaded bottom joints in that the term $\frac{h}{k}W_2$ appears instead of that part of $W_2$ which lies on the length $h$ to the left of C. The difference would generally be very small.

**82. Load Position for Maximum Shearing Force at any Section of a Beam.**—In the case of a solid beam with concentrated moving loads directly carried there will be at any given section a continuous and uniform change of shearing force as any load approaches the section, and a sudden or discontinuous change as each load passes it; hence there will be a succession of maximum shearing force values (positive and negative) for that section. The greatest of these values may easily be found by trial as in Art. 80.

*Case of a Truss.*—Using Fig. 125 and the notation given in Art. 81, for any section C, the negative shearing force at C—

$$\text{Negative } F_C = -R_A + W_1 + \frac{FD}{ED}W_2$$

$$= -\frac{W}{l}(l - x - d) + W_1 + W_2\frac{AD - x - d'}{k} \quad . \quad . \quad (1)$$

For a maximum value—

$$\frac{dF_C}{dx} = 0 = \frac{W}{l} - \frac{W_2}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or $$\frac{W}{l} = \frac{W_2}{k} \qquad W_2 = W\frac{k}{l} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or if there are $n$ equal panels of length $k$ in span $l$—

$$\text{Load on panel } (W_2) = \frac{1}{n} \text{ load on span } (W) \quad . \quad . \quad . \quad (4)$$

For a maximum shearing force at C there would generally be no load to the right of D, and for a maximum positive shearing force no load to the left of E, condition (4) giving the load on ED.

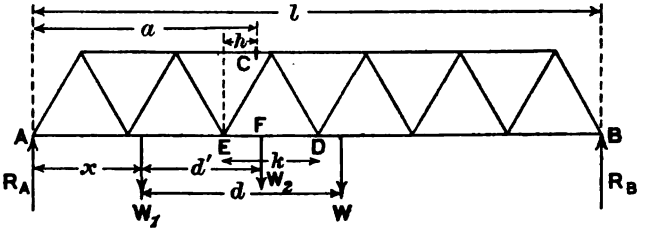



FIG. 125.

**83. Load Position for Maximum Pressure on Supports.**—When a number of axle loads at fixed distances apart traverse a series of longitudinal beams, called rail bearers or stringers, supported on cross girders which convey the load to the main girders of a bridge, it is important to consider the maximum load carried on any support or cross girder. Let AB and BC, Fig. 126, be two consecutive spans supported at A, B, and C, the beams being discontinuous at each support. In the position of loads shown in Fig. 126, let the variable distance of the first load $W_0$ from A be $x$. Then by moments about A and C the total reaction at B—

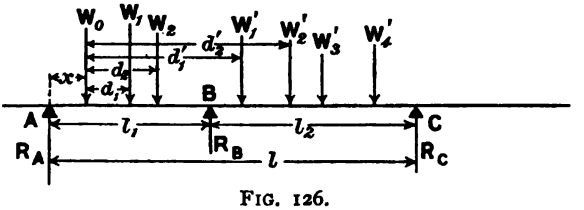



FIG. 126.

$$R_B = \frac{W_0 x + W_1(x + d_1) + W_2(x + d_2) + \text{etc.}}{l_1} + \frac{W_1'\{l - (x + d_1')\} + W_2'\{l - (x + d_2')\} + \text{etc.}}{l_2} \quad . \quad . \quad (1)$$

and for a small change in $x$—

$$\frac{dR_B}{dx} = \frac{1}{l_1}\left(W_0 + W_1 + W_2 + \text{etc.}\right) - \frac{1}{l_2}\left(W_1' + W_2' + W_3' + \text{etc.}\right) = \frac{\Sigma(W)}{l_1} - \frac{\Sigma(W')}{l_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For maximum values of $R_B$, $\frac{dR_B}{dx}$ must change from positive to negative. As in Art. 81, this can only occur when a load is passing from one span to the next, crossing B. When this load is partly on each span $\frac{dR_B}{dx} = 0$, and for this condition—

$$\frac{dR_B}{dx} = 0 = \frac{\Sigma(W)}{l_1} - \frac{\Sigma(W')}{l_2} \quad \text{or} \quad \frac{\Sigma(W)}{l_1} = \frac{\Sigma(W')}{l_2} \quad . \quad . \quad (3)$$

*i.e.* the average load on each span of the rail bearer or stringer is the same, and the same as the average on the two spans. Taking the loads as moving to the right, this condition (3) occurs as one load is crossing the support B, and so bringing the average load per foot on BC up to or beyond that on AB. During the passage of a given set of loads there may be two or more minima for positions satisfying (3); if so the pressure for each must be calculated and the greatest value found. The pressure on B when the maximum value occurs is easily calculated as in (1), which must include the weight directly over B.

If $l_1 = l_2$ condition (3) becomes—

$$\Sigma(W) = \Sigma(W') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

*i.e.* the load on the two adjacent spans must be equal.

It will be noticed that the condition (3) for a maximum reaction at the intermediate support in the length AC (Fig. 126) is the same as for maximum bending moment at any intermediate section in the length AB (Fig. 124) given at (3), Art. 81.

Further, if $M_C$ is the bending movement at C (Fig. 124), and $R_C$ is the reaction at C, if there were a support at C, AC and BC being discontinuous, by taking moments it is easy to show that—

$$R_C = -M_C \times \left(\frac{1}{a} + \frac{1}{l-a}\right) \quad \text{or} \quad -M_C \times \frac{l}{a(l-a)} \quad . \quad (5)$$

And in particular if the spans $a$ and $l - a$ are equal $\left(\frac{l}{2}\right)$

$$R_C = -M_C \times \frac{4}{l} \text{ or } -M_C \times \frac{2}{a} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

*i.e.* the maximum pressure on cross girders pitched a distance $a$ apart is $\frac{2}{a}$ times the greatest bending moment on a span $2a$ for the same moving load.

**84. Equivalent Uniformly Distributed Load.**—For designing the flanges and other parts of a girder to suit the varying bending moment, it is usual in British railway practice to find the uniformly distributed load which would give a bending moment everywhere at least equal to that caused by the actual greatest rolling load. The bending moment diagram for such an equivalent load would be a symmetrical parabola completely enveloping the diagram of maximum bending moments for rolling loads. In the single rolling load (Art. 77), or a uniformly distributed moving load shorter than the span (Art. 78), the diagrams of

maximum bending moment (Figs. 117 and 118) are parabolic, and the same parabolas will be the diagrams for the equivalent distributed load on the whole span. The load per foot $w'$ equivalent to the concentrated load W is found by equating the moments—

$$\frac{w'}{2}x(l-x)=\frac{W}{l}x(l-x)$$

as in (7) Art. 76, and (7) Art. 77, which gives—

$$w'=\frac{2W}{l}$$

And the value of $w'$ equivalent to a distributed load $w$ per foot on a length $c$ (see (9) Art. 78) is found from—

$$\frac{w'}{2}x(l-x)=wcx\frac{(l-x)}{l}\left(1-\frac{c}{2l}\right)$$

which gives—

$$w'=\frac{2c}{l}\left(1-\frac{c}{2l}\right)w$$

In more general cases the maximum (central) ordinate of an enveloping parabola is often much greater than the neighbouring maximum ordinate of the diagram of maximum bending moments for the actual rolling load, and this arises partly from the fact that the enveloping parabola includes all ordinates, including small ones close to the end supports. But this is not generally necessary, for the flanges areas, modulus of section, and resistance to bending of a girder will near the supports for practical reasons be more than ample for resisting the small bending moments. A more reasonable plan for determining the modulus of the central section of a girder is to determine the parabola (and corresponding load) through the ends of the span, and enveloping the maximum bending moment diagram for that part of the length of the beam over which the bending moment may exceed the minimum safe moment of resistance of the girder, viz. its safe moment of resistance at its ends. Experience shows about what fraction of the maximum modulus of section the minimum modulus will be for given length of span; such a fraction will decrease with increased length of span. A numerical example will make this point clear.

EXAMPLE.—Two loads of 10 tons each 12 feet apart cross a span of 24 feet. Find the equivalent uniformly distributed load if the minimum modulus of section or product of depth and flange area is at least 40 per cent. of the maximum modulus of section on a 24-foot span.

From (10), Art. 79, the maximum bending moment at a distance $x$ from one end is—

$$M_x=-\frac{x}{24}\{20(24-x)-120\}=-\frac{5x}{6}(18-x)\text{ tons-feet,}$$

which reaches zero for $x=0$ and $x=18$ feet, and a maximum negative magnitude for $x=9$ feet. When its value is—

$-7{\cdot}5\times 9=-67{\cdot}5$ tons-feet 3 feet from the centre of the span.

The diagram of maximum bending moment is shown at ADCGB,

Fig. 127. The safe moment of resistance of all parts of the girder will exceed—

$$\tfrac{40}{100} \times 67{\cdot}5 = 27 \text{ tons-feet.}$$

So we may neglect all points on the diagram below the level (27 tons-feet) of the horizontal line FG, and circumscribe the remainder of the diagram by a parabola AFHGB. This may be accomplished by drawing a parabola through A and F, having the centre line KC as axis. Let $a$ and $b$ be the co-ordinates of the intersection F of the line FG and the parabola AFC. The length $a$ may be measured from the diagram or calculated thus: $b = 27$ tons-feet, and from the above equation for $M_x$—

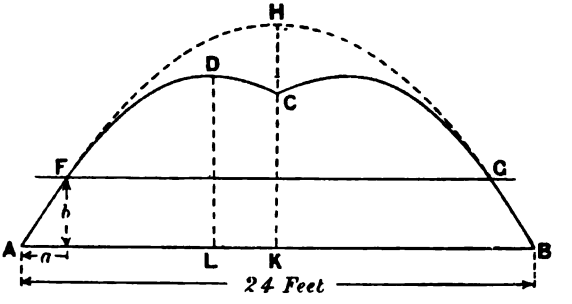


FIG. 127.

$$\tfrac{5}{6}x(18 - x) = 27 \qquad \text{hence } x \text{ or } a = 9 - \sqrt{48{\cdot}6} = 2{\cdot}03 \text{ feet.}$$

The central height $h$ or HK of the parabola is then—

$$h = b\frac{\left(\frac{l}{2}\right)^2}{a(l-a)} = 27 \times \frac{144}{44{\cdot}60} = 87{\cdot}2 \text{ tons-feet,}$$

giving the vertex H from which the parabola AFHGB can easily be drawn. If $w$ is the equivalent load per foot—

$$\frac{wl^2}{8} \text{ or } 72w = 87{\cdot}2 \qquad w = 1{\cdot}21 \text{ tons per ft. run.}$$

The parabola circumscribing the *whole* figure from A to B would have the same tangent at A as the parabola AFDC; its slope at $x = 0$ is found by differentiating $M_x$ to be $\frac{5}{6} \times 18 = 15$ tons-feet per foot. For a central ordinate $h'$ the slope at $A = \frac{4h'}{l} = \frac{4h'}{24} = 15$. Hence $h' = 90$ tons-feet. The vertex might also easily be found graphically from the fact that the vertex D bisects the projection of the tangent at A on the axis DL, and the vertex of the circumscribing parabola bisects the projection of the same tangent (produced) on the axis KC.

*Other Cases.*—The formula

$$h = b\frac{\left(\frac{l}{2}\right)^2}{a(l-a)}$$

will hold good when the co-ordinates such as $a$ and $b$ of various points are determined graphically by such a method as is given in Art. 80, trial being necessary with various points to find the greatest value of $h$, *i.e.* the height of the parabola which will envelop all the points.

The equivalent uniform load is also sometimes taken as that which would give the same maximum, and in some cases as that which would give the same average ordinate of the maximum bending moment diagram; such an equivalent will generally give maximum bending moment values for some sections which are less than the actual values, and for other sections values greater than the actual ones. In other words, the parabolic diagram will in some places fall within and in others outside the actual diagram of maximum bending moments. The errors will decrease with increase of span when the actual diagram of maximum bending moments will become more nearly a symmetrical parabola. Equality of average ordinate is the safer rule, and gives generally a greater central value. It may be approximately found by making the parabola agree with the actual diagram at $\frac{1}{4}$ span from the supports, so that if $w'$ is the uniform load per foot from (7) Art. 76—

$$\frac{w'}{2} \cdot \frac{l}{4} \cdot \frac{3l}{4} = \text{moment at } \tfrac{1}{4} \text{ span}$$

$$w' = \frac{32}{3} \times \frac{\text{M at } \frac{1}{4} \text{ span}}{l^2}$$

There is no definite and general convention as to the precise significance of the term "equivalent" uniform load.

Equivalent uniformly distributed rolling loads for shear may also be found giving the same maximum ordinate as the actual diagram, but lower ordinates near the middle of the span; these loads will be greater than the equivalent uniform load for bending moments, particularly in short spans. No uniformly distributed load will give a maximum shear diagram completely enveloping that for the actual loads, for it cannot give, say, the same maximum positive shearing force very near to the left support, as occurs there under a concentrated wheel load; an inspection of Figs. 116 and 117 will show this, since the parabola with vertical axis has a horizontal tangent at its vertex. But near the middle of the span the shearing resistance of a girder will usually be greater than is necessary, and where the positive shears are low near a support the girder has a resistance sufficient for high negative shears.

From maximum bending moment curves for a large number of locomotives of leading British railways Mr. Farr has deduced[1] the following values of the equivalent uniformly distributed loads for a single line of way, as representing the greatest values up to the year 1900. These are based on symmetrical parabolas which completely envelop the actual maximum bending moment diagrams for all the heaviest types of train headed by two of the heaviest locomotives for each span—

| Span (feet) | 5 | 7·5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Equivalent load per foot (tons) | 7·6 | 5·55 | 4·85 | 3·74 | 3·20 | 2·81 | 2·63 | 2·48 | 2·40 | 2·24 | 2·17 | 2·11 | 2·06 | 2·01 | 1·97 |

[1] "Moving Loads on Railway Underbridges," *Proc. Inst. C.E.*, vol. cxli.

The equivalent uniformly distributed loads for shearing force at the ends of a span may probably be taken at about, say, 15 per cent. greater for from 10 to 50 feet spans, 10 per cent. greater from 50 to 100 feet, and 5 per cent. greater over 100 feet.

Mr. Farr gives the following loads which may come upon a cross-girder supporting a single line of way :—

| Distance apart of cross girders . . . | Up to 7 ft. | 8 | 9 | 10 |
|---|---|---|---|---|
| Maximum load tons . . . . . | 19 | 21·5 | 23·5 | 25·11 |

*Other Equivalent Loads.*—The actual maximum bending moments and shearing forces produced by a given train load may very nearly be reproduced by a uniformly distributed load, together with a concentrated load. Thus, Dr. F. C. Lea [1] finds that for a large number of locomotive loads on British railways for spans from 100 to 200 feet (single track) for all sections the maximum shearing force may be represented by the effect of loads of 2 tons per foot, together with a load of 18 tons in the most influential position (see Arts. 88 and 89). The same loading gives close results for the maximum bending moments, but may, of course, be replaced by a uniform load of $2 + \frac{36}{l}$ tons per foot.

**85. Conventional Train Loads.**—Instead of using equivalent uniformly distributed loads for the purpose of bridge design, a common practice in America and elsewhere is to use a conventional train load, consisting of axle loads representing something like railway loads, but

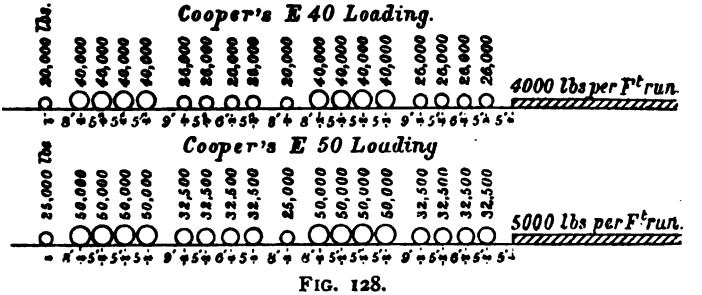



FIG. 128.

simplified in distances apart and load per axle. The best known of these is that devised by Mr. T. Cooper. In this system the spacing of the locomotive axles is in whole numbers of feet, and is kept constant for trains of different weights. Consequently, for every section in a given span the maximum bending moments and shearing forces are proportional, and if calculated for one train loading can be found for any other by multiplying by a constant factor. Fig. 128

[1] *Proc. Inst. C.E.*, vol. clxi., p. 284.

represents Cooper's E 40 and E 50 loadings for a single line of railway. Thus class E 40 consists of two locomotives (each weighing about 127 English tons) with the axle loads shown, followed by a train load uniformly distributed of 4000 lbs. per foot. Class E 50 is derived from it by multiplying all loads by $\frac{5}{4}$. For bridge design loading on this basis is chosen of such a weight as to be at least equivalent to any train load used on the railway. One advantage of this method over tables of equivalent uniformly distributed loading for bending moment is that the same loading may be applied to find the maximum shearing force.

EXAMPLE 1.—Find the maximum bending moment at the centre of a 40-ft. span when it is crossed by the load $E_{40}$, Art. 85. Marking on the edge of a strip of paper a length of 40 feet, and its middle point to scale and sliding it under load $E_{40}$, Fig. 128, it is evident that the condition (3*a*), Art. 81, is satisfied when the fourth load from the left is passing over the middle of the span. For then the load in 1000-lb. units is $20 + 40 + 40 + 40 = 140$ to the left of the middle, and $40 + 26 + 26 = 92$ to the right of the centre, and the load 40 passing to the right increases the 92 to 132 as it reduces 140 to 100, the loads on either side passing through equal values. The bending moment for this position is easily calculated; taking moments about the right-hand support for a 40-ft. span in 1000-lb. units.

Left-hand reaction $\times 40 = 20 \times 38 + 40(30 + 25 + 20 + 15) + 26(6 + 1) = 4542$.

Left-hand reaction $= \frac{4542}{40} = 113{\cdot}5$ thousand pounds.

Bending moment at centre $= -113{\cdot}5 \times 20 + (20 \times 18) + 40(10 + 5) = -1311$ units a bending moment of 1,311,000 pound-feet.

EXAMPLE 2.—Find the maximum reaction on one of a series of cross-girders spaced 20 feet apart when they are traversed by a load $E_{80}$, Art. 85.

By (6), Art. 83, this would be $\frac{2}{20}$ or $\frac{1}{10}$ times the maximum bending moment at the centre of a 40-foot span. But using the result of Example 1 above this bending moment would be

$$1{,}311{,}000 \times \tfrac{5}{4} = 1{,}638{,}750 \text{ lb.-feet,}$$

hence the maximum reaction is

$$\frac{1638750}{10} = 163{,}875 \text{ pounds.}$$

**86. Combined Shearing Force Diagrams.**—The shearing force exerted by the dead loads on a beam is at any section always the same, being positive at some sections and negative at others. At any given section the extreme values of the resultant shearing force are found by adding algebraically the constant shear due to the dead load to the maximum positive and negative shears respectively caused by the line load. The resultants will form a diagram showing the limits of the shearing force at every cross section, and whether or not at any given section any change of sign in the shearing force takes place.

Taking, for example, a span subjected to a moving load $w$ per foot run longer than the span, as in Art. 76, Fig. 116, and a dead load $w'$ per foot run, Fig. 129, shows at (1) the maximum shearing force diagram for the rolling load $w$, at (2) the shearing force diagram for the dead load.

At (3) the ordinates from A'B' to $e'f'$ are equal to those from AB to $ef$ at (2), and the ordinates vertically upwards from $e'f'$ to $A'gc'$ give the maximum positive shear due to the combined loads; the ordinate at $g$ is zero, the negative shear due to the dead load being at this section just

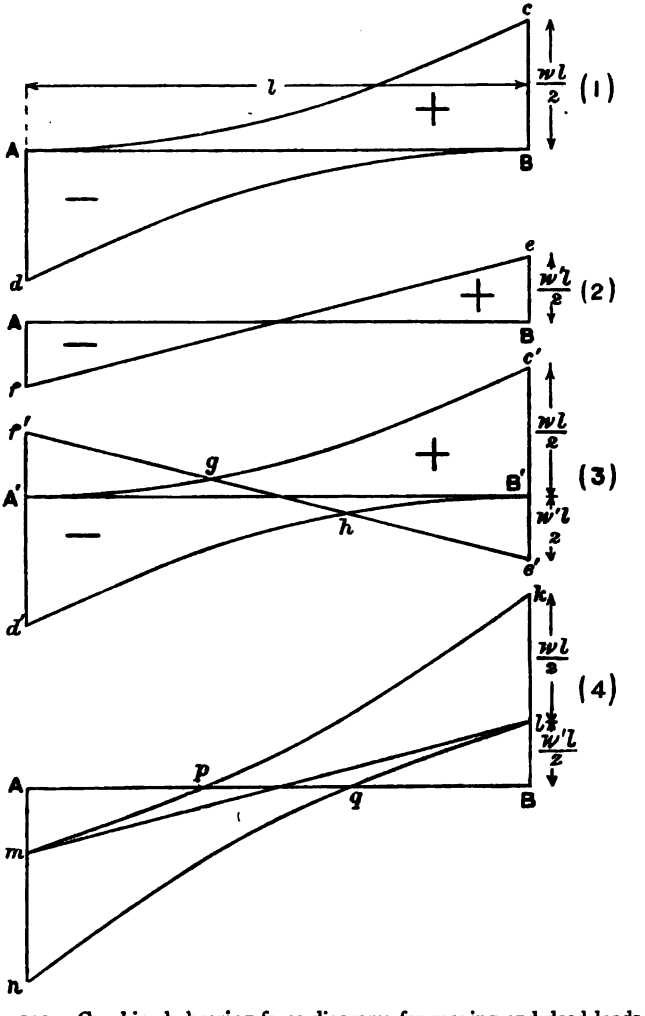



FIG. 129.—Combined shearing force diagrams for moving and dead loads.

sufficient to neutralise the maximum positive shearing force due to the live load. To the left of $g$ the negative shearing forces of the dead load exceed the maximum shear forces due to the live load; the shear to the left of $g$ is always negative, the ordinates from $gf'$ to $gA'$ giving

the minimum magnitudes of the negative shears, and the ordinates from $gf'$ to $B'd'$ giving the maximum magnitudes. Similarly to the right of $h$ the shearing force is always positive, the vertical ordinates from $he'$ to $A'gc'$ giving the maximum magnitudes and those to $hB'$ the minimum magnitudes.

Sections between $g$ and $h$ are subject to a change from positive to negative shearing force of the magnitudes given by the vertical ordinates from the line $e'f'$ or $gh$ to $A'gc'$ and to $B'hd'$ respectively.

The diagram of the extreme values of the shearing force on the span AB is again shown at (4), Fig. 129, the ordinates being measured from the horizontal base line AB, and being the algebraic sum of the ordinates in (1) and (2). The form (3) is rather easier to construct, but (4) is perhaps clearer. The length $pq$ over which the shearing force changes sign is sometimes called the *focal length* of the span. The position of the point $p$ may be found graphically from the diagram, or algebraically if general expressions for shearing force due to each kind of load separately can be written and equated.

EXAMPLE.—Span 50 feet, live load 2 tons per foot run, dead load $\frac{3}{4}$ ton per foot. Find the length over which the shearing force changes sign.

At a distance $x$ from the left end—

Maximum positive shear due to live load $= \frac{wx^2}{2l} = \frac{2x^2}{2 \times 50} = \frac{x^2}{50}$ tons

Magnitude of negative shear due to dead load $= w\left(\frac{l}{2} - x\right) = \frac{3}{4}(25 - x)$.

These magnitudes are equal when

$$\frac{x^2}{50} = \tfrac{3}{4}(25 - x) \text{ or } x^2 + 37{\cdot}5x - 937{\cdot}5 = 0$$

and neglecting the negative value of $x$ (which is not on the span)

$$x = 17{\cdot}15 \text{ feet.}$$

Distance over which shear changes sign is

$$50 - 2 \times 17{\cdot}15 = 15{\cdot}7 \text{ feet.}$$

**87. Influence Lines.**—An influence line is a curve showing for *one* section of a beam the shearing force, bending moment, stress, deflection or similar function for all positions of a moving load. It is important to distinguish between influence lines for bending moment or for shear and the diagrams previously dealt with in this chapter showing the *maximum* bending moment and shearing force at *all* sections of the beam.

**88. Influence Line for Bending Moment.**—*Single Load.*—Consider the influence line for a point C, Fig. 130, on a span AB $= l$, distant $a$ from the support A and $l - a$ from the support B, for a single load W crossing the beam AB. When the load is distant $x$ (greater than $a$) from A, the reaction at A is $W\frac{l-x}{l}$, and the bending moment at C, neglecting the negative sign, is

$$W \cdot \frac{l-x}{l} \cdot a$$

which varies from $W\frac{a(l-a)}{l}$ as $x$ increases from $a$ to $l$, *i.e.* as the load moves from C to B. The vertical ordinates of the line DB represent this varying bending moment at the fixed point C. Similarly the line AD by its ordinates gives the bending moment at C for all positions

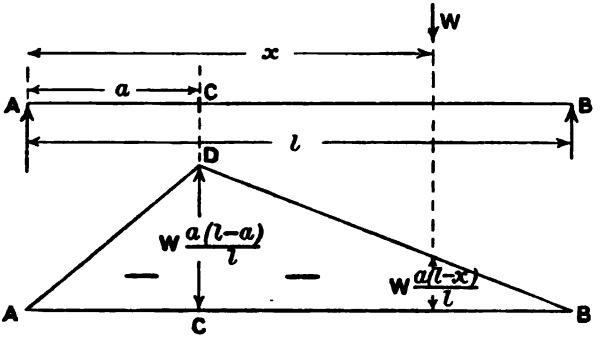



FIG. 130.—Influence line for bending moment.

of the load between A and C, and ADB is the complete influence line in this case. It is quite evident that the bending moment at C is a maximum for $x = a$, *i.e.* when the load is at C, as already mentioned in Art. 77; the maximum bending moment for two concentrated loads has been shown in Art. 79, Fig. 120, which is a modified influence line for two loads.

*Uniformly Distributed Load.*—Let the influence line ADB, Fig. 131,

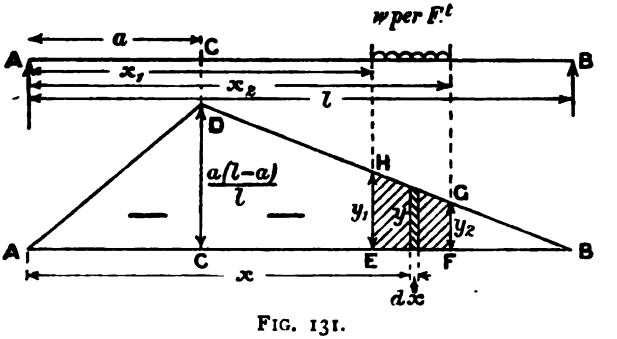



FIG. 131.

be drawn as in Fig. 130 for the point C for unit load (say 1 ton). Then any ordinate $y$ distant $x$ (greater than $a$, say) from A represents a bending moment at C

$$-\frac{a(l-x)}{l}$$

and if the moving load is $w$ per foot run, a length $dx$ distant $x$ from A would exert a bending moment

$$-wdx\frac{a(l-x)}{l}$$

at C, represented by $w$ times the strip of area $ydx$. Similarly every strip of area above a load between E and F represents a bending moment, and the total bending moment at C resulting from a uniformly distributed load from E to F would be represented by area

$$w \times \text{EFGH or } w\int_{x_1}^{x_2} ydx$$

and for all positions the bending moment at C is represented by the area over the loaded portion.

For a load longer than the span it is evident that the maximum bending moment at C occurs when the bending moment is represented

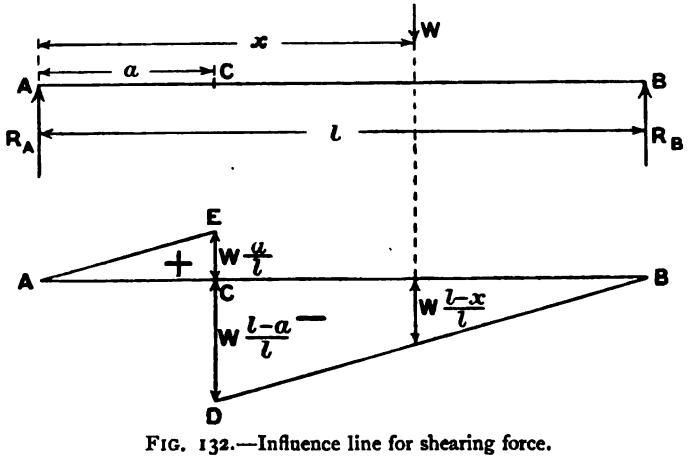



FIG. 132.—Influence line for shearing force.

by the whole triangle ADB, *i.e.* when the load covers the whole span. For loads shorter than the span it is easy to show here that condition (8) Art. 78 must be satisfied for the bending moment at C to be a maximum, and if $d$ is the length of the load the distance of the loaded portion from A is $a - \frac{a}{l}d$.

The scale on which the area under the line ADB represents the bending moment is $wqq'$ tons-feet to 1 square inch if the linear horizontal scale is $q$ feet to 1 inch and the scale of bending moment vertically for the unit load is $q'$ tons-feet to 1 inch, *i.e.* $\text{DC} = \frac{a(l-a)}{l} \div q'$ inches.

**89. Influence Lines for Shearing Force.**—*Single Load.*—Consider the influence line for shear at the point C (Fig. 132), distant $a$ from the support and $l-a$ from the support B on a beam of span $l$ crossed by a

single load W. When the load is distant $x$ greater than $a$ from the end A, the (negative) shear at C is equal to the upward reaction at A, which is found by taking moments about B

$$R_A l = W(l - x)$$

$$\text{Shearing force at C} = -R_A = -W\frac{l-x}{l} \quad . \quad . \quad . \quad (1)$$

This value varies as the ordinates of a straight line DB, from o at B, where $x = l$, to a maximum negative value $DC = -W\frac{l-a}{l}$ at C, where $x = a$. When the load is between A and C, $x$ is less than $a$, and the positive shear at C is equal to the reaction at B, viz.:

$$\text{Shearing force at C} = +W\frac{x}{l}$$

which varies as shown by the straight line AE from o at A, where $x = o$, to a maximum $EC = W\frac{a}{l}$ at C, where $x = a$. The complete influence line for shearing force at C is AEDB. The maximum shears obviously occur when the load is just reaching C.

*Uniformly Distributed Load.*—Let ADEB (Fig. 133) represent the

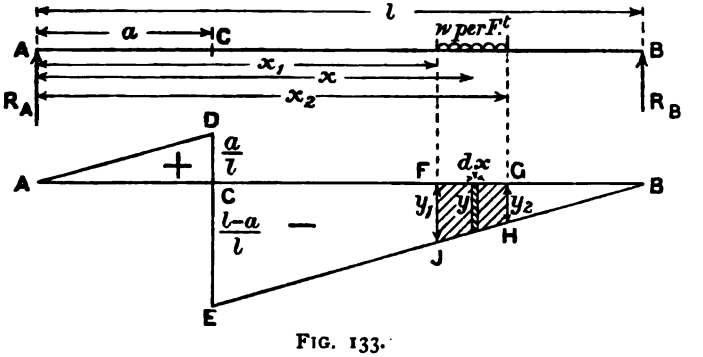



FIG. 133.

influence line for shear at C for a unit load, say 1 ton, moving over the span AB. Then the ordinate $y$, at a distance $x$ from A, represents

$$-1 \times \frac{l-x}{l} \text{ or } +1 \times \frac{x}{l}$$

according as $x$ is greater or less than $a$. If a distributed load $w$ per foot is on the span, an element $wdx$, distant $x$ from A, exerts at C a shearing $wdx$ times that represented by the ordinate $y$, and is represented by the area $y \,.\, dx$ in the figure. A similar conclusion holds for every vertical strip of area between, say, F and G, and the total shearing force at C resulting from a distributed load between F and G is represented by the area—

$$w \times \text{FGHJ, or } w\int_{x_1}^{x_2} y\,dx,$$

and for all positions the shearing force at C is represented by the area enclosed over the loaded portion by the line ADEB. The shearing force (positive or negative) at C evidently has its greatest value when the load just reaches C; if the load extends on each side of C the areas must be added with proper algebraic sign.

The scale on which the area under the line ADEB represents the shearing force is $wqq'$ tons per square inch of the line or horizontal scale is $q$ feet to 1 inch, and the scale of shearing force for the unit load is $q'$ tons to 1 inch, *i.e.* DC $= \frac{a}{l} \div q'$ inches and EC $= \frac{l-a}{l} \div q'$ inches.

**90. Influence Lines for Frame or Truss.**—(1) *For Shearing Force.*—When the rolling load is transferred to the beam at joints as in an

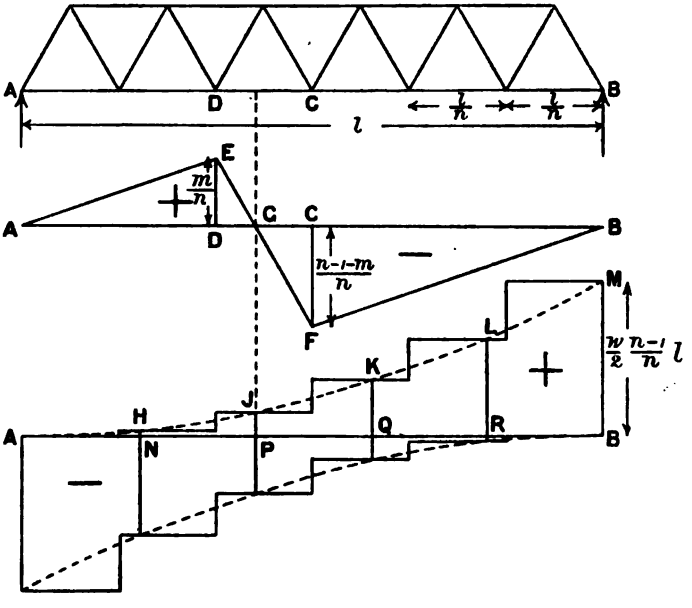



FIG. 134.—Influence line for shear in braced girder.

articulated frame or truss, or any bridge floor supported on cross girders, the shearing force at all sections in any one panel (or space between two consecutive joints) is the same. Thus in Fig. 134, if the load is carried by the lower joints, between D and C, the shearing force is the same for all points. Consider the influence line for shearing force in this panel when unit load (say 1 ton) crosses the span AB, which is divided into $n$ equal panels each $\frac{l}{n}$ long. Let there be $m$ panels to the left of D, and therefore $n - 1 - m$ panels to the right of C. When

the load is between A and D, distant $x$ from A, the shearing force in the panel DC is equal to the reaction at B, viz.

$$1 \times \frac{x}{l}$$

and the influence line is AE varying from o at A to $\frac{m}{n}$ at D. Similarly when the load is between C and B the shearing force in the panel DC is equal to minus the reaction at A, and the influence line is FB varying from $\frac{n-1-m}{n}$ at C to zero at B. As the load moves from D to C, the shearing force between D and C diminishes from the (variable) reaction at B by the amount of the load carried at the joint C, which, like the reaction at B, varies uniformly with increase of $x$: hence the shearing force in the panel varies uniformly with $x$, that is, as the ordinates of a straight line. Hence the straight line EF is the influence line. In symbols the shearing force in the panel DC, which is the reaction at B minus the load carried at C (found by moments about D), is—

$$\frac{x}{l} - \left(x - m\frac{l}{n}\right) \div \frac{l}{n} = \frac{1}{l}(x - nx + ml) \quad . \quad . \quad . \quad (1)$$

which varies as a straight line from $\frac{m}{n}$ when $x = m\frac{l}{n}$ at D to $-\frac{n-1-m}{n}$ at C when $x = (m+1)\frac{l}{n}$.

The line AEFB is then the whole influence line for shearing force in the panel DC. The maximum positive and negative shearing forces occur when the load is just to the left of D and right of C respectively.

Exactly as for the solid beam the area under the line AEFB and above any uniformly distributed load $w$ per unit length, represents the shearing force in DC due to the distributed load. Evidently if the load is longer than BG the maximum shearing forces in the panel DC occur when the load reaches from an abutment to a point G which divides DC in the ratio $m$ to $n-1-m$ so that

$$\mathrm{DG} = \frac{m}{n-1}.\mathrm{DC} = \frac{m}{n-1}.\frac{l}{n}, \quad \mathrm{CG} = \frac{n-1-m}{n-1}.\mathrm{DC} = \frac{n-1-m}{n-1}.\frac{l}{n} \quad . \quad . \quad (2)$$

$$\mathrm{AG} = \frac{m}{n-1}.l \qquad \mathrm{BG} = \frac{n-1-m}{n-1}.l \quad . \quad . \quad . \quad (3)$$

It may be noted that $w \times \mathrm{DG} = \frac{1}{n} \cdot w \cdot \mathrm{AG}$ in accordance with (4), Art. 82.

The maximum positive shearing force on the panel DC is—

$$w \times \text{area AEG} = \frac{w}{2} \times \mathrm{ED}.\mathrm{AG} = \frac{w}{2} \cdot \frac{m}{n}\left(m\frac{l}{n} + \frac{m}{n-1}.\frac{l}{n}\right)$$

$$= \frac{w}{2} \cdot \frac{m^2}{n(n-1)} \cdot l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

And in the last panel on the right $m = n - 1$, and the maximum shearing force is $\frac{w}{2} \cdot \frac{n-1}{n} \cdot l$, which would be the maximum shearing force at the abutment of a solid beam of length equal to $n - 1$ panels.

The maximum negative shearing force in the panel DC is—

$$w \times \text{area BGF} = \frac{w}{2} \times \text{CF} \cdot \text{GB} = \frac{w}{2} \cdot \frac{n-1-m}{n} \left( \frac{n-1-m}{n} \cdot l + \frac{n-1-m}{n-1} \cdot \frac{l}{n} \right) = \frac{w}{2} \cdot \frac{(n-1-m)^2}{n(n-1)} \cdot l \quad . \quad . \quad (5)$$

And on the first panel on the left $m = 0$, and the maximum negative shearing force is $\frac{w}{2} \cdot \frac{n-1}{n} l$, as for the positive value at the other end of the span.

The maximum shears in the 1st, 2nd, 3rd, 4th, etc., up to the $n$th panel are respectively—

$$0,\ 1^2,\ 2^2,\ 3^2, \text{ etc. } \ldots\ (n-1)^2 \text{ times } \frac{w}{2} \cdot \frac{l}{n(n-1)} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or—

$$0, \left(\frac{1}{n-1}\right)^2, \left(\frac{2}{n-1}\right)^2, \left(\frac{3}{n-1}\right)^2 \ldots \left(\frac{n-3}{n-1}\right)^2, \left(\frac{n-2}{n-1}\right)^2, 1 \text{ times } \frac{w}{2} \cdot \frac{l}{n} \cdot (n-1),$$

which is the value in the end panels. Thus a parabola AHJKLM, the ordinate BM being $\frac{w}{2} \cdot \frac{n-1}{n} \cdot l$, shows the maximum shearing force in each panel if the span is divided at ANPQRB into $n - 1$ equal parts, and ordinates erected to cut the parabola. The points of division N, P, Q, R, B also indicate the load positions for maximum shearing forces, for in (3) above $\text{AG} = \frac{m}{n-1} \cdot l$, the points of division for the successive panels being $0, \frac{1}{n-1} \cdot l, \frac{2}{n-1} \cdot l \ldots \frac{n-2}{n-1} \cdot l$, and $l$ from A.

For application of this influence line see Art. 144, and for extensions to the case of trusses with a curved chord see Art. 145.

(2) *For Bending Moments.*—For a vertical section which cuts *a joint* of the *loaded* chord, say AB, Fig. 135, the influence line will be as in Fig. 130, Art. 88. For other sections such as C, Fig. 135 (including joints of the top chord), when loads are carried from joints of the lower chord the influence line will be the same for say unit load crossing the span AB as for a solid beam when the load is between A and E or between F and B. Hence it is AG and HB (Fig. 135) over these respective ranges where ADB would be the influence line for a solid beam. When the load is between E and F, *i.e.* in the panel in which the vertical section through C falls, the influence line will no longer be as for a solid beam, since the load is carried by the beam at the joints E and F. But it is evident that the bending moment at C will vary uniformly with the distance advanced by the load between E and F, *i.e.* the

influence line will be a straight line joining G to H, and hence AGHB is the complete influence line for C when unit load traverses the span AB. For uniformly distributed loads we proceed as in Art. 128, *i.e.* the

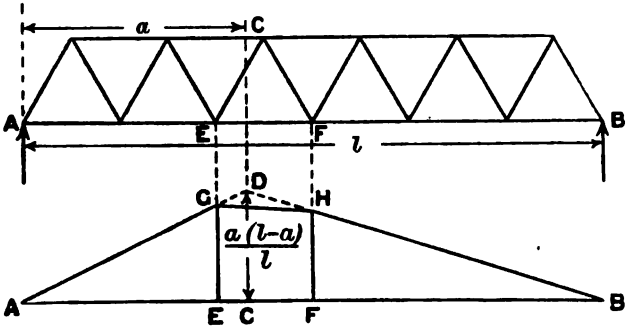



FIG. 135.—Influence line for bending moment in braced girder.

bending moment at C is represented by areas under the line AGHB and above the loaded portion, which clearly shows that the bending moment at C is a maximum when the span is fully loaded.

## EXAMPLES VI.

1. A load of 1 ton per foot and over 60 feet long crosses a span of 60 feet. Draw diagrams of maximum shearing force and bending moment. State the maximum positive and negative shearing force and maximum bending moment at 15 and 30 feet from the left-hand support.

2. A single rolling load of 2 tons crosses a girder of 30 ft. span. Draw the diagrams of maximum shearing force and maximum bending moment, and state the maximum positive and negative shearing forces and the maximum bending moment at sections 5, 10, and 15 feet from one end.

3. Draw the diagrams of maximum positive and negative shearing force and maximum bending moment for a load of ½ ton per foot and 30 feet long crossing a span of 100 feet. What is the maximum positive shearing force at 15, 40, and 50 feet from the left-hand support? State the maximum bending moment at the same sections. What load per foot extending over the whole span would produce the same maximum bending moment at every section as the above load?

4. The axle loads of a traction engine are 16 and 8 tons at a distance of 15 feet apart, and the engine crosses a girder of 50 feet span, the smaller load leading from left to right. Draw the diagram of maximum bending moments. State the greatest ordinate and its distance from the centre of the span: also the maximum bending moment midway between the supports.

5. Work problem No. 4 for a span of 25 feet. Below what span will the greatest bending moment anywhere occur when the smaller load is off the span?

6. Five loads A, B, C, D, and E, in the order given, cross a span of 60 feet. The loads are A = 12 tons, B = 20 tons, C = 20 tons, D = 8 tons, E = 8 tons. The distance between the loads in the same order are 7, 7½, 7, and 6 feet. Draw the diagrams of maximum shearing force and maximum bending moment. Where does the maximum bending moment occur and what is its amount?

7. Calculate the maximum bending moment at the centre of a span of 80 feet when crossed by a train load class $E_{80}$ (Art. 85). Also for a span of 30 feet.

8. Estimate the maximum pressure on cross girders 15 feet apart when a load of class $E_{60}$ crosses a bridge.

9. Estimate the uniformly distributed load which would give the same maximum bending moment at 20 feet from one support as the load $E_{40}$ (Art. 85), on an 80 feet span. Find the greatest maximum bending moment anywhere on the span for the uniform and for the actual load.

10. With the data of problem No. 1, find the length over which the shear changes sign if the dead load is $\frac{3}{4}$ ton per foot run.

11. By means of the influence lines find maximum bending moments and positive and negative shearing forces at a point 40 feet from the left abutment when a girder of 100 feet span is traversed by a rolling load of 1 ton per foot, extending over a length of 30 feet.

# CHAPTER VII

## *DEFLECTION OF BEAMS*

**91. Stiffness and Strength.**—It is usually necessary that a beam should be *stiff* as well as strong, *i.e.* that it should not, due to loading, deflect much from its original position. The greatest part of the deflection is generally due to bending, which produces curvature related to the intensity of stress in the manner shown in Art. 61. A knowledge of the relation between the loading and deflection of a beam is often the first step in finding the bending moments to which a beam is subjected by given loadings and methods of fixing; also in finding the stresses in other parts of a structure of which the beam forms a part. Structures in which the distribution of force due to given loads cannot be determined by the ordinary methods applicable to rigid bodies, but depends upon the relative stiffness of the various parts are called *Statically Indeterminate.* We now proceed to find the deflection of various parts of beams under a variety of different loadings and supported in various ways. The symbol $y$, a variable, will be used for deflections for different points along the neutral plane, from their original positions. This symbol is not to be confused with the variable $y$ already used for the distances of points in a cross-section from the neutral axis of that section, although it is estimated in the same direction, usually vertically. It will be assumed that all deflections take place within the elastic limit, and are very small compared to the length of the beam.

**92. Deflection in Simple Bending: Uniform Curvature.**—When a beam of constant section is subjected throughout its length to a uniform bending moment M it bends (see Arts. 61 and 63) to a circular arc of radius R, such that—

$$\frac{E}{R} = \frac{M}{I} \quad \text{or} \quad \frac{1}{R} = \frac{M}{EI}$$

where E is the modulus of direct elasticity, and I is the moment of inertia of the area of cross-section about the neutral axis. If a beam AB (Fig. 136) of length $l$, originally straight, bends to a circular arc AP'B, the deflection PP' or $y$, at the middle, can easily be found from the geometry of Fig. 136.

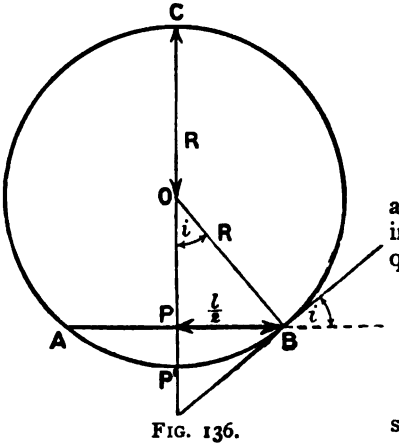


FIG. 136.

For $PP' \cdot PC = PB^2 = \left(\frac{l}{2}\right)^2$

$$PP'(2R - PP') = \frac{l^2}{4}$$

$$2PP' \cdot R - (PP')^2 = \frac{l^2}{4}$$

and for small deflections, neglecting $(PP')^2$, the square of a small quantity—

$$2PP' \cdot R = \frac{l^2}{4}$$

$$y_1 \text{ or } PP' = \frac{l^2}{8R} = \frac{1}{8}\frac{Ml^2}{EI} \quad (1)$$

since $R = \frac{EI}{M}$ (Art. 63)

In this case the whole length is subject to the maximum bending moment M as between the supports in Fig. 83. In other cases where parts of the beam are subject to less than the maximum bending moment, the factor in the above expression for maximum deflection will be less than $\frac{1}{8}$.

If $i$ is the angle of slope which the ends of the beam make with the original position AB, taking $i = \sin i$ for small deflections (in radians)—

$$i = \frac{PB}{OB} = \frac{l}{2R} = \frac{Ml}{2EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Uniform curvature is also treated in Art. 95, section (*d*).

**93. Relations between Curvature, Slope, Deflections, etc.**—Measuring distances $x$, along the (horizontal) span from any convenient origin, $y$ (vertical), deflections perpendicular to $x$, $i$ angles of slope (in radians) of the beam to some fixed direction, usually horizontal, and $s$ lengths of arc of the profile of the neutral surface of the beam when bent (Fig. 137)—

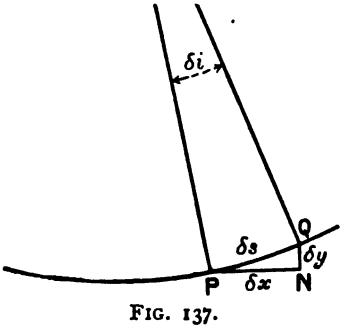


FIG. 137.

$$\frac{dy}{dx} = \tan i = i \text{ (very nearly if } i \text{ is always very small)}$$

The curvature of a line is usually defined as the change of $i$ per unit length of arc, or—

$$\frac{di}{ds}$$

and since (Fig. 137) $\delta i$ is very small, $\delta x$ is sensibly equal to $\delta s$, or $\frac{ds}{dx} = 1$,

hence the curvature $$\frac{1}{R}=\frac{di}{ds}=\frac{di}{dx}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad (1)^1$$

and $$\frac{M}{EI}=\frac{1}{R}=\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for any point $x$ along the beam, for this relation, established for uniform curvature $\frac{1}{R}$, will also hold for every elementary length $ds$ in cases where the curvature $\frac{1}{R}$ is variable.

Hence the slope

$$i \text{ or } \frac{dy}{dx}=\int\frac{d^2y}{dx^2}dx=\int\frac{M}{EI}dx \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the integration being between suitable limits.

And the deflection—

$$y=\int\frac{dy}{dx}dx=\int i\,dx \quad \text{or} \quad \iint\frac{M}{EI}dxdx \quad . \quad . \quad . \quad . \quad (4)$$

between proper limits.

Combining the above relations with those in Art. 59, viz.

$$\frac{dM}{dx}=F \quad \text{and} \quad \frac{dF}{dx}=w=\frac{d^2M}{dx^2}$$

where F is the shearing force and $w$ is the load per unit length of span at a distance $x$ from the origin, we have

$$F=\frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)=EI\frac{d^3y}{dx^3} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

when E and I are constant, and

$$w=EI\frac{d^4y}{dx^4} \quad \text{or} \quad \frac{d^4y}{dx^4}=\frac{w}{EI} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If $w$ is constant or a known integrable function of $x$, general expressions for F, M, $i$, and $y$ at any point of the beam may be found by one, two, three, or four integrations respectively of the equation

$$EI\frac{d^4y}{dx^4}=w.$$

a constant of integration being added at each integration. If sufficient

[1] The approximation may be stated in another way. The curvature—

$$\frac{1}{R}=\frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

and if $\frac{dy}{dx}$ is very small, higher powers than the first may be neglected, and $\frac{1}{R}$ reduces to $\frac{d^2y}{dx^2}$.

conditions of the fixing or supporting of the beam are given, the values of the constants may be determined. If the general expression for the bending moment at any point can be written as an integrable function of $x$, as in Art. 57, general expressions for $i$ and $y$ may be found by integrating twice the equation

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Examples of both the above methods are given in the next article.

*Signs.*—For $y$ positive vertically downwards slopes $i$ or $\frac{dy}{dx}$ will be positive downwards in the direction of $x$ positive (generally to the right); and convexity upwards corresponds to increase of $\frac{dy}{dx}$ with increase of $x$, *i.e.* to positive values of $\frac{d^2y}{dx^2}$. In Art. 59 the sign of the bending moment was so chosen that a clockwise moment of the external forces to the right was positive. Hence, if the clockwise moment of the external forces to the right of a section is written for M in equation (2) (whether positive or negative) positive curvature, *i.e.* $+\frac{d^2y}{dx^2}$, must be written on the other side of the equation. The same, of course, applies for the contra-clockwise moments to the left of a section. If the moments are estimated in the opposite senses to those stated $-\frac{d^2y}{dx^2}$ must be used in equation (2). A violation of the rule of signs will lead to an error in the signs of $i$ and $y$ resulting from integrations of (2). It may be noted that a positive clockwise moment of external forces to the right of a section gives a positive value to $\frac{d^2y}{dx^2}$, *i.e.* the beam will be convex upward at that section.

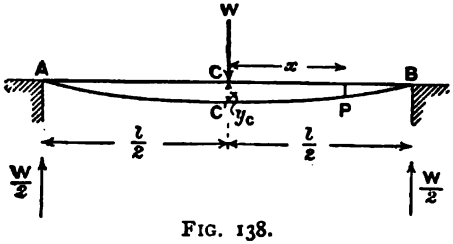



FIG. 138.

**94. Uniform Beam simply supported at its Ends with Simple Loads.**—The two following examples are worked out in considerable detail to illustrate the method of finding the constants of integration.

(*a*) Let there be a central load W (Fig. 138), and take C as origin. Then at P, distant $x$ horizontally along the half span CB from the origin C

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{1}{EI}\cdot\frac{W}{2}\left(\frac{l}{2} - x\right) \text{ (see Fig. 79)}$$

and integrating this—

$$i \text{ or } \frac{dy}{dx} = \int\frac{d^2y}{dx^2}dx = -\frac{W}{2EI}\left(\frac{l}{2}x - \frac{x^2}{2}\right) + A$$

where A is a constant.

Since $i = 0$ when $x = 0$, substituting these values, $0 = 0 + A$, therefore $A = 0$; and with this choice of origin (C) A disappears, and

$$i \text{ or } \frac{dy}{dx} = -\frac{W}{2EI}\left(\frac{l}{2}x - \frac{x^2}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating again—

$$y = \int \frac{dy}{dx}dx = -\frac{W}{2EI}\left(\frac{l}{4}x^2 - \frac{x^3}{6}\right) + B \quad . \quad . \quad . \quad . \quad (2)$$

the constant of integration, B, being $+\frac{1}{48}\frac{Wl^2}{EI}$, since $y = 0$ when $x = \frac{l}{2}$. The equations (1) and (2) give the slope and deflection anywhere on the half span, *e.g.* at the end, or $x = \frac{l}{2}$,

$$i_B = -\frac{W}{2EI}\left(\frac{l^2}{4} - \frac{l^2}{8}\right) = -\frac{Wl^2}{16EI} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and at the centre, $y_C = \frac{Wl^3}{48EI}$ . . . . . (4)

The slopes and deflections on the other half span are evidently of the same magnitude at the same distances from C.

(*b*) Let there be a uniformly spread load $w$ per unit length. Take the origin at A (Fig. 139), and use the equation $EI\frac{d^4y}{dx^4} = w$. The four integrations require four known conditions to evaluate the four added constants. The four conditions in this case are—

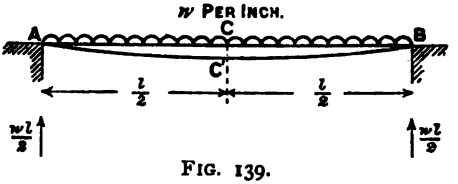


FIG. 139.

$$EI\frac{d^2y}{dx^2} = M = 0 \text{ for } x = 0$$

$$\frac{d^2y}{dx^2} = 0 \text{ for } x = l$$

$$y = 0 \text{ for } x = 0, \text{ and } y = 0 \text{ for } x = l$$

$$EI\frac{d^4y}{dx^4} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Integrating, $EI\frac{d^3y}{dx^3} = wx + A$ . . . . . (6)

Integrating again, $EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 + Ax + 0$

the added constant being zero, since both sides must reduce to 0 for $x = 0$.

Putting $\frac{d^2y}{dx^2} = 0$ when $x = l$

$$0 = \frac{1}{2}wl^2 + Al$$

hence $A = -\frac{1}{2}wl$

(a result which might also be obtained from (6), since the shearing force is zero for $x = \frac{l}{2}$).

Then substituting for A—

$$\text{EI} \cdot \frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 - \tfrac{1}{2}wlx \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Integrating this—

$$i = \frac{dy}{dx} = \frac{1}{\text{EI}}(\tfrac{1}{6}wx^3 - \tfrac{1}{4}wlx^2 + \text{B}) \quad . \quad . \quad . \quad (8)$$

Integrating again—

$$y = \frac{1}{\text{EI}}(\tfrac{1}{24}wx^4 - \tfrac{1}{12}wlx^3 + \text{B}x + \text{o})$$

the constant being zero, since $y = \text{o}$ for $x = \text{o}$.

Putting $y = \text{o}$ for $x = l$

$$\text{o} = \tfrac{1}{24}wl^4 - \tfrac{1}{12}wl^4 + \text{B}l$$

therefore $\text{B} = \tfrac{1}{24}wl^3$

which might also be found from (8), since by symmetry $i = \text{o}$ for $x = \frac{l}{2}$,

$$\left.\begin{aligned} &\text{and} && y = \frac{1}{\text{EI}}(\tfrac{1}{24}wx^4 - \tfrac{1}{12}wlx^3 + \tfrac{1}{24}wl^3x) \\ &\text{or,} && y = \frac{wx}{24\text{EI}}(x^3 - 2lx^2 + l^3) \\ &\text{or,} && y = \frac{wx(l-x)}{24\text{EI}}(l^2 + lx - x^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

(6), (7), (8), and (9) give F, M, $i$, and $y$ respectively for any point distant $x$ along the beam from the end A. E.g. $i$ is a maximum when $\frac{di}{dx} = \text{o}$ or $\text{M} = \text{o}$, *i.e.* at the ends; thus, writing $x = \text{o}$ in (8)

$$i_{\text{A}} = \frac{\text{B}}{\text{EI}} = \frac{wl^3}{24\text{EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$y$ is a maximum when $\frac{dy}{dx}$ or $i = \text{o}$, *i.e.* when $x = \frac{l}{2}$,

and then $$y_{\text{C}} = \frac{wl^4}{24\text{EI}}(\tfrac{1}{16} - \tfrac{1}{4} + \tfrac{1}{2}) = \tfrac{5}{384}\frac{wl^4}{\text{EI}} \quad . \quad . \quad . \quad . \quad (11)$$

or, if the whole load $wl = \text{W}$—

$$y_{\text{C}} = \tfrac{5}{384} \cdot \frac{\text{W}l^3}{\text{EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The signs here all agree with and illustrate the convention given at the end of Arts. 59 and 93.

*Overhanging Ends.*—For points between two supports a distance $l$ apart the work would be just as before, except that $EI\frac{d^2y}{dx^2}$ at each support would be equal to the bending moment due to the overhanging end instead of zero.

*Propped Beam.*—If this beam were propped by a central support to the same level as the ends, the central deflection becomes zero, or, in other words, the upward deflection caused by the reaction of the prop (and proportional to it) is equal to the downward deflection caused by the load at the middle of the span.

Let P be the upward reaction of the prop; then from (4) and (11)—

$$\frac{Pl^3}{48EI} = \frac{5}{384}\frac{wl^4}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and $P = \frac{5}{8}wl$, *i.e.* the central prop carries $\frac{5}{8}$ of the whole load, while the end supports each carry $\frac{3}{16}$ of the load.

*Sinking of Prop.*—If the prop is not level with the end supports, but removes $\frac{1}{n}$ of the deflection due to the downward load, the reaction of the prop will be $\frac{1}{n}$ of the above amount.

*Elastic Prop.*—If the central prop and end supports were originally at the same level, but were elastic and such that the pressure required to depress each unit distance is $e$, the compression of the prop is $\frac{P}{e}$, and of each end support $\frac{wl - P}{2e}$. Then equating the difference of levels to the downward deflection due to the load, minus the upward deflection due to P—

$$\frac{P}{e} - \frac{wl - P}{2e} = \frac{5}{384}\frac{wl^4}{EI} - \frac{Pl^3}{48EI}$$

$$P\left(\frac{3}{2e} + \frac{l^3}{48EI}\right) = wl\left(\frac{5}{384}\frac{l^3}{EI} + \frac{1}{2e}\right)$$

$$P = wl\frac{\frac{5}{8} + \frac{24EI}{el^3}}{1 + \frac{72EI}{el^3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which evidently reduces to the previous result for perfectly rigid supports for which $e$ is infinite, and approaches $\frac{1}{3}wl$ for very elastic supports. If the elasticities of the end supports and central prop are different, the modification in the above would be simple.

**Relation between Bending Stress and Deflection.**—For a beam simply supported at each end and carrying a uniformly distributed load, if $\delta$ = central deflection, $f$ = maximum bending stress in tons per square inch, and $d$ = depth of symmetrical section in inches, from (12),

$$\delta = \frac{5}{384}\frac{Wl^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

And from Art. 63

$$\tfrac{1}{8}Wl = f \cdot \frac{2I}{d} \text{ or } \frac{Wl}{I} = \frac{16f}{d}$$

hence substituting in (15)

$$l^2 = \tfrac{24}{5} \cdot \frac{E}{f} \cdot d \cdot \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The deflection $\delta$ for steel beams is commonly limited to $\frac{1}{400}$ of the span $l$ then taking E = 12,500 tons per square inch (16) becomes

$$l = 150\frac{d}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which gives the limit of span for uniform loading: if $f = 7{\cdot}5$ tons per square inch, $\frac{l}{d} = 20$. Any degree of concentration of the load with the same limitations of stress and deflection will allow a greater ratio of span to depth, *e.g.* for a central load the equation corresponding to (17) would be $l = 187{\cdot}5\,\frac{d}{f}$.

EXAMPLE 1.—A beam of 10 feet span is supported at each end and carries a distributed load which varies uniformly from nothing at one support to 4 tons per foot run at the other. The moment of inertia of the cross-section being 375 (inches)$^4$, and E 13,000 tons per square inch, find the slopes at each end and the magnitude and position of the maximum deflection.

The conditions of the ends are as before. Take the origin at the light end; then at a distance $x$ inches along the span the load per inch run is—

$$\frac{x}{120} \times \tfrac{4}{12} = \frac{x}{360} \text{ tons}$$

$$\frac{d^4y}{dx^4} = \frac{1}{360EI} \cdot x$$

$$\frac{d^3y}{dx^3} = \frac{1}{360EI}\left(\frac{x^2}{2} + A\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{360EI}\left(\frac{x^3}{2} + Ax + o\right)$$

$$\frac{d^2y}{dx^2} = o \text{ for } x = l; \text{ hence } A = -\frac{l^2}{6} \text{ and}$$

$$\frac{d^2y}{dx^2} = \frac{1}{360EI}\left(\frac{x^3}{6} - \frac{l^2x}{6}\right)$$

$$\frac{dy}{dx} = \frac{1}{360EI}\left(\frac{x^4}{24} - \frac{l^2x^2}{12} + B\right)$$

$$y = \frac{1}{360EI}\left(\frac{x^5}{120} - \frac{l^2x^3}{36} + Bx + o\right)$$

$y = 0$ for $x = l$; hence—

$$B = \frac{l^4}{36} - \frac{l^4}{120} = \frac{7l^4}{360}$$

$$\frac{dy}{dx} = \frac{1}{360EI}\left(\frac{x^4}{24} - \frac{l^2x^2}{12} + \frac{7l^4}{360}\right)$$

and

$$y = \frac{1}{360EI}\left(\frac{x^5}{120} - \frac{l^2x^3}{36} + \frac{7l^4x}{360}\right)$$

At the light end $x = 0$

$$\frac{dy}{dx} = \frac{7 \times 120^4}{360} \times \frac{1}{360 \times 13{,}000 \times 375} \text{ radians} = 0{\cdot}131^\circ$$

At the heavy end $x = 120$ inches, $\frac{dy}{dx} = 0{\cdot}150^\circ$

At the point of maximum deflection $\frac{dy}{dx} = 0$; therefore

$$\frac{x^4}{24} - \frac{l^2x^2}{12} + \tfrac{7}{360}l^4 = 0$$

hence, solving $x = 0{\cdot}52l = 62{\cdot}4$ inches
and substituting this value, $y = 0{\cdot}0925$ inch

EXAMPLE 2.—A wooden plank 12 inches wide, 4 inches deep, and 10 feet long, is suspended from a rigid support by three wires, each of which is $\frac{1}{8}$ of a square inch in section and 15 feet long, one being at each end, and one midway between them. All the wires being just drawn up tight, a uniform load of 400 lbs. per foot run is placed on the plank. Neglecting the weight of the wood, find the tension in the central and end wires, and the greatest intensity of bending stress in the plank, the direct modulus of elasticity (E) for the wires being 20 times that for the wood.

Let $E_s$ be the modulus for the wires, and $E_w$ that for the wood $= \frac{1}{20}E_s$. The force per inch stretch of the wires $(e) = \frac{E_s}{8 \times 180}$, the strain being $\frac{1}{180}$. For the wooden beam supported at the centre,

$$I = \tfrac{1}{12} \times 12 \times 64 = 64 \text{ (inches)}^4$$

The load on the central wire may be found from (14) above

$$\frac{24E_wI}{el^3} = \frac{24E_w \times 64 \times 8 \times 180}{E_s \times 120 \times 120 \times 120} = 0{\cdot}064$$

hence, by (14) the total tension in the middle wire is

$$P = 4000 \times \frac{0{\cdot}625 + 0{\cdot}064}{1 + (3 \times 0{\cdot}064)} = 4000 \times 0{\cdot}578 = 2312 \text{ lbs.}$$

In each end wire, total pull $= \frac{4000 - 2312}{2} = 844$ lbs.

The greatest bending moment may occur at the middle support, where the diagram is discontinuous, or as a mathematical maximum between the end and the middle of the beam.

At $x$ inches from one end—

$$M = 844x - \tfrac{1}{2} \cdot \tfrac{400}{12} \cdot x^2$$

$$\frac{dM}{dx} = 844 - \tfrac{100}{3}x$$

which is zero for $x = 25{\cdot}32$ inches.

Substituting this for $x$

$$M = 21{,}370 - 10{,}685 = 10{,}685 \text{ lb.-inches}$$

At the middle of the span

$$M = (844 \times 60) - (2000 \times 30) = -9360 \text{ lb.-inches}$$

this being less than that at $x = 25{\cdot}32$ inches.

The greatest intensity of bending stress is

$$\frac{My_1}{I} = \frac{10{,}685 \times 2}{64} = 334 \text{ lbs. per square inch}$$

**95. Uniform Cantilever simply loaded.**—(*a*) A concentrated load W at the free end. Take the origin O (Fig. 140) at the fixed end. Then for $x = 0$, $\frac{dy}{dx} = 0$, and $y = 0$.

At any point $x$ the bending moment—

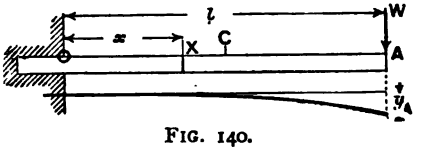



FIG. 140.

$$EI \cdot \frac{d^2y}{dx^2} = W(l - x)$$

$$EI\frac{dy}{dx} = W(lx - \tfrac{1}{2}x^2) + 0$$

$$EI \cdot y = W(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3) + 0$$

At the end A—

$$\left(\frac{dy}{dx}\right)_A \text{ or } i_A = \frac{W}{EI}(l^2 - \tfrac{1}{2}l^2) = \frac{Wl^2}{2EI} \quad \ldots \quad (1)$$

and

$$y_A = \frac{Wl^3}{EI}(\tfrac{1}{2} - \tfrac{1}{6}) = \frac{Wl^3}{3EI} \quad \ldots \quad (2)$$

Note that the upward deflection of the support relative to the centre of the beam in Fig. 138 might be found from the formula (2), viz.

$$\frac{\frac{W}{2} \cdot \left(\frac{l}{2}\right)^3}{3EI} = \frac{Wl^3}{48EI} \text{ (as in (4), Art. 94)}$$

(*b*) A concentrated load distant $nl$ from the fixed end. Origin at O (Fig. 141) at the fixed end, all conditions as above.

From O to C

$$EI\frac{d^2y}{dx^2} = W(nl - x)$$

$$EI\frac{dy}{dx} = W(nlx - \tfrac{1}{2}x^2) + o$$

$$EI \,.\, y = W(\tfrac{1}{2}nlx^2 - \tfrac{1}{6}x^3)$$

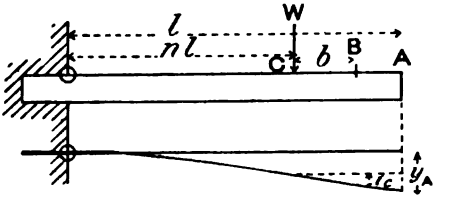



FIG. 141.

At C
$$\left(\frac{dy}{dx}\right)_C \text{ or } i_C = \frac{W(nl)^2}{2EI} \text{ (as before)} \quad . \quad . \quad . \quad . \quad (3)$$

and
$$y_C = \frac{W(nl)^3}{3EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

At any point B beyond C the slope remains the same as at C, and the deflection at B exceeds that at C by

$$b \times (\text{slope from C to B}) = b \,.\, \frac{W(nl)^2}{2EI}$$

In particular—

$$y_A = \frac{W(nl)^3}{3EI} + \frac{Wn^2l^3(1-n)}{2EI} = \frac{Wl^3n^2}{6EI}(3 - n) \quad . \quad . \quad (5)$$

The same formula would be applicable to any number of loads and for a number of different values of W and $n$ may be written

$$y_A = \frac{l^3}{6EI}\left\{3\Sigma(Wn^2) - \Sigma(Wn^3)\right\} \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

while from (3)

$$i_A = \frac{l^2}{2EI}\,\Sigma(Wn^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

The equation of upward and downward deflections as used in the previous article may be used to find the load taken by a prop at the free end or elsewhere.

(*c*) Uniformly distributed load $w$ per unit length. Origin O (Fig. 142) at the fixed end. A start may be made from relation (2) or (6) of Art. 93. Selecting the former

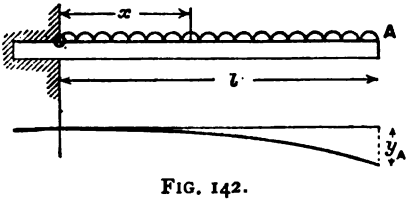



FIG. 142.

$$M = EI\frac{d^2y}{dx^2} = \frac{w}{2}(l - x)^2 = \frac{w}{2}(l^2 - 2lx + x^2)$$

$$EI\frac{dy}{dx} = \frac{w}{2}(l^2x - lx^2 + \tfrac{1}{3}x^3) + o$$

$$EI \,.\, y = \frac{w}{2}(\tfrac{1}{2}l^2x^2 - \tfrac{1}{3}lx^3 + \tfrac{1}{12}x^4) + o$$

For $x = l$—

$$i_A \text{ or } \left(\frac{dy}{dx}\right)_x = \frac{wl^3}{2EI}(1 - 1 + \tfrac{1}{3}) = \tfrac{1}{6}\frac{wl^2}{EI} \text{ or } \tfrac{1}{6}\frac{Wl^2}{EI} \quad . \quad . \quad (6)$$

where $W = wl$.

$$y_A = \frac{wl^4}{2EI}(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{12}) = \tfrac{1}{8}\frac{wl^4}{EI} \text{ or } \tfrac{1}{8}\cdot\frac{Wl^3}{EI} \quad . \quad . \quad . \quad (7)$$

The result (12), Art. 94, might be deduced from the above, for the upward deflection of the support relative to the centre of the beam is—

$$\frac{\frac{wl}{2}\cdot\left(\frac{l}{2}\right)^3}{3EI} - \frac{w\left(\frac{l}{2}\right)^4}{8EI} = \tfrac{5}{384}\cdot\frac{wl^4}{EI}$$

Partial uniformly distributed load.

If the load only extended a distance $nl$ from the fixed end, the deflection at the free end would be, by the method employed in (5) above

$$y = \tfrac{1}{8}\frac{w(nl)^4}{EI} + (1 - n)\tfrac{1}{6}\cdot\frac{w(nl)^4}{EI} = \frac{wl^4}{24EI}(7 - 4n) \quad . \quad . \quad (7a)$$

If the load extended from the free end to a distance $nl$ from the fixed end, the deflection of the free end would be found by subtracting (7a) from (7).

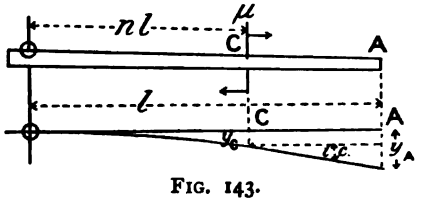



FIG. 143.

(*d*) Bending couple $\mu$ at a distance $nl$ from the fixed end where $n$ is a fraction. Origin at the fixed end O (Fig. 143).

From O to C— $\quad EI\frac{d^2y}{dx^2} = \mu$

$$EI\frac{dy}{dx} = \mu x + o$$

$$EIy = \mu\frac{x^2}{2}$$

At C and beyond—

$$\frac{dy}{dx} = i_C = \frac{\mu nl}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$y_C = \frac{\mu(nl)^2}{2EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$y_A = y_C + l(1 - n)i_C = \frac{\mu nl^2}{EI}\left(1 - \frac{n}{2}\right) \quad . \quad . \quad (10)$$

And if $n = 1$ $\quad i_A = \frac{\mu l}{EI} \quad y_A = \frac{\mu l^2}{2EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$

Which agree with (2) and (1), Art. 92, if we write M for $\mu$ and $\frac{l}{2}$ for $l$. If the couple $\mu$ consists of two opposite vertical forces at a distance $a$ apart each will be equal to $\frac{\mu}{a}$, and if the downward force is distant $nl$ from the fixed end the downward deflection at the free end due to it from (5) is

$$\frac{\mu}{a}\left\{\frac{(nl)^3}{3EI} + l(1-n)\frac{(nl)^2}{2EI}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

while the resultant deflection of the free end due to the two forces is found by subtracting from (12) the corresponding expression with $(nl - a)$ substituted for $nl$ throughout. The result will approach the value (9) as $a$ approaches zero. From (3) the slope between $x = nl$ and $x = l$ is

$$\frac{\mu}{a}\cdot\frac{1}{2EI}\left\{(nl)^2 - (nl-a)^2\right\} = \frac{\mu}{2EI}(2nl - a) \quad . \quad . \quad (13)$$

which approaches (8) as $a$ approaches zero.

*Propped Cantilever.*—From (2) and (7) it is evident, by equating upward and downward deflections, that a prop at the free end, level with the fixed end, when loaded, would carry $\frac{3}{8}$ of the whole distributed load. The bending-moment diagram may be drawn by superposing diagrams such as Fig. 75 and Fig 77, making $W = \frac{3}{8}wl$, and taking the difference of the ordinates as representing the resulting bending moments. The curve of shearing force is a straight line similar to that of Fig. 77, but raised throughout by an amount $\frac{3}{8}wl$ relative to the base-line. Other types of loading of propped cantilevers may be dealt with on similar principles. For example, if P is the pressure on an end prop for a beam loaded as in Fig. 143 from (2) and (10)

$$\frac{Pl^3}{3EI} = \frac{\mu n l^2}{EI}\left(1 - \frac{n}{2}\right)$$

hence

$$P = \frac{3}{l}\mu n\left(1 - \frac{n}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The reader should work out some simple cases fully as an exercise, noting the points of maximum deflection, contraflexure, etc., by integration of the equation $EI\frac{d^4y}{dx^4} = w$, the conditions being $y = 0$ at both ends, slope $= 0$ at the fixed end, and $\frac{d^2y}{dx^2} = 0$ at the free end.

*Sinking Prop.*—If the prop is below the level of the fixed end, the load carried by it would be proportionately reduced. If it is above that level, the load on it would be proportionally increased.

*Elastic Prop.*—If the fixed end is rigid and the support at the free end is elastic, requiring a force $e$ per unit of depression and being before loading at the same level as the fixed end, for the above simple case of

distributed load, equating the depression of the prop to the difference of deflections due to the load and the prop

$$\frac{P}{e} = \frac{1}{8}\frac{wl^4}{EI} - \frac{Pl^3}{3EI}$$

whence $$P = wl\left(\frac{\frac{3}{8}}{1 + \frac{3EI}{el^3}}\right)$$

For other types of loading or positions of prop, similar principles would hold good.

EXAMPLE 1.—A cantilever carries a concentrated load W at $\frac{3}{4}$ of its length from the fixed end, and is propped at the free end to the level of the fixed end. Find what proportion of the load is carried on the prop.

Let W be the load, and P the pressure on the prop. Then

$$\frac{1}{3}\frac{Pl^3}{EI} = \frac{1}{3}\frac{W(\frac{3}{4}l)^3}{EI} + \frac{1}{4}l.\frac{W(\frac{3}{4}l)^2}{2EI}$$

$$\frac{1}{3}P = W(\tfrac{9}{64} + \tfrac{9}{128}) = \tfrac{27}{128}W$$

$$P = \tfrac{81}{128}W$$

EXAMPLE 2.—A cantilever 10 feet long carries a uniformly spread load over 5 feet of its length, running from a point 3 feet from the fixed end to a point 2 feet from the free end, which is propped to the same level as the fixed end. Find what proportion of the load is carried by the prop.

Let $w$ = load per foot run, and P = pressure on the prop. The total load is $\frac{1}{2}wl$. Deflection of the free end if unpropped would be

$$\frac{1}{8}\frac{w(0{\cdot}8l)^4}{EI} + 0{\cdot}2l.\frac{1}{6}\frac{w(0{\cdot}8l)^3}{EI} - \left\{\frac{1}{8}\frac{w(0{\cdot}3l)^4}{EI} + 0{\cdot}7l.\frac{1}{6}\frac{w(0{\cdot}3l)^3}{EI}\right\}$$

$$= \frac{wl^4}{EI}\left\{\frac{0{\cdot}4096}{8} + \frac{0{\cdot}1024}{6} - \frac{0{\cdot}0081}{8} - \frac{0{\cdot}0189}{6}\right\} = 0{\cdot}0641\frac{wl^4}{EI}$$

Therefore $$\frac{1}{3}\frac{Pl^3}{EI} = 0{\cdot}0641\frac{wl^4}{EI}$$

$$P = 0{\cdot}1923wl \text{ or } 0{\cdot}385 \text{ of the total load}$$

Note that this is less than half the load, although the centre of gravity of the load is nearer to the propped end.

EXAMPLE 3.—A vertical stanchion 15 feet long is fixed at the lower end and hinged at the top end so as to form a cantilever propped at the free end. It is acted upon by two equal and opposite horizontal forces which form a couple of 10 tons-feet, the more distant being 10 feet and the nearer one 7·5 feet from the fixed end. Find the reaction at the hinged end and the bending moments at the loads and fixed end (Fig. 144).

Using expression (12) where $\frac{\mu}{a} = \frac{10}{2{\cdot}5} = 4$ tons, or equation (5), the deflection of the hinged end A, if free, would be

$$\frac{4\times10^3}{EI}\left\{\frac{8}{27\times3}+\frac{4}{27\times2}\right\}-\frac{4\times10^3}{EI}\left\{\frac{1}{8\times3}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\right\}=\frac{4\times10^3}{EI}\times\frac{89}{16\times81}$$

And if R = horizontal reaction at the hinge from (2), the opposite deflection counteracting that of the couple is

$$\frac{R\times10^3}{3EI}$$

hence $$R=\frac{12\times89}{16\times81}=0{\cdot}824 \text{ tons.}$$

Bending moment at B (Fig. 144) $= -5\times0{\cdot}824 = -4{\cdot}12$ tons-feet
Bending moment at C $= -7{\cdot}5\times0{\cdot}824+10 = +3{\cdot}82$ tons-feet
Bending moment at O $= -15\times0{\cdot}824+10 = -2{\cdot}36$ tons-feet

The bending moment diagram is shown to the right of Fig. 144, the positive sign corresponding to convexity towards the left. The points of inflexion could easily be calculated.

The ordinate MN exceeds the ordinate HK. The position of application of the couple to produce least bending moment might be found by equating expressions for MN and HK in terms of a variable corresponding to OB or OC.

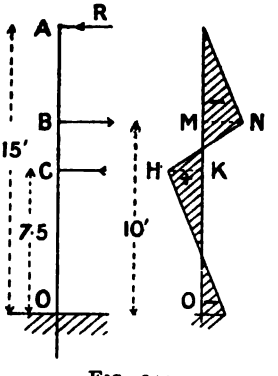



FIG. 144.

EXAMPLE 4.—A bar of steel 2 inches square is bent at right angles 3 feet from one end; the other and longer arm is firmly fixed vertically in the ground, the short (3-foot) arm being horizontal and 10 feet above the ground. A weight of $\frac{1}{4}$ ton is hung from the end of the horizontal arm. Find the horizontal and vertical deflection of the free end. E = 13,000 tons per square inch.

The bending moment throughout the long arm is sensibly the same as that at the bend, viz. $\frac{1}{4}\times36=9$ ton-inches.

It therefore bends to a circular arc, the lower end remaining vertical. A line joining the two ends of the long arm would therefore make with the vertical an angle

$$\frac{Ml}{2EI}\text{ (Art. 92 (2) or (11) Art. 95)}=\frac{9\times120}{2EI}$$

$$=\frac{9\times120\times12}{2\times13{,}000\times16}=\frac{81}{2600}\text{ radians}$$

and the horizontal deflection of the whole of the short arm will be

$$\frac{81}{2600}\times\frac{120}{1}=\frac{243}{65}=3{\cdot}74\text{ inches}$$

The inclination of the upper end of the long arm to the vertical is evidently twice the amount $\frac{81}{2600}$, which is the average inclination. The

downward slope of the short cantilever arm is therefore $\frac{81}{1300}$ at the bend. The total vertical deflection at the free end is

$$36 \times \tfrac{81}{1300} + \frac{\frac{1}{4} \times 36 \times 36 \times 36 \times 12}{3 \times 13{,}000 \times 16} = 2{\cdot}243 + 0{\cdot}224 = 2{\cdot}467 \text{ inches}$$

**96. Simply supported Beam with Non-central Load.**—Let W be a load concentrated at a distance $a$ from one support A (Fig. 145), and $b$ from the other support, B, the span being $a + b = l$. The reaction $R_A$ at A is evidently

$$\frac{b}{a+b} \cdot W$$

and $R_B = \frac{b}{a+b} \cdot W$

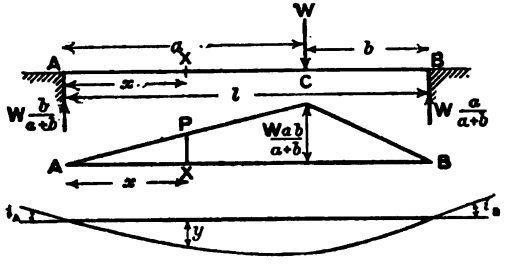



FIG. 145.

Suppose that $a$ is greater than $b$. Taking A as origin, from A to C

$$\frac{d^2y}{dx^2} = -\frac{W}{EI}\frac{b}{a+b} \cdot x \quad \ldots\ldots\ldots (1)$$

$$\frac{dy}{dx} = -\frac{Wb}{EI(a+b)} \cdot \frac{x^2}{2} + A$$

and at C $\quad \frac{dy}{dx}$ or $i_C = -\frac{Wb}{EI(a+b)} \cdot \frac{a^2}{2} + A$

or, $\quad A = i_C + \frac{Wba^2}{2EI(a+b)}$

and $\quad \frac{dy}{dx} = -\frac{Wb}{EI(a+b)} \cdot \frac{x^2 - a^2}{2} + i_C \quad \ldots (2)$

$$y = -\frac{Wb}{EI(a+b)}\left(\frac{x^3}{6} - \frac{a^2x}{2}\right) + i_C \cdot x + 0 \quad \ldots (3)$$

and at C, where $x = a$

$$y_C = \frac{Wb}{EI(a+b)} \cdot \frac{a^3}{3} + a \cdot i_C \quad \ldots\ldots (4)$$

Similarly, taking B as origin and measuring $x$ as positive towards C, making $i_C$ of opposite sign

$$y_C = \frac{Wa}{EI(a+b)} \cdot \frac{b^3}{3} - b \cdot i_C \quad \ldots\ldots (5)$$

Subtracting (5) from (4)

$$(a+b) \cdot i_C = -\frac{Wab(a^2 - b^2)}{3EI(a+b)}$$

and $\quad i_C = -\frac{Wab(a-b)}{3E(a+b)} \quad \ldots\ldots (6)$

Substituting this value of $i_c$ in (3)

$$y = -\frac{Wb}{EI(a+b)}\left(\frac{x^3}{6} - \frac{a^2x}{6} - \frac{abx}{3}\right) = \frac{Wbx}{EI(a+b)} \cdot \frac{a^2 + 2ab - x^2}{6} \cdot \quad (7)$$

and at C, when $x = a$ under load

$$y_c = \frac{Wa^2b^2}{3EI(a+b)} \quad \ldots \ldots \quad (8)$$

The maximum deflection occurs where $\frac{dy}{dx} = 0$. Substituting for $i_c$ in (2), or differentiating (7)

$$\frac{dy}{dx} = -\frac{Wb}{EI(a+b)}\left(\frac{x^2}{2} - \frac{a^2}{6} - \frac{ab}{3}\right) \ldots \ldots \quad (2a)$$

and when $\frac{dy}{dx} = 0$, $x^2 = \frac{1}{3}a(a + 2b)$

$$x = \frac{1}{\sqrt{3}} \cdot \sqrt{a^2 + 2ab} \text{ or } \frac{1}{\sqrt{3}} \cdot \sqrt{l^2 - b^2}$$

which gives the value of $x$ where the deflection $y$ is a maximum.

Note that this value of $x$ is always less than $a$ if $b$ is less than $a$. A corresponding expression for the other part of the span would not hold, for $x$ is then greater than $b$; $\frac{dy}{dx}$ is not zero within the smaller segment $b$.

Also note that as $b$ varies from $\frac{1}{2}l$ to zero, the position ($x$) of maximum deflection only varies from $\frac{1}{2}l$ to $\frac{1}{\sqrt{3}}l$, or $0{\cdot}577l$, so that the point of maximum deflection is always within 8 per cent. of the length of the beam from the middle. Substituting the above value of $x$ in (7)

$$y_{max.} = \frac{Wb(a^2 + 2ab)^{\frac{3}{2}}}{9\sqrt{3}\,EI(a+b)} \text{ or } \frac{Wb(l^2 - b^2)^{\frac{3}{2}}}{9\sqrt{3}\,E \cdot I \cdot l} \quad \ldots \quad (9)$$

Within the smaller segment $b$ the deflection at any point distant $(a + b - x)$, or, say $x'$ (less than $b$), from B, the deflection corresponding to (7) will be

$$y = \frac{Wax'}{EI(a+b)} \cdot \frac{b^2 + 2ab - x'^2}{6} \quad \ldots \ldots \quad (10)$$

and corresponding to (2a)

$$\frac{dy}{dx} \text{ or } i = \frac{Wa}{EI(a+b)}\left(\frac{x'^2}{2} - \frac{b^2}{6} - \frac{ab}{3}\right) \ldots \ldots \quad (11)$$

which is never zero when $x'$ is less than $b$.

*Several Loads.*—If there are several concentrated loads on one span the deflection at any selected point, whether directly under a load or not, may be found by adding the deflections due to the several

loads as calculated by (7) or (10) above, using (7) for points in major segments, and (10) for points in minor ones, the origins being chosen for each load so that the selected point is between the origin and the load.

The slope between any two loads might be written down in terms of $x$, the distance from A, by using the sum of such terms as (2a) and (11), writing $(a + b - x)$ instead of $x'$. If this sum vanishes for any value of $x$ lying between the two chosen loads, that value of $x$ gives the position of the maximum deflection. If not, the maximum lies between another pair of loads. The pair between which the maximum deflection lies can usually be determined by inspection, from the fact noted above, that every individual load causes its maximum deflection within a short distance of the mid-span. A simpler method is given in Art. 97.

EXAMPLE.—A beam of 20-feet span is freely supported at the ends, and is propped 9 feet from the left-hand end to the same level as the supports, thus forming two spans of 9 and 11 feet. The beam carries a load of 3 tons 5 feet from the left-hand support, and one of 7 tons 4 feet from the right-hand end. Find the reactions at the prop and at the end supports.

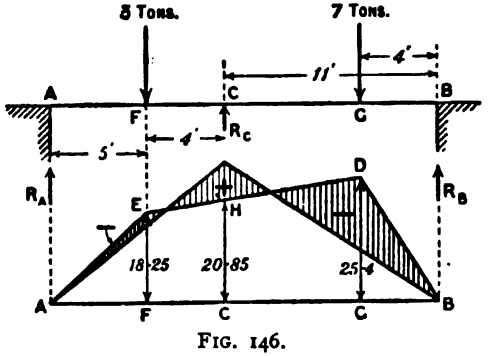


FIG. 146.

If the beam were not propped, the deflection at C (Fig. 146), 9 feet from A, would be, for the 3-ton load, taking $a = 5$, $b = 15$, $W = 3$ and $x' = 11$, in (10) above

$$y_C = \frac{3 \times 5 \times 11}{20EI}\left\{\frac{225 + (10 \times 15) - 121}{6}\right\} = \frac{349{\cdot}25}{EI}$$

and for the 7-ton load, taking $a = 16$, $b = 4$, $W = 7$, $x = 9$, in (7)

$$y_C = \frac{7 \times 4}{20EI}\left\{\frac{729 - (256 \times 9) - (2 \times 16 \times 4 \times 9)}{6}\right\} = \frac{636{\cdot}3}{EI}$$

Adding these, the downward deflection of the beam would be, if it were not propped

$$\frac{985{\cdot}55}{EI}$$

If $R_C$ is the reaction of the prop at C, the upward deflection is, by (8) above

$$y_C = \frac{R_C \times 81 \times 121}{3EI \times 20} = \frac{163{\cdot}35R_C}{EI}$$

Equating this to the above deflection at C

$$R_C = \frac{985{\cdot}5}{163{\cdot}35} = 6{\cdot}031 \text{ tons}$$

The reactions at A and B follow by taking moments about the free ends.

$$R_B = \frac{(3 \times 5) + (7 \times 16) - (6{\cdot}03 \times 9)}{20} = 3{\cdot}636 \text{ tons}$$

$$R_A = 10 - 6{\cdot}031 - 3{\cdot}635 = 0{\cdot}334 \text{ ton}$$

**97. Deflection and Slope from Bending-moment Diagrams.**

*Slopes.*—The change of slope between any two points on a beam may be found from the relation shown in (3), Art. 93

$$i \text{ or } \frac{dy}{dx} = \int \frac{d^2y}{dx^2} \cdot dx = \int \frac{M}{EI}\, dx = \frac{1}{EI} \int M dx$$

if E and I are constant.

Between two points P and Q (Fig. 147, in which the slopes and deflections are greatly exaggerated), on a beam of constant cross-section, the change of inclination $i_2 - i_1$, which is the angle between the two tangents at P and Q, may be represented by

$$i_2 - i_1 = \frac{1}{EI} \int_{x_2}^{x_1} M dx \quad . \quad (1)$$

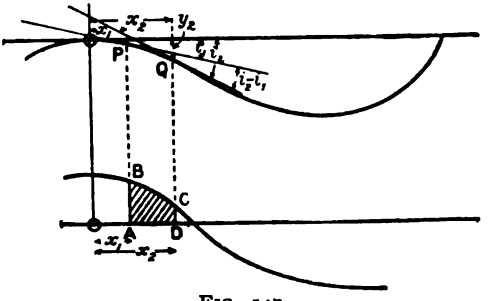



FIG. 147.

The quantity $\int_{x_1}^{x_2} M dx$ represents the area ABCD of the bending-moment diagram between P and Q. If the lower limit $x_1$ be zero, from O, where the beam is horizontal, to Q, where the slope is $i_2$, the actual slope is equal to the change of inclination, viz.

$$i_2 = \frac{1}{EI} \int_0^{x_2} M dx \text{ (which is proportional to OECD)} \quad . \quad . \quad (2)$$

Thus the change of slope between two points on a beam is proportional to the area of the bending-moment diagram between them, and *from a point of zero slope* to any other point the area under the bending-moment curve is proportional to the actual slope at the second point. Changes of sign in the bending-moment diagram must be taken into account if the curve passes through zero. One algebraic sign, generally positive, is attached to bending, which produces convexity upwards, and the opposite sign to a bending moment, producing convexity downwards (see Art. 93), but the choice is of little importance in the present chapter.

*Scales.*—If in the bending-moment diagram 1 inch horizontally represents $q$ inches, and 1 inch vertically represents $s$ lb.-inches, 1 square inch of bending-moment diagram area represents $q \cdot s$ lb.-(inches)$^2$, and

also represents $\frac{q \cdot s}{EI}$ radians slope if E is in pounds per square inch and I in (inches)$^4$ units.

*Deflection.*—From the equation—

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \text{ ((2), Art. 93)}$$

$$x\frac{d^2y}{dx^2} = \frac{Mx}{EI}$$

Integrating between $x = x_2$ and $x = x_1$, using the method of integration by parts for the left-hand side—

$$\left(x\frac{dy}{dx} - y\right)_{x=x_1}^{x=x_2} = \int_{x_1}^{x_2} \frac{Mx}{EI}\,dx = \frac{1}{EI}\int_{x_1}^{x_2} Mx\,dx \text{ (if EI is constant)} \quad (3)$$

or,
$$(x_2i_2 - y_2) - (x_1i_1 - y_1) = \frac{1}{EI}\int_{x_1}^{x_2} Mx\,dx \quad (4)$$

If the limits of integration between which the deflection is required are such that $x\frac{dy}{dx}$ is zero (from either of the factors $x$ or $\frac{dy}{dx}$ being zero) at each limit, the expression—

$$\left(x\frac{dy}{dx} - y\right)_{x=x_1}^{x=x_2} \text{becomes} - (y_2 - y_1) \quad (5)$$

and $\frac{1}{EI}\int_{x_1}^{x_2} Mx\,dx$ gives the change in level of the beam between the two points.

The quantity—

$$\int_{x_1}^{x_2} Mx\,dx$$

represents the moment *about the origin* of the area of the bending-moment diagram between $x_2$ and $x_1$. If A is this area and $\bar{x}$ is the distance of its centre of gravity or centroid from the origin, $\int_{x_1}^{x_2} Mx\,dx$ may be represented by A . .

This quantity only represents the change in level when $x \cdot \frac{dy}{dx}$ vanishes at *both limits*. The product $x \cdot \frac{dy}{dx}$ or $x \cdot i_x$ denotes the vertical projection of the tangent at $x$, the horizontal projection of which is $x$. If the lower limit is zero, and $y$ is zero at the origin, the quantity—

$$\left(x \cdot \frac{dy}{dx} - y\right)_0^x$$

represents the difference between the vertical projection of the tangent at $x$, over a horizontal length $x$, and the deflection at $x$; in other words,

the vertical deflection of the beam from its tangent. Hence, in this case, the deflection at a distance $x$ from the origin is equal to the difference between $x \,.\, i_x$ and $\frac{1}{EI} \times$ (moment of bending-moment diagram area), or—

$$x \,.\, i_x - \frac{1}{EI}\int_0^x Mx\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

where $\int_0^x Mx\,dx$ may be either positive or negative.

*Scales.*—If in the bending-moment diagram 1 inch (horizontally) represents $q$ inches, and 1 inch (vertically) represents $s$ lb.-inches, A being measured in square inches and $\bar{x}$ in inches, the product $A \,.\, \bar{x}$ represents the deflection on a scale $\frac{q^2 s}{EI}$ inches to 1 inch.

*Applications: (a) Cantilever with Load* W *at the Free End* (see Fig. 75).—If the origin be taken at the free end before or after deflection—

$$\text{for } x = 0 \quad x\frac{dy}{dx} = 0$$

and at the fixed end $x = l$ and $\frac{dy}{dx} = 0$, hence—

$$\left(x \cdot \frac{dy}{dx} - y\right)_0^l$$

gives the difference of level of the two ends $y_0 - y_l$, which is equal to—

$$\frac{A \,.\, \bar{x}}{EI}$$

where $A = \frac{1}{2} \,.\, Wl \,.\, l$ and $\bar{x} = \frac{2}{3}l$.

So that the deflection is—

$$\tfrac{1}{2}Wl^2 \times \tfrac{2}{3}l \div EI = \frac{Wl^3}{3EI}$$

which agrees with (2), Art. 95.

Similarly, if the load is at a distance $nl$ from the fixed end, $A = \frac{1}{2}W(nl)^2$, $\bar{x} = l - \frac{n}{3}l$, and the deflection of the free end is—

$$\frac{W(nl)^2}{2EI}l\left(1 - \frac{n}{3}\right) = \frac{Wn^2l^3}{6EI}(3 - n)$$

which agrees with (5), Art. 95, and might be applied to the case of any number of isolated loads.

The deflection of a cantilever carrying a uniformly distributed load might similarly be found from the diagram of bending moment (Fig. 77) if the distance of the centroid of the parabolic spandril of Fig. 77 from the free end is known. Otherwise the moment of that area may be found by integration. Taking the origin at A (Fig. 61)—

$$\frac{1}{EI}\int_0^l Mx\,dx = \frac{w}{2EI}\int_0^l x^3\,dx = \tfrac{1}{8}\frac{wl^4}{EI}$$

which agrees with (7), Art. 95.

(*b*) *Irregularly Loaded Cantilever.*—For any irregular loading of a cantilever the same method can be applied after the bending-moment diagram ABFEDA has been drawn (Fig. 148). The deflection of the free end is given by $\frac{A \cdot \bar{x}}{EI}$ as before, the scales being suitably chosen.

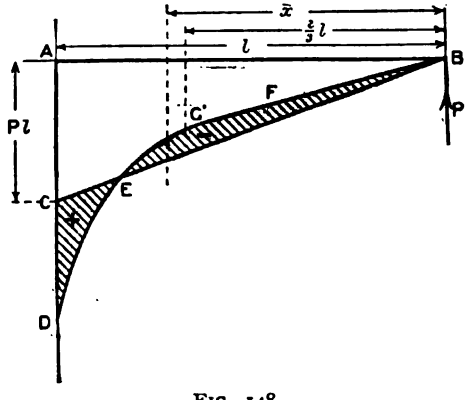



FIG. 148.

The method in such a case is a purely graphical one, consisting in drawing the bending-moment diagram to scale, measuring A and finding $\bar{x}$ by any of the various graphical methods, or finding the product $A\bar{x}$ by a derived area, as in Art. 53; the derived area corresponding to the pole B would represent the area under a curve $M \cdot x$ with origin at B.

If the irregular loading consists of a number of concentrated loads, the whole area A may be looked upon as the sum of the areas of a number of triangles, and the product $A \cdot \bar{x}$ as the sum of the products of the areas of the several triangles and the distances of their centroids from the free end.

*Propped Cantilever. Irregular Load.*—If the cantilever is propped at the end, let P be the upward reaction of the prop at B (Fig. 148). The bending-moment diagram for the irregular loading is ABFED, and that for the prop is the triangle ABC, the ordinates being of opposite sign. The moments of these two areas about B are together zero, for the quantity $\left(x\frac{dy}{dx} - y\right)$ between limits o and $l$ is zero, every term being zero, hence

$$A \cdot \bar{x} = \tfrac{1}{2} \cdot Pl \times l \times \tfrac{2}{3}l$$

$$P = \frac{3A \cdot \bar{x}}{l^3}$$

a general formula applicable to regular or irregular loads, the latter problem being worked graphically.

The resultant bending-moment diagram is shown shaded in Fig. 147, E giving the point of inflection. The parts DCE and EFB are of opposite sign. The reader should apply this method to the various cases given in Art. 95.

The deflection of any point X between A and B may be found by taking the moment about X of so much of this diagram as lies between verticals through X and A, taking account of the signs of the areas. Since the areas reckoned from A represent the slopes, the slope is zero,

and the deflection a maximum at some point to the right of E where the area to the right of E is equal to DCE.

If the cantilever is propped somewhere between A and B the above formula holds good, provided the area A and the length $\bar{x}$ refer to the portion of the diagram ABFED between A and the prop, $\bar{x}$ being measured from the prop, and $l$ refers to the distance of the prop from A.

(*c*) *Beam supported at two Points on the same Level.*—Taking the origin at one end A (Figs. 145 and 149)

$$\left(x\frac{dy}{dx} - y\right)_0^l = l \,.\, i_B = \frac{1}{EI}\int_0^l M x dx = \frac{A\bar{x}}{EI}$$

where A is the area of the bending-moment diagram, and $\bar{x}$ is the distance of its centroid from A, or $A \,.\, \bar{x}$ represents the moment of the area about the origin A, hence

$$i_B = \frac{A \,.\, \bar{x}}{EI \,.\, l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and similarly from the moment about B

$$i_A = -\frac{A(l - \bar{x})}{EI \,.\, l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and is of opposite sign to $i_B$. With the convention of signs given in Art. 93, A is negative for a beam carrying downward loads which produce convexity downwards; hence $i_A$ is positive and $i_B$ is negative.

Thus (in magnitude) the slopes at the supports are proportional to the area of the bending-moment diagram between them, and the ratio of one to the other is inversely proportional to the ratio of the distances of the supports from the centroid of that area—just the same kind of relation, it may be noted, that the reactions at the supports have to the total load.

If the area of the bending-moment diagram from A to a point X, distant $x$ to the right of A, be $A_x$, which is negative for convexity downwards, and the slope at $x$ is

$$i_x = i_A + \frac{1}{EI}\int_0^x M dx \text{ or } i_A + \frac{A_x}{EI} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is zero at the section where maximum deflection occurs, $A_x$ being negative.

Again, since $$\left(x\frac{dy}{dx} - y\right)_0^x = xi_x - y_x = \frac{1}{EI}\int_0^x M x dx$$

$$y_x = x \,.\, i_x - \frac{1}{EI}\int_0^x M x dx \quad . \quad . \quad . \quad . \quad (8a)$$

and substituting for $i_x$ from (8)—

$$y_x = x \,.\, i_A + \frac{x}{EI}\int_0^x M dx - \frac{1}{EI}\int_0^x M x dx$$

$$= xi_A + \frac{xA_x}{EI} - \frac{1}{EI}(\text{moment of } A_x \text{ about A}) \quad . \quad (9)$$

or the deflection at X is—

$$y_x = (x \times \text{slope at A}) + (\text{moment of } A_x \text{ about X})\frac{1}{EI} . \quad (10)$$

which gives the deflection anywhere along the beam, the second term being negative. And from (8*a*) we may write—

$$y_x = (x \times \text{slope at X}) - (\text{moment of } A_x \text{ about A})\frac{1}{EI} . \quad (11)$$

remembering that $A_x$ is a negative quantity.

Probably the form (10) is more convenient than (11), $i_A$ being a constant. As indicated by (8), the slope at X will be negative if X is beyond the point of maximum deflection. Note that the second term in (10) is negative, and represents the vertical displacement of the beam at X from the tangent at A, and the second term in (11) represents the vertical displacement of the beam at A from the tangent at X. In the case of convexity upwards the signs of these second terms would be changed. The reader should illustrate the geometrical meaning of the various terms on sketches of beams under various conditions.

*Overhanging Ends.*—The deflection at any point on an overhanging end, such as in Figs. 83, 84, 92, or 93, may be determined as for a cantilever, provided the deflection due to the slope at the support be added (algebraically). For points between the supports of an overhanging beam the above relations hold, provided that the signs of the areas and moments of areas, etc., be taken into account. For irregular loading these processes may be carried out graphically, and the moments of areas $(A . \bar{x})$ may be found by a "derived area," as in Art. 53 without finding the centres of gravity of the areas.

When the above expressions for slopes and deflections, which are applicable to any kind of loading, are written down symbolically in terms of dimensions of the bending-moment diagram, they give algebraic expressions, such as have already been obtained in other ways for various cases of loading, *e.g.* the deflection and slope anywhere for a beam carrying a single concentrated load may be found in this way as an alternative to the methods in Art. 96.

*Non-central Load.*—From Fig. 145 and (7) above, dividing the moment of the area of the bending-moment diagram about B into two parts—

$$i_A = \frac{1}{EI(a+b)}\left\{\left(\tfrac{1}{2} . b . \frac{Wab}{a+b} . \tfrac{2}{3}b\right) + \tfrac{1}{2}a\frac{Wab}{a+b} . \left(b + \frac{a}{3}\right)\right\}$$
$$= \frac{Wab(a+2b)}{6EI(a+b)} . \quad (12)$$

and from (8) within the range A to C—

$$i_x = i_A - \frac{1}{EI}(\text{area PAX}) = i_A - \frac{1}{EI}\left(\frac{Wbx}{a+b} . \frac{x}{2}\right)$$
$$= \frac{Wb}{EI(a+b)}\left(\frac{a^2+2ab}{6} - \frac{x^2}{2}\right) . \quad (13)$$

which agrees with (2*a*), Art. 96.

For $i_x = 0$ $\qquad x^2 = \frac{1}{3}(a^2 + 2ab)$

Also from (10), within the range A to C—

$$y = i_A \cdot x - \frac{Wbx}{EI(a+b)} \cdot \frac{x}{2} \cdot \frac{x}{3} = \frac{Wbx}{EI(a+b)}\left(\frac{a^2 + 2ab - x^2}{6}\right) \quad (14)$$

which agrees with (7), Art. 96. And when $x = a$—

$$y_C = \frac{Wa^2b^2}{3EI(a+b)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

*Several Loads.*—If there are several vertical loads $W_1$, $W_2$, $W_3$, and $W_4$, at $P_1$, $P_2$, $P_3$, and $P_4$ (Fig. 149), distant $a_1$, $a_2$, $a_3$, and $a_4$ from A, the bending-moment diagram may be drawn as in Art. 58, or calculated as in Art. 56. Let the bending moments at $P_1$, $P_2$, $P_3$, etc., be $M_1$, $M_2$, $M_3$, etc., respectively. Let the total area of the bending-moment diagram be A, and let it be divided by verticals through $P_1$, $P_2$, $P_3$, and $P_4$ (Fig. 149), into five parts, $A_1$, $A_2$, $A_3$, $A_4$, and $A_5$, as shown, so that—

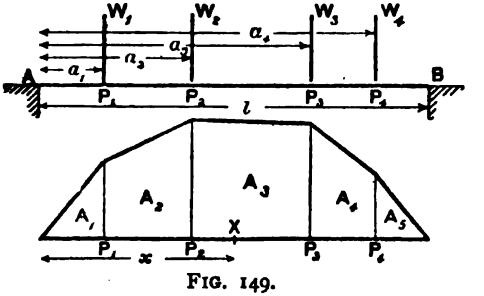


FIG. 149.

$$A_1 = \frac{a_1 M_1}{2} \qquad A_2 = \frac{M_1 + M_2}{2}(a_2 - a_1) \qquad A_3 = \frac{M_2 + M}{2}(a_3 - a_2)$$

and so on, all the areas being negative for downward loads.

Then $$i_A = -\frac{1}{EI} \cdot \frac{A(l - \bar{x})}{l}$$

where $\bar{x}$ is the distance of the centroid of the area A from the origin A, and $l - \bar{x}$ is its distance from B.

The quantity $A(l - \bar{x})$, or the moment of the area A about B, may be found by the sum of the moments of the triangular areas of the bending-moment diagrams, which might be drawn for the several weights separately, *i.e.* the quantity $i_A$ is the sum of four such terms as (12) above.

The slopes at $P_1$, $P_2$, $P_3$, etc., are then

$$i_1 = i_A + \frac{A_1}{EI} \qquad i_2 = i_A + \frac{A_1 + A_2}{EI} \qquad i_3 = i_A + \frac{A_1 + A_2 + A_3}{EI}$$

and so on, the second term in each case being negative.

The segment in which the slope passes through zero is easily found from the slope, or total area from point A to successive loads. If the

zero slope occurs between, say, $P_2$ and $P_3$, the slopes at $P_2$ and $P_3$ are of opposite sign

$$-(A_1 + A_2) \text{ is less than } -\frac{A(l-\bar{x})}{l}$$

$$-(A_1 + A_2 + A_3) \text{ is greater than } -\frac{A(l-\bar{x})}{l}$$

If the zero slope is at X, distant $x$ from A, the bending moment there is $M_2 + \frac{x-a_2}{a_3-a_2}(M_3 - M_2)$, and the slope being zero, the area from point A to the point X of zero slope is equal to $A.\frac{l-\bar{x}}{l}$, or

$$A_1 + A_2 + \left(M_2 + \frac{x-a_2}{a_3-a_2}.\frac{M_3-M_2}{2}\right)(x-a_2) = A.\frac{l-\bar{x}}{l}$$

from which quadratic equation $x$ may be found.

The magnitude of the maximum deflection is then easily found from (11) above, viz.—

$$-\frac{1}{EI}(\text{moment about point A of the bending-moment diagram over AX})$$

an expression which may conveniently be written down after dividing the area over AX into triangles, say, by diagonals from $P_2$. The deflection elsewhere may be found from equation (10). With numerical data this method will appear much shorter than in the above symbolic form. Other purely graphical methods for the same problem are given in the next article.

*Other Cases.*—Beams carrying uniformly distributed loads over part of the span might conveniently be dealt with by these methods, the summation of moments of the bending-moment diagram area being split up into separate parts with proper limits of integration at sudden changes or discontinuities in the rate of loading.

EXAMPLE 1.—The example at the end of Art. 96 may be solved from the bending-moment diagram as follows:—

Let the bending-moment diagram be drawn by the funicular polygon (see Art. 58), or by calculation (see Art. 57). It is shown in Fig. 146, AEDB being the diagram for the two loads on the unsupported span AB. Then from (7)—

$$i_A = -\frac{1}{EI}(\text{moment of area AEDB about B}) \div AB$$

Divide the negative area AEDB into four triangles by joining DF for convenience in calculating the above moment. Using ton and feet units

$$i_A = \frac{1}{20EI}\left[\left(\frac{25{\cdot}4 \times 4}{2}.\frac{2}{3}.4\right) + \left\{\frac{25{\cdot}4 \times 11}{2} \times \left(4 + \tfrac{11}{3}\right)\right\} + \left\{\left(\frac{18{\cdot}25 \times 11}{2}\right) \times \left(4 + \tfrac{22}{3}\right)\right\} + \left\{\left(\frac{18{\cdot}25 \times 5}{2}\right)\left(15 + \tfrac{5}{3}\right)\right\}\right]$$

$$i_A = \frac{155{\cdot}2}{EI}$$

And from (10), dividing EHCF by a diagonal FH

$$y_0 = \frac{155{\cdot}2}{EI} \times 9 - \frac{1}{EI}\left[\left(\frac{20{\cdot}85 \times 4}{2} \cdot \frac{4}{3}\right) + \left(\frac{18{\cdot}25 \times 4}{2} \cdot \frac{8}{3}\right) + \left\{\left(\frac{18{\cdot}25 \times 5}{2}\right)\left(4 + \frac{5}{3}\right)\right\}\right]$$

$$y_0 = \frac{1397 - 411{\cdot}5}{EI} = \frac{985{\cdot}5}{EI} \text{ (downward)}$$

For an upward load $R_0$ at C, by (15)—

$$y_0 = \frac{R_0 \times 81 \times 121}{3EI \times 20} = \frac{163{\cdot}35 R_0}{EI} \text{ (upward)}$$

Equating this to the downward deflection at C—

$$R_0 = \frac{985{\cdot}5}{163{\cdot}35} = 6{\cdot}03 \text{ tons}$$

$$R_A = \frac{(7 \times 4) + (15 \times 3) - (6{\cdot}03 \times 11)}{20} = 0{\cdot}334 \text{ ton}$$

$$R_B = 10 - 0{\cdot}334 - 6{\cdot}03 = 6{\cdot}636 \text{ tons}$$

The above methods might now be applied to the resultant bending-moment diagram, shown shaded in Fig. 146, to determine the deflection anywhere between A and C, or between C and B, and the position of the maximum deflection, etc.

EXAMPLE 2.—Find the deflection of the free ends of the beam in Fig. 84. From (6) and (7) above, slopes downward towards the right—

$$i_A = -i_B = -\frac{1}{EI} \cdot \frac{l_2}{2} \cdot \frac{1}{l_2} \int_0^{l^2} \left\{\frac{wl_1^2}{2} - \frac{w}{2}(l_2 x - x^2)\right\} dx$$

or,

$$-\frac{1}{2EI}\left(\frac{wl_1^2 l_2}{2} - \frac{wl_2^2}{8} \times \frac{2}{3} \times l_2\right) = -\frac{wl_2}{24EI}(6l_1^2 - l_2^2)$$

which is negative if $l_2$ is less than $l_1\sqrt{6}$.

Downward deflection at the free end is—

$$-i_A l_1 + \frac{wl_1^4}{8EI} = \frac{wl_1}{24EI}(6l_1^2 l_2 - l_2^3 + 3l_1^3)$$

Upward deflection at the centre consists of

(upward deflection due to end loads) – (downward deflection due to load between supports)

which, using (11) for the first term, is

$$\frac{1}{EI}\left(0 + \frac{wl_1^2}{2} \cdot \frac{l_2}{2} \cdot \frac{l_2}{4}\right) - \frac{5}{384}\frac{wl_2^4}{EI} = \frac{wl_2^2}{16EI}\left(l_1^2 - \frac{5}{24}l_2^2\right)$$

which is positive if $l_2$ is less than $\sqrt{4{\cdot}8}l_1$.

EXAMPLE 3.—Find the deflection at B and midway between A and C in Ex. 2 of Art. 58 (see Fig. 92).

Taking the origin at A, $R_A$ being 10 tons, by (6), downwards towards B

$$i_C = \frac{1}{16EI}\int_0^{16}\left(10x + \frac{x^3}{2}\right)x\,dx = \frac{1}{16EI}\left(\tfrac{10}{3}x^3 + \frac{x^4}{8}\right)_0^{16} - \frac{21{,}845{\cdot}3}{16EI}$$

E being in tons per square foot, and I in (feet)$^4$—

$$\text{Deflection at B} = \left(8 \times \frac{21{,}845}{16EI}\right) + \frac{32 \times 8 \times 8 \times 8}{3EI} + \frac{8 \times 8 \times 8 \times 8}{8EI}$$

$$= \frac{16{,}896}{EI} \text{ feet}$$

(If E and I are in inch units, deflection at B $= 1728 \times \frac{16{,}896}{EI}$ inches.)

Taking an origin midway between A and C and $x$ positive towards C

$$M = 10(8 + x) + \tfrac{1}{2}(8 + x)^2 = \frac{x^2}{2} + 18x + 112 \text{ tons-feet}$$

and using (*4a*) over the range from the origin to C, the deflection upward at the origin is

$$8 \times \frac{21{,}845{\cdot}3}{16EI} - \frac{1}{EI}\int_0^8\left(\frac{x^3}{2} + 18x^2 + 112x\right)dx$$

$$= \frac{1}{EI}(10{,}922 - 7168) = \frac{3754}{EI} \text{ feet}$$

or, $1728 \times \frac{3754}{EI}$ inches if E is in tons per square inch and I in (inches)$^4$.

**98. Other Graphical Methods.**

*First Method.*—The five equations of Art. 93 immediately suggest a possible graphical method of finding deflections, slopes, etc., from the curve showing the distribution of load on the beam. If the five quantities $w$, F, M, $i$, and $y$ (see Art. 93) be plotted successively on the length of the beam as a base-line, each curve will represent the integral of the one preceding it, *i.e.* the difference between any two ordinates of any curve will be proportional to the area included between the two corresponding ordinates of the preceding curve. Hence, if the first be given, the others can be deduced by measuring areas, *i.e.* by graphical integration. Five such curves for a beam simply supported at each end are shown in Fig. 149. At the ends the shearing forces and slopes are not zero, but the methods of finding their values have already been explained, and are shown in Fig. 150, G and G′ being the centroids of the loading- and bending-moment diagrams respectively. The reader should study the exact analogies between the various curves. In carrying into practice this graphical method the various scales are of primary importance; the calculation of these is indicated below.

In the case of a cantilever, the F and M curves corresponding to (*b*) and (*c*), Fig. 150, must start from zero at the free end (unless there is a concentrated end load), and the *i* and *y* curves corresponding to (*d*) and (*e*), Fig. 150, must start from zero at the fixed end.

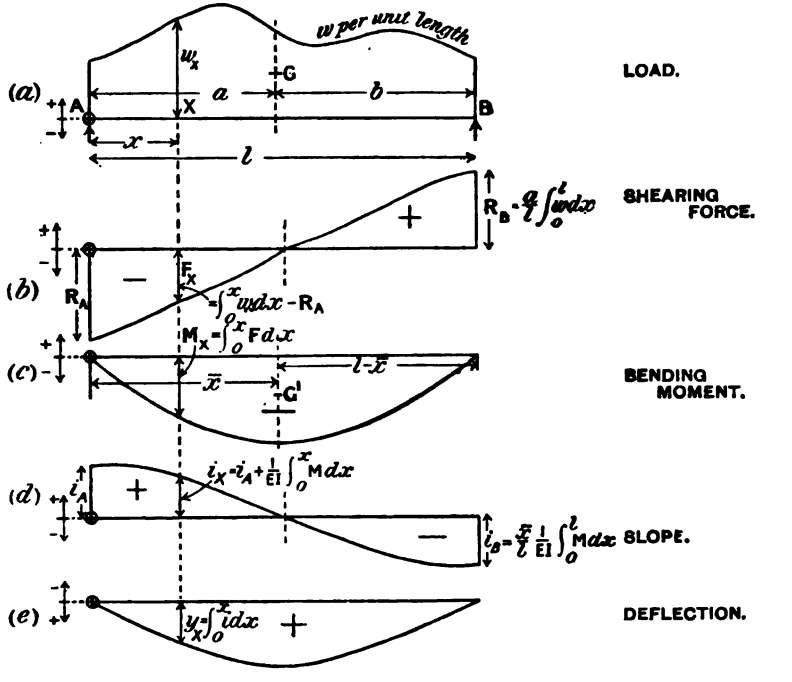



FIG. 150.

*Scales for Fig.* 150.—Linear scale along the span, $q$ inches to 1 inch; E in pounds per square inch; I in (inches)$^4$.

(*a*) Ordinates, $p$ lbs. per inch run = 1 inch.
Therefore 1 square inch area represents $p \cdot q$ lb. load.

(*b*) Ordinates, $n$ square inches from (*a*) = 1 inch = $n \cdot p \cdot q$ lbs.
Areas 1 square inch represent $n \cdot p \cdot q^2$ lb.-inches.

(*c*) Ordinates, $m$ square inches from (*b*) = 1 inch = $mnpq^2$ lb.-inches.
Areas 1 square inch represent $mnpq^3$ lb.-(inches)$^2$

(*d*) Ordinates, $n'$ square inches from (*c*) = 1 inch = $\frac{n'mnpq^3}{EI}$ radians.
Areas 1 square inch represent $\frac{n'mnpq^4}{EI}$ inches.

(*e*) Ordinates, $m'$ square inches from (*d*) = 1 inch = $\frac{m'n'mnpq^4}{EI}$ inches.

If instead of $p$ lbs. per inch run to 1 inch the force scale is $p$ lbs. to 1 inch, the deflection scale would be $\frac{m'n'mnpq^3}{EI}$ inches to 1 inch.

*Second Method.*—This is probably the best method for irregular types of loading. The equations

$$\frac{d^2y}{dx^2} = \frac{1}{EI} \cdot M \text{ and } \frac{d^2M}{dx^2} = w$$

or the diagrams in Fig. 150 show that the same kind of relation exists between bending moment (M) and deflection ($y$) as between the load per

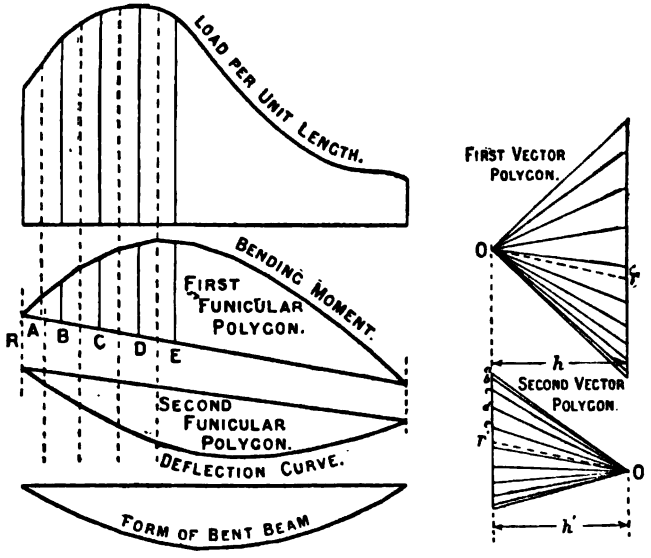



FIG. 151.

unit of span ($w$) and the bending moment. Hence, the curve showing $y$ on the span as a base-line can be derived from the bending-moment diagram in the same way that the bending-moment diagram is derived from the diagram of loading, viz. by the funicular polygon (see Art. 58). If the bending-moment diagram be treated as a diagram of loading, the funicular polygon derived from it will give the polygon, the sides of which the curve of deflection touches internally, and which approximates to the curve of deflection with any desired degree of nearness.

With a distributed load it was necessary (Art. 58) to divide the loading diagram into parts (preferably vertical strips), and take each part of the load as acting separately at the centroid of these parts. Similarly the bending-moment diagram, whether derived from a distributed load or from concentrated loads, must be divided into parts (see Fig. 151), and each part of the area treated as a force at its centre of gravity or centroid. A second pole O' is chosen, and the distances

*ab*, *bc*, *cd*, *de*, etc., set off proportional to the areas of bending-moment diagram, having their centroids on the lines AB, BC, CD, DE, etc. The second funicular polygon, with sides parallel to lines radiating from O', gives approximately the curve of deflection; the true curve is that inscribed within this polygon, for the tangents to the deflection curve at any two cross-sections must intersect vertically below the centroid of that part of the bending-moment diagram lying between those two sections.

To show the form of the beam when deflected the deflection curve must be drawn on a base parallel to the beam, *i.e.* horizontal. This can be done by drawing the second vector polygon again with a pole on the same level as $r'$, and drawing another funicular polygon corresponding to it, or by setting off the ordinates of the second funicular polygon from a horizontal base-line.

This method is applicable to other cases than that of the simply supported beam here illustrated, provided the bending-moment diagram has been determined. When different parts of a beam have opposite curvature, *i.e.* when the curvature changes sign, *e.g.* in a overhanging or in a built-in beam (see Chap. VIII.), the proper sign must be attached to the vertical vectors in the vector polygon. If bending-moment diagram areas of one kind are represented by downward vectors, those of opposite kind (or sign) must be represented by upward vectors.

*Scales.*—If the linear horizontal scale is $q$ inches to 1 inch and the force scale is $p$ lbs. to 1 inch, the horizontal polar distance of the first vector polygon being $h$ inches, the scale of the bending-moment diagram ordinates is $p \cdot q \cdot h$ lb.-inches to 1 inch, as in Art. 58. One square inch area of the bending-moment diagram represents $p \cdot q^2 \cdot h$ lb.-(inches)$^2$; and if the (horizontal) polar distance of the second vector polygon is $h'$ inches, and the vector scale used for it is $m$ square inches of bending-moment diagram to 1 inch, the deflection curve represents $EI \cdot y$ on a scale $m \cdot p \cdot q^3 \cdot h \cdot h'$ lb.-(inches)$^3$ to 1 inch, and therefore represents $y$ on a scale

$$\frac{m \cdot p \cdot q^3 \cdot h \cdot h'}{EI} \text{ inches to 1 inch}$$

E being in pounds per square inch, and I in (inches)$^4$.

If instead of a force $p$ lbs. to 1 inch a scale of $p$ lbs. *per inch run* to 1 inch be used on a diagram of continuous loading, as shown in Fig. 122, the final scale would be $\frac{mpq^4hh'}{EI}$ inches to 1 inch. If the forces are in tons, E should be expressed in tons, and the other modifications in the above are obvious.

**99. Beams of Variable Cross-Section.**—The slopes and deflections so far investigated have been those for beams of constant section, so that the relation (3) of Art. 93—

$$i = \int \frac{M}{EI} dx \text{ has become } \frac{1}{EI} \int M dx$$

If, however, I is not constant, but E is constant, this becomes

$$i = \frac{1}{E}\int \frac{M}{I}\,dx$$

and the equation (1), Art. 97, becomes

$$i_2 - i_1 = \frac{1}{E}\int_{x_1}^{x_2} \left(\frac{M}{I}\right) dx$$

and the equation (3), Art. 81, becomes

$$\left(x\frac{dy}{dx} - y\right)_{x=x_1}^{x=x_2} = \frac{1}{E}\int_{x_1}^{x_2} \frac{Mx}{I}\,dx$$

The methods of finding the slopes and deflections employed in Arts. 94, 95, 97, and 98 may therefore be applied to beams of variable section, provided that the quantity $\frac{M}{I}$ is used instead of M throughout.

Where I and M are both expressed as simple algebraic functions of $x$ (distance along the beam), analytical methods can usually be employed (see Ex. 1 below), but when either or both vary in an irregular manner, the graphical methods should be used. Thus equation (3) of Art. 97 may be written

$$\left(x\frac{dy}{dx} - y\right)_{x_1}^{x_2} = \frac{A\bar{x}}{E}$$

where A or $\int_{x_1}^{x_2} \frac{M}{I}\,dx$ = area under the curve $\frac{M}{I}$ and $\bar{x}$ is the distance of its centroid from the origin. The moment $A.\bar{x}$ may of course be found conveniently by a derived area (see Art. 53). When the quantity I varies suddenly at some section of the beam, but is a simply expressed quantity over two or more ranges, neglecting the effects of a discontinuity in the cross-section, ordinary integration may be employed if the integrals are split up into ranges with suitable limits (see Ex. 2 below). The solution of problems on propped beams of all kinds by equating the upward deflection at the prop caused by the reaction of the prop to the downward deflection of an unpropped beam caused by the load, is still valid, the deflections being calculated for the varying section as above. For example, the equation giving the load carried by a prop at the end of a cantilever, with any loading, as in Fig. 148, may be stated as follows. If M is the bending moment in terms of the distance from the free end B

$$\int_0^l \frac{M}{I}x\,dx = \int_0^l \frac{Px}{I}x\,dx = P\int_0^l \frac{x^2}{I}\,dx$$

and

$$P = \int_0^l \frac{Mx}{I}\,dx \div \int_0^l \frac{x^3}{I}\,dx$$

For a graphical solution, let A be the area enclosed by the curve $\frac{M}{I}$, and $\bar{x}$ the distance of its centroid from B. Assume any load $p$ on the prop, and let $P = \alpha p$. Draw the bending-moment diagram (a straight line) for the end load $p$; divide each ordinate $(p \,.\, x)$ by I, giving a curve $\frac{p \,.\, x}{I}$. Let A′ be the area enclosed by this curve, and $\bar{x}'$ the distance of its centroid from B. Then the above equation in graphical form becomes—

$$A \,.\, \bar{x} = \alpha \,.\, A' \,.\, \bar{x}'$$

$$\alpha = A\bar{x} \div A'\bar{x}' \text{ and } P = \alpha p$$

The moments $A \,.\, \bar{x}$ and $A' \,.\, \bar{x}'$ may be most conveniently found graphically by the derived area method of Art. 53, with B as pole; the bases ($l$) being the same for each diagram, the equation $A \,.\, \bar{x} = \alpha A' \bar{x}'$ becomes—

first derived area of A = $\alpha$(first derived area of A′)

The scales are not important, $\alpha$ being a mere ratio; it is only necessary to set off the ordinate $pl$ in the bending-moment diagram for the assumed reaction $p$, on the same scale as the bending-moment diagram for the loading. A more general application of these methods to other cases is referred to in Arts. 103 and 106.

EXAMPLE 1.—A cantilever of circular section tapers in diameter uniformly with the length from the fixed end to the free end, where the diameter is half that at the fixed end. Find the slope and deflection of the free end due to a weight W hung there.

Let D be the diameter at the fixed end at O, which is taken as origin (Fig. 140). Then diameter at a distance $x$ from O is

$$D\left(1 - \frac{x}{2l}\right) \text{ or } \frac{D}{2l}(2l - x)$$

At O about the neutral axis, $I_0 = \frac{\pi}{64} D^4$ (see Arts. 52 and 66); hence at a distance $x$ from O—

$$I = \frac{\pi}{64} D^4 \left(1 - \frac{x}{2l}\right)^4 \text{ or } \frac{I_0}{16l^4}(2l - x)^4$$

and $M = W(l - x)$ (see Fig. 75).

Then $$\frac{dy}{dx} \text{ or } i = \frac{1}{E}\int_0^x \frac{M}{I} dx = \frac{16Wl^4}{EI_0}\int_0^x \frac{l - x}{(2l - x)^4} dx$$

or in partial fractions—

$$i = \frac{16Wl^4}{EI_0}\int_0^x \left\{\frac{-l}{(2l - x)^4} \times \frac{1}{(2l - x)^3}\right\} dx$$

$$= \frac{16Wl^4}{EI_0}\left\{-\tfrac{1}{3} \cdot \frac{l}{(2l - x)^3} + \tfrac{1}{2}\frac{1}{(2l - x)^2} - \frac{1}{12l^2}\right\}$$

the constant term $-\frac{1}{12l^2}$ being such that $i = 0$ for $x = 0$.

Then, for $x = l$—

$$i_A = \tfrac{4}{3} \cdot \frac{Wl^2}{EI_0}$$

Also

$$y = \int_0^x i\,dx = \frac{16Wl^4}{EI_0}\left\{-\tfrac{1}{6}\frac{l}{(2l-x)^2} + \frac{1}{2(2l-x)} - \frac{x}{12l^2} - \frac{5}{24l}\right\}$$

and for $x = l$ $\qquad y_A = \tfrac{2}{3} \cdot \frac{Wl^3}{EI_0}$

If the deflection only were required, it might be obtained by a single integration by modifying (3), Art. 97, taking the origin at the free end A, Fig. 140—

$$\left(x\frac{dy}{dx} - y\right)_0^l = y_A = \frac{1}{E}\int_0^l \frac{Mx}{I}dx$$

$$y_A = \frac{16Wl^4}{EI_0}\int_0^l \frac{x^2}{(l+x)^4}dx = \frac{16Wl^4}{EI_0}\int_0^l \left\{\frac{l^2}{(l+x)^4} - \frac{2l}{(l+x)^3} + \frac{1}{(l+x)^2}\right\}dx$$

$$= \frac{16Wl^4}{EI_0}\left\{-\tfrac{1}{3}\frac{l^2}{(l+x)^3} + \frac{l}{(l+x)^2} - \frac{1}{(l+x)}\right\}_0^l = \tfrac{2}{3}\frac{Wl^3}{EI_0} \text{ (as before)}$$

EXAMPLE 2.—A cantilever of circular section is of constant diameter from the fixed end to the middle, and of half that diameter from the middle to the free end. Estimate the deflection at the free end due to a weight W there.

If $I_0$ = moment of inertia of the thick end (fixed)

$\frac{1}{16}I_0$ = " " " thin " (free).

As in Art. 95, taking the origin at the fixed end O (Fig. 140), from O to C (the middle point) —

$$\frac{d^2y}{dx^2} = \frac{W}{EI_0}(l - x)$$

$$i \text{ or } \frac{dy}{dx} = \frac{W}{EI_0}(lx - \tfrac{1}{2}x^2) + o$$

and at $x = \frac{l}{2}$ $\qquad i_0 = \tfrac{3}{8}\frac{Wl^2}{EI_0}$

$$y = \int i\,dx = \frac{W}{2EI_0}(lx^2 - \tfrac{1}{3}x^3) + o$$

and at $x = \frac{l}{2}$ $\qquad y_0 = \frac{5Wl^3}{48EI_0}$

From C to A (free end)—

$$i = \frac{16W}{EI_0}(lx - \tfrac{1}{2}x^2) + A$$

at $x = \frac{l}{2}$ $\quad i = \frac{3}{8}\frac{Wl_2}{EI_0}$ (above) $\qquad$ hence $A = -\frac{45}{8}\frac{Wl^2}{EI_0}$

$$y = \int i dx = \frac{W}{EI_0}\{8(lx^2 - \tfrac{1}{3}x^3) - \tfrac{45}{8}l^2x + B\}$$

at $x = \frac{l}{2}$ $\quad y = \frac{5}{48}\frac{Wl^3}{EI_0}$ (above) $\qquad$ hence $B = \frac{5}{4}l^3$

$$y = \frac{W}{EI_0}\{8(lx^2 - \tfrac{1}{3}x^3) - \tfrac{45}{8}l^2x + \tfrac{5}{4}l^3\}$$

and at $x = l$ $\quad y_A = \frac{23}{24}\frac{Wl^3}{EI^0}$

To find the deflection only, the method of Art. 97 might be used taking the origin at A, the free end (Fig. 140). Then $M = Wx$, and splitting the integration into two ranges, over which I is $I_0$ and $\frac{1}{16}I_0$—

$$y = \frac{1}{E}\int_0^l \frac{Mx}{I}dx = \frac{1}{EI_0}\int_{\frac{l}{2}}^{l} Wx^2dx + \frac{16}{EI_0}\int_0^{\frac{l}{2}} Wx^2dx$$

$$= \frac{W}{EI_0}\left[\tfrac{1}{3}\left\{l^3 - \left(\frac{l}{2}\right)^3\right\} + \tfrac{16}{3}\left(\frac{l}{2}\right)^3\right] = \frac{23}{24}\frac{Wl^3}{EI_0}$$

## EXAMPLES VII.

1. A beam of I section 14 inches deep, is simply supported at the ends of a 20-feet span. If the moment of inertia of the area of cross-section is 440 (inches)$^4$, what load may be hung midway between the supports without producing a deflection of more than $\frac{1}{4}$ inch, and what is the intensity of bending stress produced? What total uniformly distributed load would produce the same deflection, and what would then be the maximum intensity of bending stress? (E = 13,000 tons per square inch.)

2. A beam is simply supported at its ends and carries a uniformly distributed load W. At what distance below the level of the end supports must a rigid central prop be placed if it is to carry half the total load? If the prop so placed is elastic and requires a pressure $e$ to depress it unit distance, what load would it carry, the end supports remaining rigid?

3. A beam rests on supports 20 feet apart and carries a distributed load which varies uniformly from 1 ton per foot at one support to 4 tons per foot at the other. Find the position and magnitude of the maximum deflection if the moment of inertia of the area of cross-section is 2654 (inches)$^4$, and E is 13,000 tons per square inch.

4. A cantilever carries a load W at the free end and is supported in the middle to the level of the fixed end. Find the load on the prop and the deflection of the free end.

5. A cantilever carries a load W at half its length from the fixed end. The free end is supported to the level of the fixed end. Find the load carried by this support, the bending moment under the load and at the fixed end, and the position and amount of the maximum deflection.

If this cantilever is of steel, the moment of inertia of cross-section being 20 (inches)$^4$, and the length 30 inches, find what proportion of the load would be carried by an end support consisting of a vertical steel tie-rod

10 feet long and $\frac{1}{2}$ a square inch in section, if the free end is just at the level of the fixed end *before* the load is placed on the beam.

6. A cantilever carries a uniformly spread load W, and is propped to the level of the fixed end at a point $\frac{3}{4}$ of its length from the fixed end. What proportion of the whole load is carried on the prop?

7. A cantilever carries a distributed load which varies uniformly from $w$ per unit length at the fixed end to zero at the free end. Find the deflection at the free end.

8. A girder of I section rests on two supports 16 feet apart and carries a load of 6 tons 5 feet from one support. If the moment of inertia of the area of cross-section is 375 (inches)$^4$, find the deflection under the load and at the middle of the span, and the position and amount of the maximum deflection. (E = 13,000 tons per square inch.)

9. If the beam in the previous problem carries an additional load of 8 tons 8 feet from the first one, and is propped at the centre to the level of the ends, find the load on the prop. By how much will it be lessened if the prop sinks 0·1 inch?

10. A girder of 16 feet span carries loads of 7 and 6 tons 4 and 6 feet respectively from one end. Find the position of the maximum deflection and its amount if the moment of inertia of the cross-section is 345 (inches)$^4$ and E = 13,000 tons per square inch.

11. A steel beam 20 feet long is suspended horizontally from a rigid support by three vertical rods each 10 feet long, one at each end and one midway between the other two. The end rods have a cross-section of 1 square inch and the middle one has a section of 2 square inches, and the moment of inertia of cross-section of the beam is 480 (inches)$^4$. If a uniform load of 1 ton per foot run is placed on the beam, find the pull in each rod.

12. A cantilever carries a uniformly distributed load throughout its length and is propped at the free end. What fraction of the load should the prop carry if the intensity of bending stress in the cantilever is to be the least possible, and what proportion does this intensity of stress bear to that in a beam propped at the free end exactly to the level of the fixed end?

13. At what fraction of its length from the free end should a uniformly loaded cantilever be propped to the level of the fixed end in order that the intensity of bending stress shall be as small as possible, and what proportion does this intensity of stress bear to that in a beam propped at the end to the same level? What proportion of the whole load is carried by the prop?

14. A cast-iron girder is simply supported at its ends and carries a uniformly distributed load. What proportion of the deflection at mid-span may be removed by a central prop without causing tension in the compression flange? What proportion of the deflection at $\frac{1}{4}$ span may be removed by a prop there under a similar restriction?

15. A beam, AB, carries a uniform load of 1 ton per foot run and rests on two supports, C and D, so that AC = 3 feet, CD = 10 feet, and DB = 7 feet. Find the deflections at A, B, and F from the level of the supports, F being midway between C and D. I = 375 (inches)$^4$. E = 13,000 tons per square inch. How far from A is the section at which maximum upward deflection occurs?

16. If the beam in the previous problem carries loads of 5, 3, and 4 tons at A, F, and B respectively, and no other loads, find the deflections at A, F, and B, and the section at which maximum deflection occurs.

17. A cantilever is rectangular in cross-section, being of constant breadth and depth, varying uniformly from $d$ at the wall to $\frac{1}{2}d$ at the free end. Find the deflection of the free end of the cantilever due to a load W placed there, the moment of inertia of section at the fixed end being $I_0$.

18. A vertical steel post is of hollow circular section, the lower half of the length being 4 inches external and $3\frac{1}{2}$ inches internal diameter, and the upper half 3 inches external and $2\frac{1}{2}$ inches internal diameter. The total length of the post is 20 feet, the lower end being firmly fixed. Find the deflection of the top of the post due to a horizontal pull of 125 lbs., 4 feet from the top. ($E = 30{,}000{,}000$ lbs. per square inch.)

19. A beam rests on supports at its ends and carries a load W midway between them. The moment of inertia of its cross-sectional area is $I_0$ at mid-span, and varies uniformly along the beam to $\frac{1}{2}I_0$ at each end. Find an expression for the deflection midway between the supports.

20. Find the deflection midway between the supports of the beam in the previous problem if the load W is uniformly spread over the span.

# CHAPTER VIII

## *ELASTICITY OF BEAMS (continued)*

**100. Built-in or Encastré Beams.**—By this term is understood a beam firmly fixed at each end so that the supports completely constrain the inclination of the beam at the ends, as in the case of the "fixed" end of a cantilever. The two ends are usually at the same level, and the slope of the beam is then usually zero at each end if the constraint is effectual. The effect of this kind of fastening on a beam of uniform section is to make it stronger and stiffer, *i.e.* to reduce the maximum intensity of stress and to reduce the deflection everywhere. When the beam is loaded the bending moment is not zero at the ends as in the case of a simply supported beam, the end fastening imposing such "fixing moments" as make the beam convex *upwards* at the ends, while it is convex downwards about the middle portion, the bending moment passing through zero and changing sign at two points of contraflexure.

If the slope is zero at the ends, the necessary fixing couples at the ends are, for distributed loads, the greatest bending moments anywhere on the beam. Up to a certain degree, relaxation of this clamping, which always takes place in practice when a steel girder is built into masonry, tends to reduce the greatest bending moment by decreasing the fixing moments and increasing the moment of opposite sign about the middle of the span. In a condition between perfect fixture and perfect freedom of the ends, the beam may be subject to smaller bending stresses than in the usual ideal form of a built-in beam with rigidly fixed ends. The conditions of greatest strength will be realized when the two greatest convexities are each equal to the greatest concavity, the greatest bending moments of opposite sign being equal in magnitude.

Simple cases of continuous loading of built-in beams where the rate of loading can be easily expressed algebraically may be solved by integration of the fundamental equation—

$$\mathrm{EI}\frac{d^4y}{dx^4} = w \text{ (Art. 93)}$$

Taking one end of the beam as origin, the conditions will usually be $\frac{dy}{dx} = 0$ for $x = 0$ and for $x = l$, and $y = 0$ for $x = 0$ and for $x = l$.

For example, suppose that the load is uniformly distributed, being $w$ per unit length of span, integrating the above equation—

$$\mathrm{EI}\,.\,\frac{d^3y}{dx^3} = wx + \mathrm{A}$$

$$\mathrm{EI}\,.\,\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 + \mathrm{A}x + \mathrm{B}$$

$$\mathrm{EI}\,.\,\frac{dy}{dx} = \tfrac{1}{6}wx^3 + \tfrac{1}{2}\mathrm{A}x^2 + \mathrm{B}x + \mathrm{o}$$

since $\frac{dy}{dx} = \mathrm{o}$ for $x = \mathrm{o}$, and putting $\frac{dy}{dx} = \mathrm{o}$ for $x = l$—

$$\mathrm{o} = \tfrac{1}{6}wl^2 + \tfrac{1}{2}\mathrm{A}l + \mathrm{B} \quad \text{and} \quad \mathrm{B} = -\tfrac{1}{6}wl^2 - \tfrac{1}{2}\mathrm{A}l$$

$$\mathrm{EI}\,.\,\frac{dy}{dx} = \tfrac{1}{6}wx^3 + \tfrac{1}{2}\mathrm{A}x^2 - \tfrac{1}{6}wl^2x - \tfrac{1}{2}\mathrm{A}lx$$

$$\mathrm{EI}\,.\,y = \tfrac{1}{24}wx^4 + \tfrac{1}{6}\mathrm{A}x^3 - \tfrac{1}{12}wl^2x^2 - \tfrac{1}{4}\mathrm{A}lx^2 + \mathrm{o}$$

since $y = \mathrm{o}$ for $x = \mathrm{o}$, and putting $y = \mathrm{o}$ for $x = l$, and dividing by $l^3$—

$$\mathrm{o} = \tfrac{1}{24}wl - \tfrac{1}{12}wl + \tfrac{1}{6}\mathrm{A} - \tfrac{1}{4}\mathrm{A}$$

hence $$\mathrm{A} = -\tfrac{1}{2}wl \quad \text{and} \quad \mathrm{B} = \tfrac{1}{12}wl^2$$

Substituting these values in the above equations, the values of the shearing force, bending moment, slope, and deflection everywhere are found, viz.—

$$\mathrm{F} = \mathrm{EI}\frac{d^3y}{dx^3} = w(x - \tfrac{1}{2}l)$$

$$\mathrm{M} = \mathrm{EI}\frac{d^2y}{dx^2} = \tfrac{1}{12}w(6x^2 - 6lx + l^2)$$

which reaches a zero value for $x = l(\frac{1}{2} \pm \mathrm{o}{\cdot}289)$, *i.e.* $\mathrm{o}{\cdot}289l$, on either side of mid-span. Also for $x = \mathrm{o}$, or $x = l$, $\mathrm{M} = \frac{1}{12}wl^2$, and for $x = \frac{l}{2}$ $\mathrm{M} = -\frac{1}{24}wl^2$,

$$i = \frac{dy}{dx} = \frac{w}{12\mathrm{EI}}(2x^3 - 3lx^2 + l^2x)$$

which reaches zero for $x = \mathrm{o}$, $x = l$, $x = \frac{l}{2}$

$$y = \frac{w}{24\mathrm{EI}} \cdot x^2(l - x)^2$$

and at the centre, where $x = \frac{l}{2}$, the deflection is—

$$\tfrac{1}{24}\frac{w}{\mathrm{EI}} \cdot \left(\frac{l}{2}\right)^2 \cdot \left(\frac{l}{2}\right)^2 = \tfrac{1}{384}\frac{wl^4}{\mathrm{EI}}$$

or $\frac{1}{5}$ of that for a freely supported beam (see (12), Art. 94).

The bending-moment diagram is shown in Fig. 152; it should be noticed that the bending moment varies in the same way as if the ends were free, varying from $+\frac{1}{12}wl^2$ to $-\frac{1}{24}wl^2$, a change of $\frac{1}{8}wl^2$, as in the freely supported beam (see Fig. 81), but the greatest bending moment to which the beam is subjected is only $\frac{1}{12}wl^2$ instead of $\frac{1}{8}wl^2$, so that with the same cross-section the greatest intensity of direct bending stress will be reduced in the ratio 3 to 2. The greatest bending moment and greatest shearing force ($\frac{1}{2}wl$) here occur at the same section. Evidently, to attain the greatest flexural strength the bending moment at the centre should be equal to that at the ends, each being half of $\frac{1}{8}wl^2$. In this case the equation to the bending-moment curve would be, from (7), Art. 94

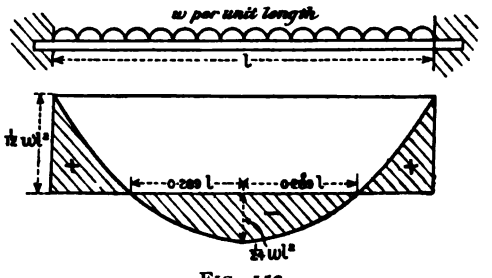


FIG. 152.

$$M = EI\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 - \tfrac{1}{2}wlx + \tfrac{1}{16}wl^2$$

the last or constant term alone differing from the equation used above. Integrating this twice and putting $y = 0$ for $x = 0$ and for $x = l$, or integrating once and putting $\frac{dy}{dx} = 0$ for $x = \frac{l}{2}$ because of the symmetry, the necessary slope at the ends is found to be $\frac{1}{96}\frac{wl^3}{EI}$ or $\frac{1}{4}$ of that in a beam freely supported at its ends (see (10), Art. 94).

Other types of loading where $w$ is a simple function of $x$ may be easily solved by this method.

As another example, suppose that $w = 0$, but one end support sinks a distance $\delta$, both ends remaining fixed horizontally. Taking the origin at the end which does not sink

$$EI\,.\frac{d^4y}{dx^4} = 0$$

$$EI\,.\frac{d^3y}{dx^3} = F$$

where F is the (constant) shearing force throughout the span,

$$EI\,.\frac{d^2y}{dx^2} = Fx + m$$

where $m$ is the bending moment for $x = 0$,

$$EI\,.\frac{dy}{dx} = \tfrac{1}{2}Fx^2 + mx + 0$$

and putting $\frac{dy}{dx} = 0$ for $x = l$,

$$m = -F\frac{l}{2}$$

and $$EI \cdot \frac{dy}{dx} = \tfrac{1}{2}F(x^2 - lx)$$

$$EI \cdot y = \tfrac{1}{2}F\left(\frac{x^3}{3} - \frac{lx^2}{2} + 0\right)$$

and putting $y = \delta$ for $x = l$—

$$EI \cdot \delta = \frac{Fl^3}{2}(\tfrac{1}{3} - \tfrac{1}{2}) = -\tfrac{1}{12}Fl^3$$

$$F = -\frac{12EI\delta}{l^3} \qquad m = \frac{6EI\delta}{l^2}$$

and the bending moment anywhere is—

$$\frac{6EI \cdot \delta}{l^2} - \frac{12EI \cdot x \cdot \delta}{l^3}$$

a straight line reaching the value $-\frac{6EI\delta}{l^2}$ at $x = l$. The equal and opposite vertical reactions at the supports are each of magnitude F.

**101. Effect of Fixed Ends on the Bending-Moment Diagram.**—In a built-in beam the effect of the fixing moments applied at the walls or piers when a load is applied, if acting alone, would be to make the beam convex upwards throughout. Suppose only these "fixing couples" act on the beam, the bending moment due to them at any point of the span may easily be found by looking on the beam as one simply supported, but overhanging the supports at each end and carrying such loads on the overhanging ends as would produce at the supports the actual fixing moments of the built-in beam. If these fixing moments are equal they produce a bending moment of the same magnitude throughout the span (see Fig. 83). If the fixing moments at the two ends are unequal, being say $M_A$ at one end A (Fig. 153) and $M_B$ at the other end B, the bending moment throughout the span varies from $M_A$ to $M_B$ as a straight-line diagram, *i.e.* at a constant rate along the span, as the reader will find by sketching the diagram of bending moments for a beam overhanging its two supports and carrying end

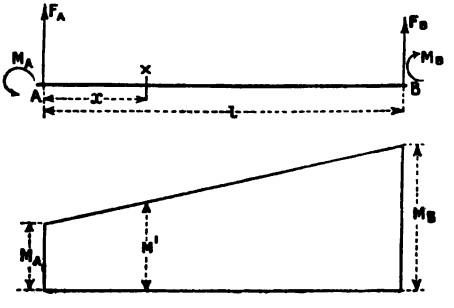



FIG. 153.—Effect of fixing couples.

loads. At a distance $x$ from A the bending moment due to fixing couples will be

$$M' = M_A + \frac{x}{l}(M_B - M_A) \text{ (see Fig. 153)}$$

The actual bending moment at any section of a built-in beam will be the algebraic sum of the bending moment which would be produced by the load on a freely supported beam, and the above quantity M'.

Without any supposition of the case of an overhanging beam, we may put the result as follows for any span of a beam not "free" at the ends.

Let $F_A$ (Fig. 153) be the shearing force just to the right of A, and $F_B$ the shearing force just to the left of B, $M_A$ and $M_B$ being the moments imposed by the constraints at A and B respectively. Let $w$ be the load per unit length of span whether constant or variable. Then, as in Art. 93, with A as origin

$$\frac{d^2M}{dx^2} = w \quad \ldots \quad (1)$$

$$F \text{ or } \frac{dM}{dx} = \int_0^x w dx + F_A \quad \ldots \quad (2)$$

$F_A$ being the value of F for $x = 0$.

Then $$M = \int_0^x \int_0^x w dx dx + F_A . x + M_A \quad \ldots \quad (3)$$

$M_A$ being the value of $EI\frac{d^2y}{dx^2}$ for $x = 0$. Putting $x = l$

$$M_B = \int_0^l \int_0^l w dx dx + F_A l + M_A$$

hence $$F_A = \frac{M_B - M_A}{l} - \frac{1}{l}\int_0^l \int_0^l w dx dx \quad \ldots \quad (4)$$

Note that the term $\frac{1}{l}\int_0^l \int_0^l w dx dx$ is the value of the reaction at A if $M_B = M_A$, or if both are zero as in the freely supported beam.

Substituting the value of $F_A$ in (3)—

$$EI\frac{d^2y}{dx^2} \text{ or } M = \int_0^x \int_0^x w dx dx + (M_B - M_A)\frac{x}{l} + M_A - \frac{x}{l}\int_0^l \int_0^l w dx dx \quad (5)$$

or re-arranging—

$$M = M_A + (M_B - M_A)\frac{x}{l} + \int_0^x \int_0^x w dx dx + \int_0^l \int_0^l w dx dx \quad (6)$$

With free ends $M_A = M_B = 0$, and—

$$M = \int_0^x \int_0^x w dx - \frac{x}{l}\int_0^l \int_0^l w dx dx$$

and if the ends are not free there is the additional bending moment, which may be written

$$M' = M_A + (M_B - M_A)\frac{x}{l} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or,

$$M' = M_A \cdot \frac{l-x}{l} + M_B \cdot \frac{x}{l} \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

a form which will be used in Arts. 103 and 104.

With this notation (5) may be written—

$$M = EI\frac{d^2y}{dx^2} = \mu + M' = \mu + M_A + (M_B - M_A)\frac{x}{l} \quad . \quad (8)$$

where $\mu$ is the bending moment at any section for a freely supported beam similarly loaded, and M′ is the bending moment (Fig. 153) at that section due to the fixing moments $M_A$ and $M_B$ at the ends. Usually $\mu$ and M′ will be of opposite sign; if the magnitudes of $\mu$ and M′ are then plotted on the same side of the same base-line, the actual bending moment M at any section is represented by the ordinates giving the difference between the two curves (see Fig. 154). The conventional algebraic signs used in the above integrations (see Art. 93) make M negative for concavity upwards. The reactions $R_A (= -F_A)$ and $R_B$ may be found from equation (4). If $M_B - M_A$ is positive, the reaction at A is less (in magnitude) than it would be for a simply supported beam by $\frac{1}{l}(M_B - M_A)$ and the reaction at B is greater than for a simply supported beam by the same moment.

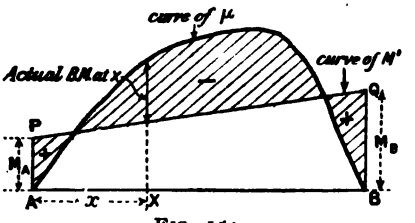


FIG. 154.

**102. Built-in Beam with any Symmetrical Loading.**—For a symmetrically loaded beam of constant cross-section the fixing couples at the supports are evidently equal, and Fig. 83 shows that equal couples at the ends of a span cause a bending moment of the same amount throughout. Or, from (7), Art. 101, if $M_B = M_A$, $M' = M_A = M_B$ at every section. Hence, the resulting ordinates of the bending-moment diagram (see Art. 101) will consist of the difference in ordinates of a rectangle (the trapezoid APQB, Fig. 154, being a rectangle when $M_A = M_B$) and those of the curve of bending moments for the same span and loading with freely supported ends. And since between limits—

$$\frac{dy}{dx} \text{ or } i = \int \frac{M}{EI} . dx \text{ (see (3), Art. 93)}$$

if E and I are constant, the change of slope $\frac{I}{EI}\int_0^l M dx$ between the two ends of the beam is—

$$\frac{1}{EI}\int_0^l(\mu + M')dx$$

with the notation of the previous article, where $l$ is the length of span and the origin is at one support. Now in a built-in beam, if both ends are fixed horizontally, the change of slope is zero, hence

$$\left.\begin{aligned} \int_0^l(\mu + M')dx &= 0 \\ \text{or,} \quad -\int_0^l M'dx &= \int_0^l \mu dx \\ \text{or,} \quad -M' &= \frac{1}{l}\int_0^l \mu dx \end{aligned}\right\} \quad \ldots\ldots (1)$$

This may also be written—

$$A + A' = 0 \quad \ldots\ldots\ldots (2)$$

where A stands for the area of the $\mu$ curve, and A' stands for the area of the trapezoid APQB or M' curve (Fig. 154), which in this special case is a rectangle, AA'BB' (Fig. 155).

$\int_0^l(\mu + M')dx$ represents the area of the bending-moment diagram for the whole length of span, and equation (1) shows that the total area is zero. Hence the rectangle of height $M_A$ (or M'), and the bending-moment diagram $\mu$ for the simply supported beam have the same area $-A$, and the constant value ($M_A$) of M' is $-\frac{1}{l}\int_0^l \mu dx$; the ordinate representing it is $-\frac{A}{l}$, A and $\mu$ being generally negative.

Hence, to find the bending-moment diagram for a symmetrically loaded beam, first draw the bending-moment diagram as if the beam were simply supported (ACDC'B, Fig. 155), and then reduce all ordinates by the amount of the average ordinate, or, in other words, raise the base-line AB by an amount $M_A$, which is represented by the mean ordinate of the diagram ACDC'B, or (area ACDC'B) ÷ (length AB). The points N and N' vertically under C and C' are points of contraflexure or zero bending moment, and the areas AA'C and BB'C' are together equal to the area CDC' and of opposite sign. With downward load, the downward slope from A to N increases and is at N proportional to the area AA'C. From N towards mid-span the slope decreases, becoming zero at mid-span when the net area of the bending-moment diagram from A is zero, *i.e.* as much area is positive as negative.

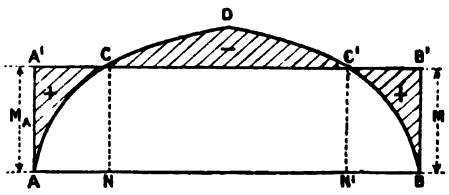


FIG. 155.

The slopes and deflections may be obtained from the resulting

bending-moment diagram by the methods of Art. 97, taking account of the sign of the areas. Or the methods of Art. 98 may be employed, remembering the opposite signs of the different parts of the bending-moment diagram area, and that the slope and deflection are zero at the ends. Another possible method is to treat the portion NN′ between the points of contraflexure (or virtual hinges) as a separate beam supported at its ends on the ends of two cantilevers, AN and BN′.

If the slopes at the ends A and B are not zero, but are fixed at equal magnitudes $i$ and of opposite sign, both being downwards towards the centre, slopes being reckoned positive downwards to the right, equation (1) becomes

$$\int_0^l (\mu + \mathrm{M}')dx = -2i \,.\, \mathrm{EI}$$

and $$\int_0^l \mathrm{M}'dx = -\int_0^l \mu dx - 2i \,.\, \mathrm{EI} \quad \text{or} \quad \mathrm{M}' = -\frac{1}{l}\int_0^l \mu dx - \frac{2i \,.\, \mathrm{EI}}{l}$$

$\mu$ being usually negative, and for minimum intensity of bending stress this value of M′ should be equal in magnitude to half the maximum value of $\mu$.

EXAMPLE 1.—Uniformly distributed load $w$ per unit span on a built-in beam. The area of the parabolic bending-moment diagram for a simply supported beam (see Fig. 81) is

$$\tfrac{2}{3} \times \tfrac{1}{8}wl^2 \times l = \tfrac{1}{12}wl^3$$

The mean bending moment is therefore $\frac{1}{12}wl^2$. By reducing all ordinates of Fig. 81 by the amount $\frac{1}{12}wl^2$, we get exactly the same diagram as shown in Fig. 152.

EXAMPLE 2.—Central load W on a built-in beam.

The bending-moment diagram for the simply supported beam is shown in Fig. 79. Its mean height is proportional to $\frac{1}{2} \,.\, \frac{Wl}{4}$ or $\frac{Wl}{8}$. Hence for the built-in beam the bending-moment diagram is as shown in Fig. 156. The points of contraflexure are evidently $\frac{1}{4}l$ from each end, and the bending moments at the ends and centre are $\frac{Wl}{8}$.

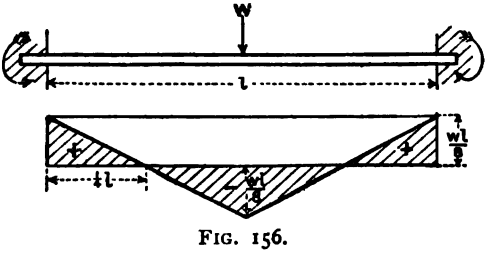



FIG. 156.

Taking the origin at the centre or either end, using the method of Art. 97 (3) and taking account of the signs, $\frac{dy}{dx}$ vanishes at both limits and $y$ at one limit, and the central deflection under the load is

$$\frac{1}{\mathrm{EI}}\left\{\left(\tfrac{1}{2} \,.\, \frac{Wl}{8} \times \frac{l}{4}\right)\left(\frac{l}{2} - \tfrac{1}{3}\frac{l}{4}\right) - \left(\tfrac{1}{2} \,.\, \frac{Wl}{8} \,.\, \frac{l}{4}\right)\left(\tfrac{1}{3} \,.\, \frac{l}{4}\right)\right\} = \frac{Wl^3}{192\mathrm{EI}}$$

**103. Built-in Beams with any Loading.**—As in the previous article, and with the same notation, if I and E are constant

$$\left.\begin{aligned}\int_0^l(\mu + M')dx &= 0\\ A + A' &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or,

or substituting for M′ its value from (7), Art. 101

$$\int_0^l\left\{\mu + M_A + (M_B - M_A)\frac{x}{l}\right\}dx = 0 \quad . \quad . \quad . \quad . \quad (2)$$

The loading being not symmetrical, $M_B$ is not necessarily equal to $M_A$, and the area A′ is not a rectangle but a trapezoid (Fig. 157), and the equation of areas A and A′ is insufficient to determine the *two* fixing couples $M_A$ and $M_B$. We may, however, very conveniently proceed by the method used in Art. 97 to establish a second relation. Thus, taking one end of the span, say A, Fig. 157, as origin

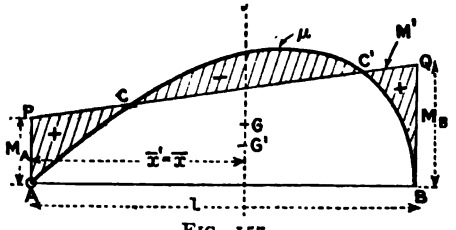


FIG. 157.

$$\frac{d^2y}{dx^2} = \frac{\mu + M'}{EI}$$

and multiplying by $x$ and integrating (by parts), with limits $l$ and 0

$$\left(x\frac{dy}{dx} - y\right)_0 = \frac{1}{EI}\int_0^l(\mu + M')xdx = \frac{1}{EI}\left(\int_0\mu xdx + \int_0^l M'xdx\right)$$

or,

$$EI\left(x\frac{dy}{dx} - y\right)_0^l = A\bar{x} + A\bar{x}''$$

where $\bar{x}$ and $\bar{x}'$ are the respective distances of the centres of gravity or centroids of the areas A and A′ from the origin. Further, the term

$$\left(x\frac{dy}{dx} - y\right)_0^l$$

is obviously zero, since each part of it vanishes at both limits $x = l$ and $x = 0$; hence

$$Ax + A'\bar{x}' = 0 = \int_0^l\mu xdx + \int_0^l M'xdx \quad . \quad . \quad . \quad (3)$$

or the moments about either support of the areas A and A′ are equal in magnitude, in addition to the areas themselves being equal, or, in other words, their centroids are in the same vertical line (see Fig. 157).

Evidently, from Fig. 157, the area APQB or $A' = \frac{M_A + M_B}{2} \times$

hence from (1)

$$\frac{M_A + M_B}{2} \cdot l = -A \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and, taking moments about the point A (Fig. 157), dividing the trapezoid into triangles by a diagonal PB

$$A\bar{x} = (\tfrac{1}{2}M_A \,.\, l \,.\, \tfrac{1}{3}l) + (\tfrac{1}{2}M_B \,.\, l \,.\, \tfrac{2}{3}l) = \tfrac{1}{6}l^2(M_A + 2M_B) \quad . \quad (4a)$$

or from (3), $\tfrac{1}{6}l^2(M_A + 2M_B) = -A\bar{x}$ . . . . . (5)

or, $$M_A + 2M_B = -\frac{6}{l^2} \,.\, A\bar{x}$$

and from (4), $$M_A + M_B = -\frac{2}{l} \,.\, A$$

from which $$M_B = \frac{2A}{l} - \frac{6A\bar{x}}{l^2} \quad \text{or} \quad \frac{2A}{l}\left(1 - \frac{3\bar{x}}{l}\right) \quad . \quad (6)$$

$$M_A = 6\frac{A\bar{x}}{l^2} - 4\frac{A}{l} \quad \text{or} \quad 2\frac{A}{l}\left(\frac{3\bar{x}}{l} - 2\right) \quad . \quad (7)$$

Thus the fixing moments are determined in terms of the area of the bending-moment diagram (A) and its moment ($A\bar{x}$) about one support, or the distance of its centroid from one support. The trapezoid APQB (Fig. 157) can then be drawn, and the difference of ordinates between it and the bending-moment diagram for the simply supported beam gives the bending moments for the built-in beam. The resultant diagram is shown shaded in Fig. 157. With the convention as to signs used in Art. 93 the area A must be reckoned negative for values of M producing concavity upwards. With loading which gives a bending moment the area of which and its moment are easily calculated, $M_B$ and $M_A$ may be found algebraically or arithmetically from (6) and (7), and then the bending moment elsewhere found from the equation (8) of Art. 101. With irregular loading the process may be carried out graphically; the quantity $A \,.\, \bar{x}$ may then conveniently be found by a "derived area," as in Art. 53, Fig. 64, using the origin A as a pole, without finding $\bar{x}$.

When the resultant bending-moment diagram has been determined, either of the graphical methods of Art. 98 may be used to find the deflections or slopes at any point of the beam, taking proper account of the difference of sign of the areas and starting both slope and deflection curves from zero at the ends. Or the methods of Art. 97, (*b*) and (*c*), may be employed, taking account of the different signs in calculating slopes from the areas of the bending-moment diagram or deflections from the moments of such areas. When the bending moment has been determined, the problem of finding slopes, deflections, etc., for the built-in beam is generally simpler than for the merely supported beam, because the end slopes are generally zero. The shearing-force diagram for the built-in beam with an unsymmetrical load changes from point to point just as for the corresponding simply supported beam $\left(\text{since } \frac{dF}{dx} = w\right)$, but the reactions at the ends are different, as shown by (4), Art. 101, one ($R_B$) being greater in magnitude, and the other ($R_A$)

being less by the amount $\frac{1}{l}(M_B - M_A)$, which may be positive or negative.

If the ends of the beam are built in so that the end slopes are not zero, equation (1) becomes

$$A + A' = EI(i_B - i_A) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $i_B$ and $i_A$ are the fixed slopes at the ends B and A, and are reckoned positive if downward to the right (usually they will have opposite signs). Equation (3) then becomes

$$A\bar{x} + A'\bar{x}' = EI \,.\, l \,.\, i_B \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the values of $M_B$ and $M_A$ are

$$M_B = \frac{2A}{l} - \frac{6A\bar{x}}{l^2} + \frac{2(2i_B + i_A)EI}{l} \quad . \quad . \quad . \quad (10)$$

$$M_A = \frac{6A\bar{x}}{l^2} - \frac{4A}{l} - 2\frac{(i_B + 2i_A)EI}{l} \quad . \quad . \quad . \quad (11)$$

quantities which will be less in magnitude (the area A being negative) than (6) and (7) when both ends slope downwards towards the centre, unless $i_B$ and $i_A$ are very unequal in magnitude. To secure the greatest possible flexural strength from a given section it would be necessary to make the two fixing moments $M_B$ and $M_A$ equal, and opposite to half the maximum bending moment for the freely supported beam. The necessary end slopes could more easily be calculated than secured in practice. And in the case where A = o

$$i_A = -\frac{l}{6EI}(2M_A + M_B), \qquad i_B = +\frac{l}{6EI}(M_A + 2M_B) \quad (12)$$

while if A is not zero

$$i_A = -\frac{l - \bar{x}}{l}\cdot\frac{A}{EI} - \frac{l}{6EI}(2M_A + B), \quad i_B = \frac{A\bar{x}}{lEI} + \frac{l}{6EI}(M_A + 2M_B) \quad (13)$$

which reduces to (7) and (6), Art. 97, when $M_A = o = M_B$. Also

$$i_A - i_B = -\frac{A}{EI} - \frac{l}{2EI}(M_A + M_B) \quad . \quad . \quad . \quad (13a)$$

another form of equation (8).

*An Alternative Method.*

A very simple method of dealing with a beam *encastré* at its ends is to look upon it as a cantilever fixed at one end, A, say (Fig. 158), the otherwise free end, B, being propped by a force $R_B$ (the reaction at B) and subject to a couple, $M_B$, that of the wall. Then principles, similar to those used in Art. 95, readily give $R_B$ and $M_B$. Thus let the slope and deflection produced

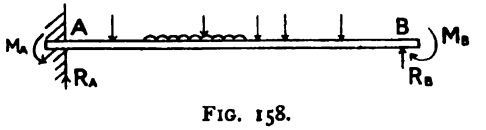



FIG. 158.

at B if free by the loads be $i$ and $\delta$ respectively. Then using (1) and (11) Art. 95, and equating the resultant upward slope to the right at B to zero, say

$$\frac{R_B l^2}{2EI} - \frac{M_B l}{EI} - i = 0, \text{ or } \tfrac{1}{2}R_B \cdot l^2 - M_B \cdot l - EI \cdot i = 0 \quad . \quad (14)$$

And using (2) and (11), Art. 95, and equating the resultant deflection to zero

$$\frac{R_B l^3}{3EI} - \frac{M_B \cdot l^2}{2EI} - \delta = 0, \text{ or } \tfrac{1}{3}R_B l^3 - \tfrac{1}{2}M_B l^2 - EI \cdot \delta = 0 \quad . \quad (15)$$

And from (14) and (15)

$$R_B = \frac{6EI}{l^3}(2\delta - li) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$M_B = \frac{2EI}{l^2}(3\delta - 2li) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Then $R_A$ = whole load $- R_B$ . . . . . (18)

And $M_A = M_B +$ moment of whole load about $A - R_B \cdot l$ . (19)

Given slopes $i_A$ at A and $i_B$ at B may easily be taken into account in equations (14) and (15), and any given difference in levels of the ends in equation (15).

Also equations (14) and (15) might have been written in the notation of Art. 97, application ($b$), $\frac{A\bar{x}}{EI}$ replacing $\delta$ and $\frac{A}{EI}$ replacing $i$. The factor $\frac{I}{EI}$ then disappears from the result, and (16) becomes

$$R_B = \frac{6A(2\bar{x} - l)}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and (17) becomes

$$M_B = \frac{2A(3\bar{x} - 2l)}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Values of F, M, $i$, and $y$ anywhere may be found by methods and expressions used for the cantilever combining the effects of $R_B$, $M_B$ and the loads, or otherwise.

EXAMPLE 1.—A built-in beam of span $l$ carries a load W at $\frac{1}{4}l$ from one end. Find the bending-moment diagram, points of inflection, deflection under the load, and the position and magnitude of the maximum deflection.

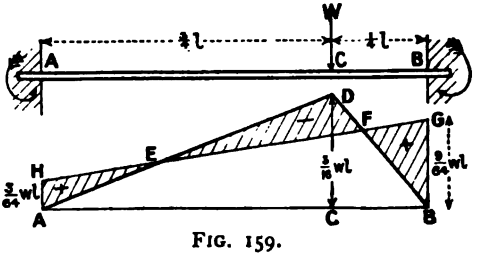



FIG. 159.

Taking the beam as a cantilever having the fixed end A (Fig. 159),

and an upward force $R_B$ and moment $M_B$ applied at B, from (5), Art. 95, the deflection at the end is

$$\delta = \frac{27}{128} \cdot \frac{Wl^3}{EI}$$

and from (3), Art. 95, the slope at the end is

$$i = \frac{9}{32} \cdot \frac{Wl^2}{EI}$$

hence from (16)

$$R_B = 6(\tfrac{27}{64} - \tfrac{9}{32})W = \tfrac{27}{32}W, \qquad \text{and } R_A = \tfrac{5}{32}W$$

and from (17)

$$M_B = 2l(\tfrac{81}{128} - \tfrac{9}{16})\,W = \tfrac{9}{64}Wl$$

and hence the bending moment at A

$$M_A = \tfrac{9}{64}Wl + \tfrac{3}{4}Wl - \tfrac{27}{32}Wl = \tfrac{3}{64}Wl$$

For the longer segment A to C, taking moments of forces to the left of a section with A as origin

$$M = M_A - R_A \cdot x = \tfrac{3}{64}Wl - \tfrac{5}{32}Wx$$

which vanishes for $x = \frac{3}{10}l$, giving the position of the point of inflexion E. For the shorter segment C to B, taking moments to the right of a section

$$M = M_B - R_B(l - x) = \tfrac{9}{64}Wl - \tfrac{27}{32}W(l - x) = -\tfrac{45}{64}Wl + \tfrac{27}{32}Wx$$

which vanishes for $x = \frac{5}{6}l$ giving the point of inflection F. The deflections and slopes might be found from the general expressions for deflection in Art. 95, sections (*a*), (*b*), and (*d*), but it is simpler, since $i$ and $y$ vanish for $x = 0$, to use the relations (3) and (4) of Art. 94 directly thus.

Slopes from A to C reckoned positive downwards to the right

$$i = \frac{W}{EI}\int_0^x (-\tfrac{5}{32}x + \tfrac{3}{64}l)dx = \frac{W}{64EI}(-5x^2 + 3lx)$$

This vanishes for $x = \frac{3}{5}l$, which gives the position of the point of maximum deflection. That its distance from A is twice that of the point of inflection under E is evident from a glance at the bending-moment diagram, Fig. 159.

For $x = \frac{3}{4}l$ at C

$$i_C = \frac{Wl^2}{64EI}(-5 \times \tfrac{9}{16} + \tfrac{9}{4}) = -\tfrac{9}{1024} \cdot \frac{Wl^2}{EI}$$

Slopes from C to B

$$i = i_C + \frac{W}{EI}\int_{\frac{3}{4}l}^{x} (-\tfrac{45}{64}l + \tfrac{27}{32}x)dx = -\tfrac{9}{1024}\frac{Wl^2}{EI} + \frac{W}{64EI}(-45lx + 27x^2)_{\frac{3}{4}l}^{x}$$

which does not reach zero for any value of $x$ between $\frac{3}{4}l$ and $l$.

Deflections from A to C

$$y = \int i dx = \frac{W}{64EI}\int_0^x (-5x^2 + 3lx)dx = \frac{W}{64EI}(-\tfrac{5}{3}x^3 + \tfrac{3}{2}lx^2)$$

and at C, where $x = \frac{3}{4}l$,

$$y_C = +\tfrac{9}{4096}\cdot\frac{Wl^3}{EI}$$

and at $x = \frac{3}{8}l$, $\qquad y_{max.} = +\tfrac{9}{3200}\cdot\frac{Wl^3}{EI}$

at $x = \frac{l}{2}$, $\qquad y = +\tfrac{1}{384}\cdot\frac{Wl^3}{EI}$

Deflections from C to B

$$y = y_C + \int_{\frac{3}{4}l}^{x} i\,.\,dx = +\tfrac{9}{4096}\cdot\frac{Wl^3}{EI} + \frac{W}{64EI}\int_{\frac{3}{4}l}^{x}(18l^2 - 45lx + 27x^2)dx$$

$$= \frac{W}{64EI}\left\{+\tfrac{9}{64}l^3 + (18l^2x - \tfrac{45}{2}lx^2 + 9x^3)\Big|_{\frac{3}{4}l}^{x}\right\}$$

$$= \frac{9W}{128EI}(2x^3 - 5lx^2 + 4l^2x - l^3)$$

EXAMPLE 2.—The more general problem of a load W on a built-in beam, placed at distances $a$ from one support A and $b$ from the other B, may be solved in just the same way.

If $a$ is greater than $b$, and A is the origin (Fig. 160)

$$M_A = \frac{Wab^2}{(a+b)^2}$$

$$M_B = \frac{Wa^2b}{(a+b)^2}$$

$$R_A = W\frac{b^2(3a+b)}{(a+b)^3}$$

$$R_B = W\frac{a^2(a+3b)}{(a+b)^3}$$

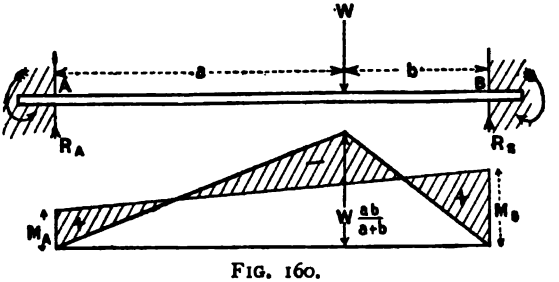



FIG. 160.

The points of inflection are at

$$x = \frac{a}{3a+b}(a+b) \quad \text{and} \quad x = \frac{a+2b}{a+3b}(a+b)$$

The slope under the load is

$$i = -\frac{Wa^2b^2(a-b)}{2EI(a+b)^3}$$

The zero slope and maximum deflection occurs at

$$x = \frac{2a}{3a+b}(a+b)$$

and when $b = 0$ this becomes $\frac{2}{3}(a + b)$, so that the maximum deflection is for the built-in beam always within the middle third of the span.

The deflection under the load is

$$y = \frac{Wa^3b^3}{3EI(a + b)^3}$$

which is $\frac{ab}{(a + b)^2}$ times that for a freely supported beam.

The maximum deflection is—

$$\frac{2}{3}\frac{Wa^3b^2}{(3a + b)^2EI}$$

and the deflection at mid-span is—

$$\frac{Wb^2(3a - b)}{48EI}$$

*Built-in Beams of Variable Section.*—Having considered in Art. 99 how simple beam-deflection problems are affected by a variable section, and in the present article the case of built-in beams of constant section, it will be sufficient to say that the modification in the work when the quantity I is not a constant consists in using $\frac{M}{I}$ instead of M as a variable throughout. The matter is treated fully from its algebraic and graphical aspects in the author's "Strength of Materials."

**104. Continuous Beams. Theorem of Three Moments.**—A beam resting on more than two supports and covering more than one span

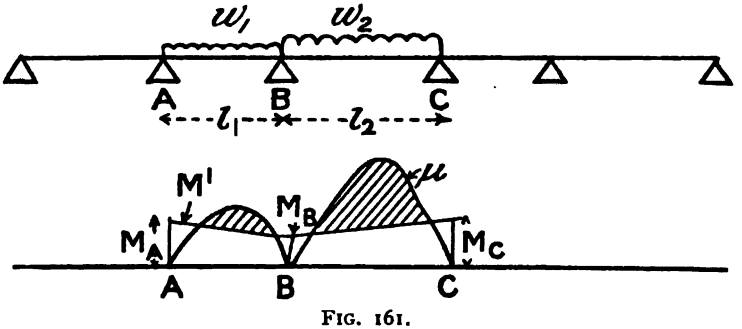



FIG. 161.

is called a continuous beam. Beams supported at the ends and propped at some intermediate point have already been noticed (Arts. 94 and 96), and form simple special cases of continuous beams.

Considering first a simple case of a continuous beam, let AB and BC, Fig. 161, be two consecutive spans of length $l_1$ and $l_2$ of a continuous beam, the uniformly spread loads on $l_1$ and $l_2$ being $w_1$ and $w_2$ per unit length respectively. Then for either span, as in Art. 101, the bending moment is the algebraic sum of the bending moment for a

freely supported beam of the same span and that caused by the fixing moments at the supports, or, as in Art. 101 (8)

$$M = EI\frac{d^2y}{dx^2} = \mu + M'$$

M′ being generally of opposite sign to $\mu$. First apply this to the span BC, taking B as origin and $x$ positive to the right, $\mu$ being equal to $-\frac{w_2}{2}(l_2x - x^2)$, being reckoned negative when producing concavity upwards, by (7) and (8), Art. 101—

$$EI\frac{d^2y}{dx^2} = -\frac{w_2}{2}l_2x + \frac{w_2}{2}x^2 + M_B + (M_C - M_B)\frac{x}{l_2} \quad . \quad . \quad (1)$$

and integrating—

$$EI\frac{dy}{dx} = -\frac{w_2}{4}l_2x^2 + \frac{w_2}{6}x^3 + M_B . x + (M_C - M_B)\frac{x^2}{2l_2} + EI . i_B \quad (2)$$

where $i_B$ is the value of $\frac{dy}{dx}$ at B, where $x = 0$.

Integrating again, $y$ being 0 for $x = 0$

$$EI . y = -\frac{w_2l_2}{12} . x^3 + \frac{w_2}{24}x^4 + \frac{M_B}{2} . x^2 + (M_C - M_B)\frac{x^3}{6l_2} + EI . i_B . x + 0 \quad (3)$$

and when $x = l_2$, $y = 0$, hence dividing by $l_2$

$$EI . i_B = \frac{w_2l_2^3}{24} - \frac{M_Bl_2}{2} - \frac{(M_C - M_B)l_2}{6}$$

or,
$$6EI . i_B = \frac{w_2l_2^3}{4} - 2M_Bl_2 - M_Cl_2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, taking B as origin, and dealing in the same way with the span BA, $x$ being positive to the left, we get similarly (changing the sign of $i_B$)

$$-6EI . i_B = \frac{w_1l_1^3}{4} - 2M_Bl_1 - M_A . l_1 \quad . \quad . \quad . \quad (5)$$

and adding (4) and (5)

$$M_Al_1 + 2M_B(l_1 + l_2) + M_C . l_2 - \tfrac{1}{4}(w_1l_1^3 + w_2l_2^3) = 0 \quad . \quad (6)$$

This is Clapeyron's Theorem of Three Moments for the simple loading considered. If there are $n$ supports and $n - 1$ spans, or $n - 2$ pairs of consecutive spans, such as ABC, $n - 2$ equations, such as (6), may be written down. Two more will be required to find the bending moments at $n$ supports, and these are supplied by the end conditions of the beam: *e.g.* if the ends are freely supported, the bending moment at each end is zero.

If an end, say at A, were fixed horizontally, $i_A = 0$ and an equation similar to (5) for the end span would be

$$2M_A + M_B - \frac{w_1 l_1^2}{4} = 0$$

When the bending moment at each support is known, the reactions at the supports may be found by taking the moments of internal and external forces about the various supports, or from Art. 101 (4), the shearing force on a section just to the right of $A_1$,

$$F_A = \frac{M_B - M_A}{l} - \frac{wl}{2}$$

The shearing force immediately to each side of a support being found, the pressure on that support is the algebraic difference of the shearing forces on the two sides. As the shearing force generally changes sign at a support, the magnitude of the reaction is generally the sum of the magnitudes of the shearing forces on either side of the support without regard to algebraic sign.

EXAMPLE 1.—A beam rests on five supports, covering four equal spans, and carries a uniformly spread load. Find the bending moments, reactions, etc., at the supports.

Since the ends are free (Fig. 162), $M_A = 0$, and $M_E = 0$.

And from the symmetry evidently $M_D = M_B$.

Applying the equation of three moments (6) to the portions ABC and BCD

$$0 + 2M_B \cdot 2l + M_C \cdot l - \tfrac{1}{2}wl^3 = 0$$

and
$$M_B \cdot l + 2M_C \cdot 2l + M_B l - \tfrac{1}{2}wl^3 = 0$$

hence
$$4M_B l + M_C l - \tfrac{1}{2}wl^3 = 0$$

and
$$4M_B l + 8M_C l - wl^3 = 0$$

$$7M_C \cdot l = \tfrac{1}{2}wl^3 \qquad M = \tfrac{1}{14}wl^2 \qquad M_B = \tfrac{3}{28}wl^2 = M_D$$

Taking moments about B

$$-R_A \cdot l + \frac{wl^2}{2} = \tfrac{3}{28}wl^2 \qquad R_A = \tfrac{11}{28}wl = R_E$$

Taking moments about C

$$\tfrac{22}{28}wl^2 + R_B \cdot l - 2wl^2 = -\tfrac{1}{14}wl^2 \qquad R_B = \tfrac{8}{7}wl = R_D$$

$$R_C = 4wl - \tfrac{11}{14}wl - \tfrac{16}{7}wl = \tfrac{13}{14}wl$$

The shearing-force diagram for Fig. 162 may easily be drawn by setting up $\frac{11}{28}wl$ at A, and decreasing the ordinates uniformly by an amount $wl$ to $-\frac{17}{28}wl$ at B, increasing there by $\frac{8}{7}wl$, and so on, changing at a uniform rate over each span, and by the amount of the reactions at the various supports.

The bending-moment diagram (Fig. 162) may conveniently be drawn by drawing parabolas of maximum ordinate $\frac{1}{8}wl^2$ on each span, and erecting ordinates $M_B$, $M_C$, . $M_D$, and joining by straight lines. The algebraic sum of $\mu$ and M′ is given by vertical ordinates across the

shaded area in Fig. 162. An algebraic expression for the bending moment in any span may be written from (8) Art. 101 as follows:—
Span AB, origin A—

$$M = -\frac{w}{2}(lx - x^2) + \tfrac{3}{28}wlx = -\frac{wx}{2}(\tfrac{11}{14}l - x)$$

Span BC, origin B—

$$M = -\frac{w}{2}(lx - x^2) + \tfrac{3}{28}wl^2 - \tfrac{1}{28}wlx = -\frac{w}{2}(\tfrac{15}{14}lx - \tfrac{3}{14}l^2 - x^2)$$

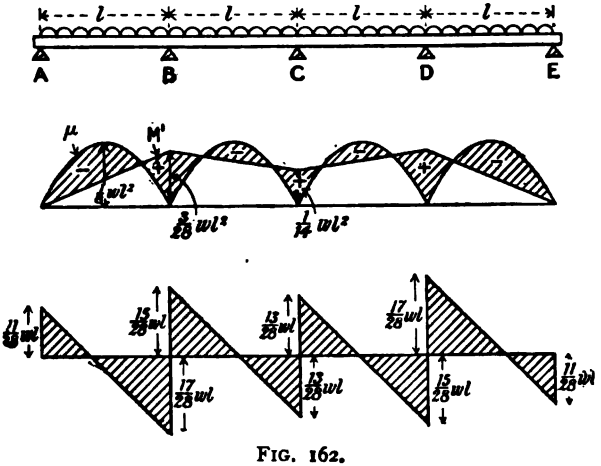



FIG. 162.

EXAMPLE 2.—A continuous girder ABCD covers three spans, AB 60 feet, BC 100 feet, CD 40 feet. The uniformly spread loads are 1 ton, 2 tons, and 3 tons per foot-run on AB, BC, CD respectively. If the girder is of the same cross-section throughout, find the bending moments at the supports B and C, and the pressures on each support

For the spans ABC—

$$0 + 320M_B + 100M_C = \tfrac{1}{4} \times 1000(216 + 2000) = 554{,}000$$

hence $16M_B + 5M_C = 27{,}700$ tons-feet.

For the spans BCD—

$$100M_B + 280M_C + 0 = \tfrac{1}{4} \times 1000(2000 + 192)$$

hence $5M_B + 14M_C = 27{,}400$ ton-feet.

From which $M_B = 1260{\cdot}3$ ton-feet $\quad M_C = 1507{\cdot}0$ ton-feet.

Taking moments about B, $R_A \times 60 - 60 \times 30 = -1260{\cdot}3$

$R_A = 9$ tons

" " C, $9 \times 160 + 100R_B - 60 \times 130 - 200 \times 50 = -1507$

$R_B = 148{\cdot}5$ tons

Taking moments about C, $40R_D - 120 \times 20 = -1507$

$R_D = 22{\cdot}3$ tons

" " B, $22{\cdot}3 \times 140 + 100R_C - 120 \times 120 - 200 \times 50 = -1260$ $R_C = 200{\cdot}1$ tons

**105. Continuous Beams; any Loading.**—Let the diagrams of bending moment APB and BQC be drawn for any two consecutive spans AB or $l_1$, and BC or $l_2$ (Fig. 163), of a continuous beam as if each span were bridged by independent beams freely supported at their ends. Let the area APB be $A_1$, and the distance of its centroid from the point A be $\bar{x}_1$, so that $A_1\bar{x}_1$ is the moment of the area about the point A. Let the area under BQC be $A_2$, and the distance of its centroid from C be $\bar{x}_2$, the moment about C being $A_2\bar{x}_2$. (In accordance with the

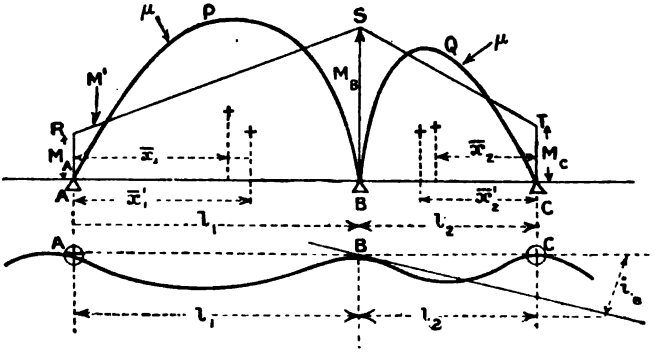



FIG. 163.

signs adopted in Art. 93, and used subsequently, the areas $A_1$ and $A_2$ will be negative quantities for downward loading, bending moments which produce upwards convexity being reckoned positive.) Draw the trapezoids ARSB and BSTC as in Art 101, to represent M', the bending moments due to the fixing couples. Let $A_1'$ and $A_2'$ be the areas of ARSB and BSTC respectively, and $\bar{x}_1'$ and $\bar{x}_2'$ the distances of their centroids from A and C respectively.

From A as origin, $x$ being measured positive towards B, using the method of Art. 97 equation (3) between limits $x = l_1$ and $x = 0$, the supports at A and B being at the same level

$$\left(x\frac{dy}{dx} - y\right)_0^{l_1} = l_1 i_B = \frac{1}{EI}\int_0^{l_1} (\mu + M')x\,dx = \frac{1}{EI}(A_1\bar{x}_1 + A_1'\bar{x}_1') \quad (1)$$

$i_B$ being the slope $\left(\frac{dy}{dx}\right)$ at B.

From C as origin, $x$ being measured positive toward B, C and B being at the same level—

$$\left(x\frac{dy}{dx} - y\right)_0^{l_2} = l_2 i_B = \frac{1}{EI}(A_2\bar{x}_2 + A_2'\bar{x}_2'). \quad . \quad . \quad . \quad (2)$$

Equating the slope at B from (1) and (2) with sign reversed on account of the reversed direction of $x$—

$$\frac{A_1\bar{x}_1 + A_1'\bar{x}_1'}{l_1} = -\frac{A_2\bar{x}_2 + A_2'\bar{x}_2'}{l_2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And as in Art. 103 (4$a$), by joining AS and taking moments about A—

$$A_1\bar{x}_1' = \frac{l_1^2}{6}(M_A + 2M_B)$$

and similarly $$A_2'\bar{x}_2' = \frac{l_2^2}{6}(M_C + 2M_B)$$

hence (3) becomes—

$$\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2} + \tfrac{1}{6}M_A \cdot l_1 + \tfrac{1}{3}M_B(l_1 + l_2) + \tfrac{1}{6}M_C l_2 = 0$$

or,
$$\frac{6A_1\bar{x}_1}{l_1} + \frac{6A_2\bar{x}_2}{l_2} + M_A \cdot l_1 + 2M_B(l_1 + l_2) + M_C l_2 = 0 \quad . \quad (4)$$

This is a general form of the Equation of Three Moments, of which equation (6) of the previous article is a particular case easily derived by writing $A_1 = -\frac{2}{3} \cdot \frac{wl_1^2}{8} \cdot l_1$, and $x_1 = \frac{l_1}{2}$, etc., the areas $A_1$ and $A_2$ being negative for bending producing concavity upwards. For a beam on $n$ supports this relation (4) provides $n - 2$ equations, and the other necessary two follow from the manner of support at the ends. If either end is fixed horizontally, an equation of moments for the adjacent span follows from the method of Art. 103. If A is an end fixed horizontally, and AB the first span, from area moments *about* B, an equation similar to (5), Art. 103, is—

$$2M_A + M_B + \frac{6A_1(l_1 - \bar{x}_1)}{l_1^2} = 0 \quad (A_1 \text{ being generally negative})$$

If both ends are fixed horizontally, a similar equation holds for the other end. If, say, the end A is fixed at a downward slope $i_A$ towards B, the right-hand side of this equation would be $-\frac{6EIi_A}{l_1}$ instead of zero. If either end overhangs an extreme support the bending moment at the support is found as for a cantilever.

If some or all the supports sink, the support B falling $\delta_1$ *below* A and $\delta_2$ *below* C, a term corresponding to $y$ appears in (1) and (2), so that (3) becomes—

$$\frac{A_1\bar{x}_1 + A_1'\bar{x}_1' + EI\delta_1}{l_1} = -\frac{A_2\bar{x}_2 + A_2'\bar{x}_2' + EI\delta_2}{l_2} \quad . \quad (3a)$$

and (4) becomes—

$$\frac{6A_1\bar{x}_1}{l_1} + \frac{6A_2\bar{x}_2}{l_2} + M_A \cdot l_1 + 2M_B(l_1 + l_2) + M_C \cdot l_2 + 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right) = 0 \quad (5)$$

*Wilson's Method.*—A simple and ingenious method of solving general problems on continuous beams, published by the late Dr. George Wilson,[1] consists of finding the reactions at the supports by equating the upward deflections caused at every support by all the supporting forces, to the downward deflections which the load would cause at those various points if the beam were supported at the ends only. This provides sufficient equations to determine the reactions at all the supports except the end ones. The end reactions are then found by the usual method of taking moments of all upward and downward forces about one end, and in the case of free ends, equating the algebraic sum to zero. To take a definite case, suppose the beam to be supported at five points A, B, C, D, and E, Fig. 164, all at the same level. Let the distances of B, C, D, and E from A be $b$, $c$, $d$, and $e$ respectively. Let the deflections at A, B, C, D, and E due to the load on the beam if simply supported at A and E be $o$, $y_B$, $y_C$, $y_D$, and $o$ respectively. These may be calculated by the methods of Arts. 94, 96, 97, 98, according to the manner in which the beam is loaded.

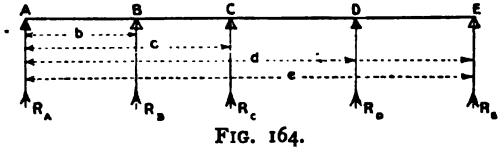


FIG. 164.

Now let the upward deflection at B, C, and D, if the beam were supported at the ends, due to 1 lb. or 1 ton or other unit force at B be

$$ {}_b\delta_B,\ {}_b\delta_C, \text{ and } {}_b\delta_D \text{ respectively,} $$

and those at B, C, and D due to the unit force at C be

$$ {}_c\delta_B,\ {}_c\delta_C, \text{ and } {}_c\delta_D \text{ respectively,} $$

and due to unit force at D be

$$ {}_d\delta_B,\ {}_d\delta_C, \text{ and } {}_d\delta_D \text{ respectively.} $$

Then all the supports being at zero level, if $R_B$, $R_C$, and $R_D$ are the reactions at B, C, and D respectively, equating downward and upward deflections at B, C, and D for the beam supported at the ends A and E only

$$ y_B = (R_B \times {}_b\delta_B) + (R_C \times {}_c\delta_B) + (R_D \times {}_d\delta_B) \quad . \quad . \quad . \quad (6) $$
$$ y_C = (R_B \times {}_b\delta_C) + (R_C \times \delta_C) + (R_D \times {}_d\delta_C) \quad . \quad . \quad . \quad (7) $$
$$ y_D = (R_B \times {}_b\delta_D) + (R_C \times {}_c\delta_D) + (R_D \times {}_d\delta_D) \quad . \quad . \quad . \quad (8) $$

Note that ${}_c\delta_B = {}_b\delta_C$, ${}_d\delta_B = {}_d\delta_D$, ${}_c\delta_D = {}_d\delta_C$, which becomes apparent by changing $b$ into $x$, $x$ into $b$ and $a$ into $a + b - x$ in (7), Art. 96.

From three simple simultaneous equations (6), (7), and (8), $R_B$, $R_C$, and $R_D$ can be determined. $R_E$ may be found by an equation of moments about A.

$$ R_E \times e = (\text{moment of whole load about A}) - b \,.\, R_C - cR_C - dR_D $$
and $$ R_A = \text{whole load} - R_B - R_C - R_D - R_E $$

[1] *Proc. Roy. Soc.*, vol. 62, Nov., 1897.

The exercise at the end of Art. 96 is a simple example of this method, there being only one support, and therefore only one simple equation for solution.

Wilson's method may be used for algebraic calculations when the loading is simple, so that the upward and downward deflections may be easily calculated, but it is equally applicable to irregular types of loading where downward deflections at several points are all determined in one operation graphically.

When the reactions are all known, the bending moment and shearing force anywhere can be obtained by direct calculation from the definitions (Art. 56).

Sinking of any support can evidently be taken into account in this method very simply. If the support at B, for example, sinks a given amount, that amount of subsidence must be subtracted from the left-hand side of equation (6).

If one end of the beam is fixed, the deflections must be calculated as for a propped cantilever (Arts. 95 and 97). If both ends, they must be calculated as indicated in Arts. 102 and 103.

EXAMPLE 1.—Find the reactions in Ex. 1 of Art. 104 by Wilson's Method. Using Fig. 162 the beam being supported at A and E only, and A being the origin, by (9) Art. 94

$$y_B = \frac{wl^4}{24EI}(1 - 8 + 64) = \frac{57}{24}\frac{wl^4}{EI} = y_D \text{ from the symmetry}$$

And by (11), Art. 94

$$y_C = \frac{5}{384}\cdot\frac{256l^4}{EI} = \frac{10}{3}\frac{wl^4}{EI}$$

And using (7) and (8), Art 96, the upward deflections due to the props are, at B

$$\frac{l^3}{EI}\left\{\frac{R_B \times 9 \times 1}{3 \times 4} - \frac{R_C \times 2}{4}\left(\tfrac{1}{6} - \tfrac{4}{6} - \tfrac{4}{3}\right) - \frac{R_D}{4}\left(\tfrac{1}{6} - \tfrac{9}{6} - \tfrac{3}{3}\right)\right\}$$
$$= \frac{l^3}{EI}\left(\tfrac{4}{3}R_B + \tfrac{11}{12}R_C\right), \text{ since by symmetry } R_B = R_D$$

And at C

$$\frac{l^3}{EI}\left\{-\frac{2R_B \cdot 2}{4}\left(\tfrac{4}{3} - \tfrac{9}{6} - \tfrac{3}{3}\right) + \frac{R_C \cdot 16}{12}\right\} = \frac{l^3}{EI}\left(\tfrac{11}{6}R_B + \tfrac{4}{3}R_C\right)$$

Equating upward and downward deflections at B and C

$$\tfrac{57}{24}wl = \tfrac{4}{3}R_B + \tfrac{11}{12}R_C$$
$$\tfrac{10}{3}wl = \tfrac{11}{6}R_B + \tfrac{4}{3}R_C$$

from which $R_B = R_D = \frac{8}{7}wl$ and $R_C = \frac{13}{14}wl$.

$$R_A = R_E = \tfrac{1}{2}(4wl - 2 \times \tfrac{8}{7}wl - \tfrac{13}{14}wl) = \tfrac{11}{28}wl$$
$$M_B = -\tfrac{11}{28}wl^2 + \frac{wl^2}{2} = \tfrac{3}{28}wl^2$$
$$M_C = 2wl^2 - \tfrac{8}{7}wl^2 \times \tfrac{11}{28}wl \times 2l = \tfrac{1}{14}wl^2$$

The bending moment anywhere can be simply stated, the diagrams of bending moment and shearing being as shown in Fig. 162.

EXAMPLE 2.—A continuous beam 30 feet long is carried on supports at its ends, and is propped to the same level at points 10 feet and 22 feet from the left-hand end. It carries loads of 5 tons, 7 tons, and 6 tons at distances of 7 feet, 14 feet, and 24 feet respectively from the left-hand end. Find the bending moment at the props, the reactions at the four supports, and the points of contraflexure.

*Firstly, by the General Equation of Three Moments.*—For the spans ABC, Fig. 165, with the notation of Art. 105.

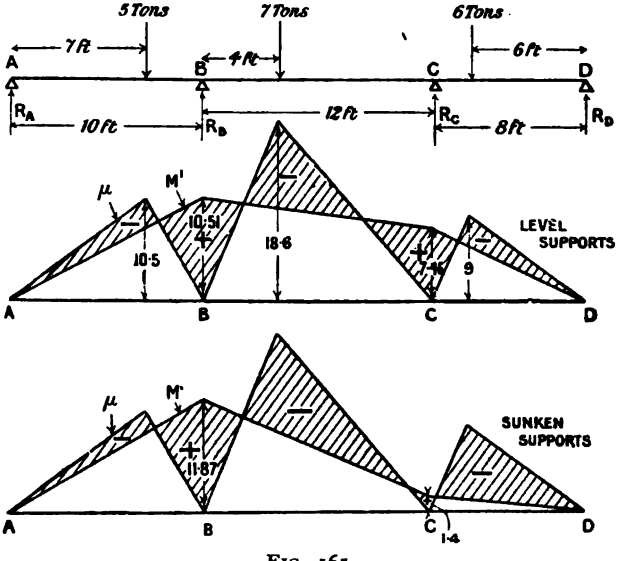



FIG. 165.

Moment of the bending-moment diagram area on AB about A

$$A_1\bar{x}_1 = (\tfrac{1}{2}\cdot 7\cdot\tfrac{21}{2}\cdot\tfrac{2}{3}\cdot 7) + (\tfrac{1}{2}\cdot 3\cdot\tfrac{21}{2}\cdot 8) = \tfrac{343}{2} + 126 = 297{\cdot}5 \text{ ton-(feet)}^3$$

Moment of the bending moment diagram on BC about C

$$A_2\bar{x}_2 = (\tfrac{1}{2}\cdot 4\cdot\tfrac{56}{3}\cdot\tfrac{23}{3}) + (\tfrac{1}{2}\cdot 8\cdot\tfrac{56}{3}\cdot\tfrac{16}{3}) = \tfrac{3136}{9} + \tfrac{3584}{9} = 746{\cdot}\dot{6} \text{ ton-(feet)}^3$$

This must be taken as negative in accordance with the signs adopted at the end of Art. 93. Then from (4) of Art. 105, since $M_A = 0$

$$-(6\times 29{\cdot}75) - \frac{6\times 746{\cdot}6}{12} + 0 + 2M_B\times 22 + 12M_C = 0$$

or
$$44M_B + 12M_C = 551{\cdot}83 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For the spans BCD

About B, $A_1\bar{x}_1 = (\tfrac{1}{2}\cdot 4\cdot\tfrac{56}{3}\cdot\tfrac{2}{3}\cdot 4) + (\tfrac{1}{2}\cdot 8\cdot\tfrac{56}{3}\cdot\tfrac{20}{3}) = 597{\cdot}\dot{3}$

About D, $A_2\bar{x}_2 = (\tfrac{1}{2}\cdot 2\cdot 9\cdot\tfrac{20}{3}) + (\tfrac{1}{2}\cdot 6\cdot 9\cdot 4) = 168$

Taking these as negative, from (4), $M_D$ being 0

$$-\frac{6 \times 597\dot{3}}{12} - \frac{6 \times 168}{8} + 12M_B + 2M_C \times 20 + 0 = 0$$

or, $$12M_B + 40M_C = 424\dot{6} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

And from the equations (9) and (10)

$$M_B = 10{\cdot}51 \text{ ton-feet} \qquad M_C = 7{\cdot}46 \text{ ton-feet}$$

Taking moments to the left of B

$$5 \times 3 - 10R_A = 10{\cdot}51 \qquad R_A = 0{\cdot}449 \text{ ton}$$

Taking moments to left of C

$$5 \times 15 + 7 \times 8 - 22 \times 0{\cdot}45 - 12R_B = 7{\cdot}46 \qquad R_B = 9{\cdot}471 \text{ ton}$$

Taking moments to right of C

$$6 \times 2 - 8R_D = 7{\cdot}46 \qquad R_D = 0{\cdot}567 \text{ ton}$$

$$R_C = 5 + 7 + 6 - 0{\cdot}45 - 9{\cdot}47 - 0{\cdot}57 = 7{\cdot}51 \text{ tons}$$

*Inflections.*—Taking A as origin and taking convexity upward as positive bending. From 5 ton load to B

bending moment $= 5(x - 7) - 0{\cdot}449x = 4{\cdot}551x - 35$, which vanishes, for $x = 7{\cdot}9$ feet.

From B to 7 ton load, bending moment is

$4{\cdot}551x - 35 - 9{\cdot}471(x - 10) = 59{\cdot}71 - 4{\cdot}92x$, which vanishes, for $x = 12{\cdot}14$ feet.

From 7 ton load to C the bending moment is

$59{\cdot}71 - 4{\cdot}92x + 7(x - 14) = 2{\cdot}08x - 38{\cdot}29$, which vanishes, for $x = 18{\cdot}5$ feet.

From C to 6 ton load the bending moment is

$2{\cdot}08x - 38{\cdot}29 - 7{\cdot}51(x - 22) = 126{\cdot}9 - 5{\cdot}43x$, which vanishes, for $x = 23{\cdot}4$ feet.

*Secondly by Wilson's Method.*—With end supports only, the downward deflections by (7) and (10) of Art. 96, are, at B

$$\frac{1}{6EI \times 30}[\{5 \times 7 \times 20(529 + 322 - 400)\}$$
$$+\{7 \times 16 \times 10(196 + 448 - 100)\} + \{6 \times 6 \times 10(576 + 288 - 100)\}]$$

or $$y_B = \frac{1}{180EI}(315{,}700 + 609{,}280 + 275{,}040) = \frac{1{,}200{,}020}{180EI}$$

$$y_C = \frac{1}{6EI \times 30}[\{5 \times 7 \times 8(529 + 322 - 64)\}$$
$$+\{7 \times 14 \times 18(256 + 448 - 64)\} + \{6 \times 6 \times 22(576 + 288 - 484)\}]$$

or $$y_C = \frac{1}{180EI}(220{,}360 + 501{,}760 + 300{,}960) = \frac{1{,}023{,}080}{180EI}$$

With end supports only, the upward deflections due the props at B and C are

$$\text{At B, } \frac{1}{6EI \times 30}[\{2R_B \times 100 \times 400\} + \{R_C \times 8 \times 10(484+352-100)\}]$$

$$= \frac{1}{180EI}(80{,}000R_B + 58{,}880R_C)$$

$$\text{At C, } \frac{1}{6EI \cdot 30}[\{R_B \times 10 \times 8\,(400+400-64)\} + \{2R_C \times 64 \times 484\}]$$

$$= \frac{1}{180EI}(58{,}880R_B + 61{,}952R_C)$$

Equating the upward and downward deflections at B and C

$$80{,}000R_B + 58{,}880R_C = 1{,}200{,}020 \quad . \quad . \quad . \quad (11)$$
$$58{,}880R_B + 61{,}952R_C = 1{,}023{,}080 \quad . \quad . \quad . \quad (12)$$

which equations give the values

$$R_B = 9{\cdot}47 \text{ tons} \quad R_C = 7{\cdot}51 \text{ tons}$$

confirming the previous results. The reactions at the ends, bending moments at the supports, and position of the points of inflection follow by direct calculation very simply (see Fig. 165).

EXAMPLE 3.—If the cross-section of the continuous beam in Example 2 above has a moment of inertia of 300 inch units, and the support B sinks $\frac{1}{20}$ inch and the support C sinks $\frac{1}{10}$ inch, find the bending moments and reactions at the supports, E being 13,000 tons per square inch.

*Firstly, by Wilson's Method.*—The downward deflection at B due to the load would be

$$\frac{1}{EI}\left(\frac{1{,}200{,}020}{180}\right) \text{ feet} \quad \text{or} \quad \frac{\text{ton-(feet)}^3}{\text{ton-(feet)}^2}$$

if E and I are in foot and ton units. If E and I are in inch units the deflection at B would be

$$\frac{1728}{EI} \times \frac{1{,}200{,}020}{180} \text{ inches, the dimensions being } \frac{\text{ton-(inches)}^3}{\text{ton-(inches)}^2}$$

The upward deflection at B due to the props has to balance 0·05 inch less than this amount, hence

$$\frac{1728}{180EI}(80{,}000R_B + 58{,}880R_C) = \frac{1728}{180EI}(1{,}200{,}020) - 0{\cdot}05$$

or corresponding to (11), putting I = 300 and E = 13,000

$$80{,}000R_B + 58{,}880R_C = 1{,}200{,}020 - 20{,}312 = 1{,}179{,}708 \quad (13)$$

and corresponding to (12) with 0·1 inch subsidence at C

$$58{,}880R_B + 61{,}952R_C = 1{,}023{,}080 - 40{,}625 = 982{,}455 \quad (14)$$

From the simple equations (13) and (14)

$$R_C = 6{\cdot}13 \text{ tons} \qquad R_B = 10{\cdot}23 \text{ tons}$$

And by an equation of moments about A, $R_D = 1{\cdot}33$ tons,
and by an equation of moments about D, $R_A = 0{\cdot}31$ ton.

*Secondly, by the General Equation of Three Moments.*—From equation (5), Art. 105, an equation corresponding to equation (9), the units of which are ton-(feet)$^2$, may be formed. Using such units, this becomes

$$144(44M_B + 12M_C) + 6 \times 13{,}000 \times 300\left(\frac{0{\cdot}05}{120} - \frac{0{\cdot}05}{144}\right) = 551{\cdot}83 \times 144$$

or, $$44M_B + 12M_C = 551{\cdot}83 - 11{\cdot}3 \quad . \quad . \quad . \quad (15)$$

And corresponding to (10)

$$12M_B + 40M_C = 424{\cdot}6 - \frac{6 \times 13{,}000 \times 300}{144}\left(\frac{0{\cdot}05}{144} + \frac{0{\cdot}1}{96}\right)$$

or, $$12M_B + 40M_C = 199 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

And from (15) and (16)

$$M_B = 11{\cdot}87 \text{ ton-feet} \qquad M_C = 1{\cdot}404 \text{ ton-feet}$$

From an equation of moments to the left of B, $R_A = 0{\cdot}31$ ton
" " " " right of C, $R_D = 1{\cdot}33$ tons
" " " " right of B, $R_C = 6{\cdot}13$ "
" " " " left of C, $R_B = 10{\cdot}23$ "

confirming the previous results.

The diagram of bending moments is shown in the lower part of Fig. 165. The serious changes in the magnitude of the bending moments at B, C, and under the 6-ton load may be noted; also the change in position of the points of inflection to the right and left of C, involving change in signs of the bending moment over some length of the beam: all these changes arise from the slight subsidence of the two supports at B and C.

**106. Continuous Beams of Varying Section.**—The methods of the previous article may be applied to cases where the moment of inertia of cross-section (I) varies along the length of span. The modifications in the first method will consist in dividing all bending-moment terms by the variable I before making the summation of the various terms in $\int_0 Mxdx$ and writing E in place of EI. The complete method is more fully explained in the author's "Strength of Materials."

Fixing of the girder ends at any inclination may also be taken into account as indicated in Arts. 103 and 105.

*Wilson's Method* of solving problems in continuous beams by equating the downward deflections produced by the load to the upward deflections produced by the supporting forces, supposing the beam to be supported at the ends only, may be applied in cases where the value

of I varies, provided the deflections for the necessary equations are determined in accordance with the principles in Art. 99. Generally, a graphical method will be the simplest for determining the deflections. Full details of a numerical example will be found in Dr. Wilson's paper already referred to, where the deflections are found by a novel graphical method.

**107. Advantages and Disadvantages of Continuous Beams.**—An examination of Figs. 162 and 165 and other diagrams of bending moment for continuous girders which the reader may sketch, shows that generally (1) the greatest bending moment to which the beam is subjected is less than that for the same spans if the beam were cut at the supports into separate pieces; (2) disregarding algebraic sign, the average bending moment throughout is smaller for the continuous beam, and less material to resist bending is therefore required; (3) in the continuous beam the bending moment due to external load is not greatest at points remote from the supports, but at the supports; hence, in girders of variable cross-section, the heavy sections are not placed in positions where their effect in producing bending stress is greatest.

On the other hand, a small subsidence of one or more supports may cause serious changes in the bending moment and bending stresses at particular sections, as well as changes of sign in bending moment and bending stresses over considerable lengths, with change in position of the points of contraflexure. These changes, resulting from very small changes in level of a support, form serious objections to the use of continuous girders. Another practical objection in the case of built-up girders is the difficulty in attaining the conditions of continuity during construction or renewal, or of determining to what degree the conditions are attained. In a loaded continuous girder two points of contraflexure usually occur between two consecutive supports; if at these two points the girder is hinged instead of being continuous, the bending moment there remains zero, and changes in load or subsidence of a support do not produce changes in sign of the bending moment and bending stresses. This is the principle of the cantilever bridge (see Art. 150), although the girder is not solid, but of the braced type dealt with in later chapters: the portions between the hinges are under the conditions of a beam simply supported at its ends, and the portions adjoining the piers are practically cantilevers which carry the simply supported beams at their ends. The points of zero bending moment being fixed, the bending-moment diagrams become very simple. For cantilever bridges and continuous braced girders, see Chap. XIII.

**108. Resilience of Beams.**—When a beam is bent within the elastic limits, the material is subjected to varying degrees of tensile and compressive bending stress, and therefore possesses elastic strain energy (Art. 34), *i.e.* it is a spring, although it may be a stiff one. The total flexural resilience (see Art. 34) may be calculated in various ways; it may conveniently be expressed in the form

$$c \times \frac{p^2}{E} \times \text{volume of the material of the beam} \quad . \quad . \quad (1)$$

where $p$ is the maximum intensity of direct stress to which the beam is subjected anywhere, and $c$ is a coefficient depending upon the manner in which the beam is loaded and supported, but which is always less than the value $\frac{1}{2}$, which is the constant for uniformly distributed stress (see Art. 34). If $f$ is the intensity of stress at the elastic limit of the material, then

$$c \times \frac{f^2}{E} \times \text{volume}$$

is the proof resilience of the beam.

For a beam of any kind supporting only a concentrated load W, the resilience is evidently

$$\tfrac{1}{2}.\mathrm{W} \times (\text{deflection at the load}) \quad . \quad . \quad . \quad . \quad (2)$$

*e.g.* a cantilever carrying an end load W has a deflection

$$\frac{\mathrm{W}l^3}{3\mathrm{EI}} \text{ (see (2), Art. 95)}$$

hence the resilience is

$$c \times \frac{p^2}{E} \times \text{volume} = \tfrac{1}{2} \cdot \frac{\mathrm{W}^2 l^3}{3\mathrm{EI}}$$

If the beam is of rectangular section, the breadth being $b$ and the depth $d$

$$p = \mathrm{W}l \div \tfrac{1}{6}bd^2$$
$$\text{volume} = bdl$$

and $$\mathrm{I} = \tfrac{1}{12}bd^3$$

hence $$c = \tfrac{1}{18}, \text{ or resilience} = \tfrac{1}{18} \cdot \frac{p^2}{E} \cdot bdl \quad . \quad . \quad . \quad . \quad (3)$$

For any shape of cross-section, if the radius of gyration about the neutral axis is $k$—

since $$p = \mathrm{W}l \div \frac{2\mathrm{I}}{d} \text{ and area of section} = \mathrm{I} \div k^2$$

from (1)

$$\text{resilience} = c \times \frac{\mathrm{W}^2 l^2 d^2}{4\mathrm{I}^2 E} \times \frac{\mathrm{I}}{k^2} \times l = \tfrac{1}{2} \times \frac{\mathrm{W}^2 l^3}{3\mathrm{EI}}$$

hence $$c = \tfrac{2}{3}\left(\frac{k}{d}\right)^2 \text{ and resilience} = \tfrac{2}{3} \cdot \frac{k^2}{d^2} \cdot \frac{p_2}{E} \times \text{volume}$$

*e.g.* for the rectangular section $\left(\frac{k}{d}\right)^2 = \frac{1}{12}$, for standard I sections $\frac{k}{d}$ is usually about 0·4.

The same coefficients, etc., as those above will evidently hold for a beam simply supported at its ends, and carrying a load midway between them.

If all the dimensions are in inches and the loads in tons, the resilience will be in inch-tons.

If with the notation of Art. 93, in a short length of beam $dx$, over which the bending moment is M, the change of slope is $di$, the elastic strain energy of that portion is

$$\tfrac{1}{2} \cdot M \cdot di \quad \ldots \ldots \ldots \quad (4)$$

and over a finite length the resilience is—

$$\tfrac{1}{2}\int M di \quad \ldots \ldots \ldots \quad (5)$$

which may also be written

$$\tfrac{1}{2}\int M\frac{di}{dx}dx = \tfrac{1}{2}\int M\frac{d^2y}{dx^2}dx = \tfrac{1}{2}\int \frac{M^2}{EI}dx \quad \ldots \ldots \quad (6)$$

or, if EI is constant—

$$\frac{1}{2EI}\int M^2 dx. \quad \ldots \ldots \ldots \quad (7)$$

From these forms the resilience of any beam may be found when the bending-moment diagram is known. For a beam of uniform section and length $l$, subjected to "simple bending" (see Arts. 61 and 92), for which the bending moment and curvature are constant, the resilience, from (4) or (7), is—

$$\tfrac{1}{2}M \times \text{change in inclination of extreme tangents} = \tfrac{1}{2}\frac{M^2 l}{EI}. \quad (8)$$

If such a beam is rectangular in section, the breadth being $b$ and the depth $d$, $p = M \div \tfrac{1}{6}bd^2$, and in the form (1), the resilience, from (7), is

$$c \times \frac{p^2}{E} \times \text{volume or } c \times \frac{36M^2}{Eb^2d^4} \times bdl = \tfrac{1}{2} \cdot \frac{M^2 l \times 12}{Ebd^3}$$

hence $c = \tfrac{1}{6}$, and the resilience $= \frac{1}{6}\frac{p^2}{E}\, bdl$

The same coefficient ($\frac{1}{6}$) will hold for any of the rectangular beams of uniform bending strength, in which the same maximum intensity of skin stress $p$ is reached at every cross-section, and which bend in circular arcs. For circular sections the corresponding coefficient is $\frac{1}{8}$.

In the case of a distributed load $w$ per unit length of span, the resilience corresponding to (2) may be written

$$\tfrac{1}{2}\int wydx \quad \ldots \ldots \ldots \quad (9)$$

where $y$ is the deflection at a distance $x$ from the origin.

*Beam Deflections calculated from Resilience.*—In equation (2) the deflection has been used to calculate the elastic strain energy. Similarly, if the resilience is calculated from the bending moments by (5) or (7), the deflections may be obtained from the resilience. For example, in the case given in Art. 96, of a non-central load W on a simply supported beam, using the notation of Art. 96 and Fig. 145,

taking each end as origin in turn, and integrating over the whole span, using (7)

$$\tfrac{1}{2} \cdot y_c \cdot W = \tfrac{1}{2}\int \frac{M^2}{EI} dx = \frac{1}{2EI}\int_0^a \left(\frac{bW}{a+b}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{aW}{a+b}x\right)^2 dx$$

hence
$$y_c = \frac{Wa^2b^2}{3(a+b)EI}$$

which agrees with (8), Art. 96.

Taking as a second example the case (*b*), Art. 94, and Fig. 139, of a uniformly spread load $w$ per unit span on a beam simply supported at each end, at a distance $x$ from either support

$$M = \frac{w}{2}(lx - x^2) \text{ (see Fig. 81)}$$

To find the deflection at a distance $a$ from one end, consider the effect of a very small weight W placed at that section. It would cause an additional bending moment

$$EI\frac{d^2y}{dx^2} \text{ or } EI\frac{di}{dx} = \frac{l-a}{l} \cdot W \cdot x$$

at a distance $x$ from the end anywhere over the range of length $a$; hence over this portion

$$di = \frac{l-a}{l} \cdot \frac{Wx}{EI} dx$$

and similarly for the remainder at a distance $x$ from the other end

$$di = \frac{a}{l} \cdot \frac{Wx}{EI} \cdot dx$$

Hence from (5) the total increase of strain energy in the whole beam due to W would be—

$$\tfrac{1}{2}\int M di = \tfrac{1}{2}\frac{W}{EIl} \cdot \frac{w}{2}\left\{(l-a)\int_0^a (lx^2 - x^3)dx + a\int_0^{l-a} (lx^2 - x^3)dx\right\}$$
$$= \tfrac{1}{2} \cdot W \cdot y$$

Reducing this

$$y = \frac{wa(l-a)}{24EI}(l^2 + la - a^2)$$

which agrees with (9), Art. 94, when $x$ is written instead of $a$.

Generalising this for any type of beam, take W = 1, and let $m$ be the bending moment at any section due to unit weight at the particular section the deflection at which is $y$, then $di = \frac{m}{EI} \cdot dx$.

$$\tfrac{1}{2} \times 1 \times y = \tfrac{1}{2}\int M di = \tfrac{1}{2}\int \frac{Mm}{EI} dx \quad \text{or} \quad y = \int \frac{Mm}{EI} dx \quad . \quad (10)$$

the integration being over the whole length of the beam and if necessary divided into separate ranges with convenient origins. In the particular case of the deflection under a load W, $M = Wm$, and—

$$y = W\int \frac{m^2}{EI}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

EXAMPLE.—A beam of rectangular section is supported at its ends, and carries a uniformly distributed load. Find the resilience in terms of the greatest intensity of stress, and the volume of the beam.

Using the notation of Fig. 81

$$-M = \frac{w}{2}(lx - x^2)$$

the total resilience from (7) is

$$\frac{1}{2EI}\int M^2dx = \frac{1}{2EI}\cdot\frac{w^2}{4}\int_0^l (l^2x^2 - 2lx^3 + x^4)dx = \frac{w^2l^5}{240EI}$$

If the breadth of section is $b$ and the depth $d$, the greatest intensity of stress $p$ occurring at mid-span is $\frac{1}{8}wl^2 \div \frac{1}{6}bd^2 = \frac{3}{4}\frac{wl^2}{bd^2}$

and $$c.\frac{p^2}{E}.\text{volume or } c.\frac{9}{16}.\frac{w^2l^4}{Eb^2d^4}.bdl = \frac{w^2l^5 \times 12}{240Ebd^3}$$

hence $$c = \frac{4}{45} \text{ and resilience} = \frac{4}{45} \times \frac{p^2}{E} \times \text{volume}$$

This might also be obtained as the sum

$$\tfrac{1}{2}\int_0^l wydx$$

using the expression (9) of Art. 94 for $y$.

**109. Elastic Energy in Shear Strain; Shearing Resilience.**—When material suffers shear strain within the elastic limit, elastic strain energy is stored just as in the case of direct stress and strain. For simple distributions of shear stress the resilience or elastic strain energy is easily calculated. Let Fig. 9 represent a piece of material of length $l$ perpendicular to the plane of the diagram, having uniform shear stress of intensity $q$ on the face BC, causing shear strain $\phi$ and deflection BB″.

Then the resilience evidently is

$$\tfrac{1}{2} \times (\text{force}) \times (\text{distance}) = \tfrac{1}{2} \times (BC.l.q) \times BB'' = \tfrac{1}{2}.BC.l.q.AB\phi$$

$$= \tfrac{1}{2}.BC.l.AB.\frac{q^2}{N}$$

$$= \tfrac{1}{2}.\frac{q^2}{Nl} \times \text{volume or } \tfrac{1}{2}\frac{q^2}{N} \text{ per unit of volume}$$

where N is the modulus of rigidity.

Note the similarity to the expression $\frac{1}{2}\frac{p^2}{E}$ per unit volume, which is the resilience for uniformly distributed direct stress (Art. 34).

**110. Deflection of a Beam due to Shearing.**—In addition to the ordinary deflections due to the bending moment calculated in Chap. VI., there is in any given case other than "simple bending" (Art. 64) a further deflection due to the vertical shear stress on transverse sections of a horizontal beam. This was not taken into account in the calculations of Chap. VI., and the magnitude of it in a few simple cases may now be estimated.

In the case of a cantilever of length $l$ carrying an end load W (Fig. 75), if the shearing force F ( = W) were uniformly distributed over vertical sections, the deflections due to shear at the free end would be

$$l \times \text{(angle of shear strain)}$$

or

$$\phi . l = \frac{q}{N} l \text{ or } \frac{Wl}{AN}$$

where A is the area of cross-section. If the section were rectangular, of breadth $b$ and depth $d$, the deflection with uniform distribution would be $\frac{Wl}{bd.N}$.

But we have seen (Art. 72) that the shear stress is not uniformly distributed over the section, but varies from a maximum at the neutral surface to zero at the extreme upper and lower edges of the section. The consequence is that the deflection will be rather more than $\frac{Wl}{AN}$. We can get some idea of its amount in particular cases from the distribution of shear stress calculated in Art. 72. But it should be remembered that such calculations are based on the simple theory of bending (see Art. 64), and are approximate only. While the simple (or Bernoulli-Euler) theory gives the deflections due to the bending moment with sufficient accuracy, the portion of the total deflection which is due to shearing cannot generally be estimated with equal accuracy from the distribution of shear stress deduced in Art. 72. It becomes desirable, then, to check the results by those given in the more complex theory of St. Venant (see Art. 64) if a very accurate estimate of shearing deflection is required. In a great number of practical cases, however, the deflection due to shearing is negligible in comparison with that caused by the bending moment. Assuming the distribution of sheer stress to be as calculated in Art. 72, and constant over a narrow strip of the cross-section parallel to the neutral axis of the section, a few deflections due to shear will now be calculated for cases where the shearing force is uniform, and for which the simple theory of bending is approximately correct (see Art. 64).

*Cantilever of Rectangular Section with End Load.*—The breadth being $b$ and the depth $d$, a longitudinal strip of length $l$, with $b$, and thickness $dy$, parallel to the neutral surface and distant $y$ from it, will store strain energy—

$$\tfrac{1}{2} \cdot \frac{q^2}{N} \cdot b . l . dy \text{ (see Art. 109)}$$

due to shear strain. And from (4), Art. 72

$$q = \frac{6F}{bd^3}\left(\frac{d^2}{4} - y^2\right)$$

where F = W, the end load.

Hence $$q^2 = \frac{36W^2}{b^2d^2}\left(\frac{d^4}{16} + y^4 - \frac{d^2y^2}{2}\right)$$

The total shearing resilience in the cantilever is

$$\frac{bl}{2N}\int_{-\frac{d}{2}}^{\frac{d}{2}} q^2 dy = \frac{18W^2l}{Nbd^6}\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(\frac{d^4}{16} - \frac{d^2y^2}{2} + y^4\right)dy \quad . \quad . \quad (1)$$

or $$\frac{36W^2l}{Nbd^6}\left(\frac{yd^4}{16} - \frac{y^3d^2}{6} + \frac{y^5}{5}\right)_0^{\frac{d}{2}} = \frac{3}{5}\frac{W^2l}{Nbd}$$

If $\delta$ be the deflection at the free end due to shearing, the shearing resilience is $\frac{1}{2} \cdot W \cdot \delta = \frac{3}{5}\frac{W^2l}{Nbd}$, hence

$$\delta = \frac{6}{5}\frac{Wl}{Nbd} = \frac{6}{5} \times \left(\frac{\text{mean value of } q}{N}\right) \times (l)$$

which is 20 per cent. greater than it would be with uniformly distributed shear stress.

Similarly, for a beam simply supported at its ends and of length $l$, carrying a central load W, putting $\frac{l}{2}$ for $l$, and $\frac{W}{2}$ for W, the shearing deflection is

$$\frac{3}{10} \cdot \frac{Wl}{Nbd}$$

or the total deflection due to bending and shearing is

$$\frac{Wl^3}{48EI} + \frac{3}{10}\frac{Wl^3}{Nbd} = \frac{Wl}{4Ebd^3}\left\{1 + \frac{6}{5}\frac{E}{N}\left(\frac{d}{l}\right)^2\right\}$$

or if $\frac{E}{N} = \frac{5}{2}$, this becomes—

$$\frac{Wl^3}{4Ebd^3}\left\{1 + 3\left(\frac{d}{l}\right)^2\right\}$$

or for the cantilever—

$$\frac{4Wl^3}{Ebd^3}\left\{1 + \frac{3}{4}\left(\frac{d}{l}\right)^2\right\}$$

The second term is negligible if $\left(\frac{l}{d}\right)$ is large, which is generally the case in practice. This expression for the shearing deflection is in fair

agreement with the more exact expression deduced by St. Venant,[1] provided the breadth is not great compared with the depth.

*Distributed Loads.*—With a distributed load the simple theory of bending does not hold with the same accuracy as when the vertical shearing force on the cross-sections is constant throughout the length (see Art. 64). Neglecting this, however, the resilience due to shear strain of an element of length $dx$ would be—

$$\tfrac{1}{2}\frac{q^2}{\mathrm{N}} . s . dy . dx$$

If $s$ and $y$ are not functions of $x$, *i.e.* if the sections of the beam are constant throughout its length, the effect of integrating the energy expressions throughout with respect to $x$ will be to multiply the previous values for the cantilever by the ratio of $\int_0^l \mathrm{F}^2 dx$ to $l\mathrm{W}^2$. For example, in the case of a uniformly distributed load $w$ per unit length at a distance $x$ from the free end $\mathrm{F}^2 = w^2x^2$, hence the above ratio is $\frac{1}{3}w^2l^3$ to $l\mathrm{W}^2$ or $\frac{1}{3}\left(\frac{wl}{\mathrm{W}}\right)^2$, the effect of a distributed load being $\frac{1}{3}$ that of the same load concentrated at the end. The same coefficient will evidently hold good for a beam freely supported at its ends, and uniformly loaded, compared to similar beam carrying the same load concentrated midway between the supports.

**I**-*Section Girders.*—The cases in which the shearing deflections are of more importance are the various built-up sections of which girders are made, particularly when the depth is great in proportion to the length. In an **I**-girder section, for example, the intensity of shear stress in the web is (see Art. 72) much greater than the mean intensity of shear stress over the section. A common method of roughly estimating the total deflection of large built-up girders is to calculate for ordinary bending deflection, using a value of E about 25 per cent. below the usual value to allow for shearing, etc.

*Any Section.*—For any solid section instead of (1) the elastic energy $\frac{1}{2}\mathrm{W}\delta$ would be—

$$\tfrac{1}{2}\mathrm{W}\delta = \frac{l}{2\mathrm{N}}\int_{-\frac{d}{2}}^{\frac{d}{2}} q^2 z dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $z$ is the breadth of the section at a depth $y$, as in Art. 71, and $q = \frac{\mathrm{F}}{\mathrm{I}z}\int_y^{\frac{d}{2}} yzdy$, as in Art. 72, hence the strain energy—

$$\tfrac{1}{2} . \mathrm{W} . \delta = \frac{l}{2\mathrm{N}}\int_{-\frac{d}{2}}^{\frac{d}{2}} \left\{\frac{\mathrm{F}^2}{\mathrm{I}^2 z}\left(\int_y^{\frac{d}{2}} yzdy\right)^2\right\} dy \quad . \quad . \quad . \quad (3)$$

[1] See Todhunter and Pearson's "History of Elasticity," vol. ii. Arts. 91 and 96.

or for the cantilever symmetrical about the neutral axes of the sections with end load W, where F = W—

$$\tfrac{1}{2}.W.\delta = \frac{W^2 l}{I^2 N}\int_0^{\frac{d}{2}} \frac{1}{z}\left(\int_y^{\frac{d}{2}} yz\,dy\right)dy \text{ and } \delta = \frac{2Wl}{I^2 N}\int_0^{\frac{d}{2}} \frac{1}{z}\left(\int_y^{\frac{d}{2}} yz\,dy\right)^2 dy$$

For a simply supported beam of span $l$ and central load W, the deflection would be $\frac{1}{4}$ of the above expression.

For sections the width ($z$) of which cannot be simply expressed as a function of the distance ($z$) from the neutral surface, a graphical method will be most convenient. The values of $q$ may be found as in Art. 72 and Fig. 110. A diagram, somewhat similar to Fig. 110, may then be plotted, the ordinates of which are proportional to $q^2 \times z$, by squaring

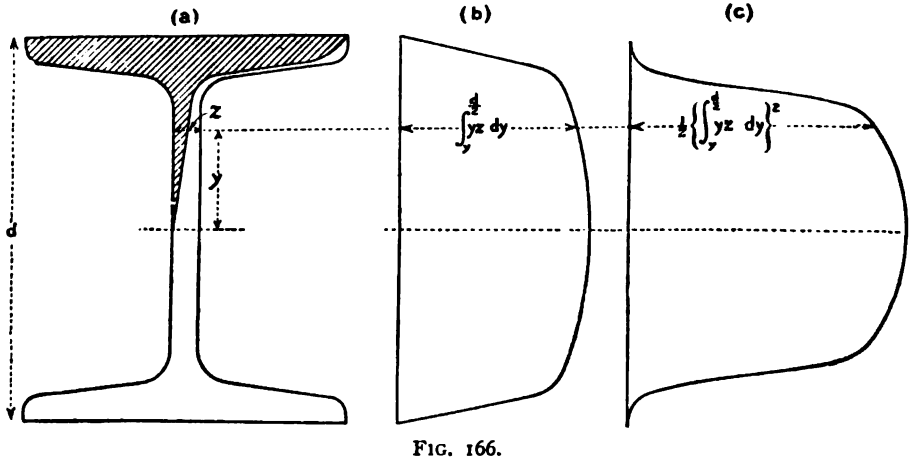



FIG. 166.

the ordinates of Fig. 110 and multiplying each by the corresponding width of the section. The total area of this diagram would represent $\int_{-\frac{d}{2}}^{\frac{d}{2}} q^2 z\,dy$, and the deflection of, say, a cantilever may be found from it by multiplying by $\frac{l}{N}$ and dividing by W. If the diagram of $q$ is not required it is rather more convenient to proceed as follows (see Fig. 166). Draw the ordinary modulus figure for the section as shown at ($a$), and plot a diagram ($b$) showing $q \,.\, z$ instead of $q$, on the depth of the beam as a base line. Equation (3), Art. 72, shows that at any height $y$ from the neutral axis

$$qz = \frac{W}{I} \times \left(\text{area of modulus figure between } y \text{ and } \frac{d}{2}\right)$$

from which equation the ordinates of (*b*) may be found by measuring areas on Fig. (*a*). Square the ordinates of this diagram (*b*), and *divide* each by the width $z$ and plot the results as ordinates of the diagram (*c*) on the depth $d$ as a base. The area of the resulting figure (*c*) represents $\int_{-\frac{d}{2}}^{\frac{d}{2}} q^2 z dy$ as before, and the deflection (see (*a*) above) is found by multiplying by $\frac{l}{N}$ and dividing by W for a cantilever with an end load, and is $\frac{1}{4}$ of this for a beam of length $l$ supported at its ends and carrying a central load W, provided W is used as above in finding $qz$, or $\frac{1}{2}$ this if $\frac{W}{2}$, the actual shearing force, is used in finding $qz$.

It is, of course, not necessary to actually plot the diagram (*b*).

*Scales.*—Fig. 166 (*a*) being drawn full size, the width of the modulus figure represents $\frac{2y}{d} \times z$. If $p$ square inches of modulus figure area at (*a*) are represented by 1-inch ordinates on (*b*), the ordinates represent $\int_{y}^{\frac{d}{2}} yzdy$ on a scale of 1 inch $= p \times \frac{d}{2}$ (inches)$^3$. If the ordinates of (*b*) in inches are square and divided by $n$, say, for convenience, and then plotted in inches, on Fig. (*c*), the area of Fig. (*c*) represents $\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{1}{z}\left(\int_{y}^{\frac{d}{2}} yzdy\right)^2 dy$ on a scale of 1 square inch $= n\left(p \cdot \frac{d}{2}\right)^3$, the units being (inches)$^6$.

To obtain, say, the cantilever deflection, it is only necessary to multiply the result in (inches)$^6$ by $\frac{Wl}{I^2N}$, the unit of which are (inches)$^{-5}$, when inch units are used for $l$, I, and N, to obtain the deflection in inches. For the centrally loaded beam the factor would be $\frac{1}{4}\frac{Wl}{I^2N}$. Fig. 166, when drawn full size, represents the British Standard Beam section, No. 10, for which $d = 6$ inches, I $= 43{\cdot}61$ (inches)$^4$, and the web is $0{\cdot}41$ inch thick: the area of the diagram (*c*) represents 761 (inches)$^6$, and the shearing deflection of a cantilever would be $0{\cdot}416\frac{Wl}{N}$ inches.

The deflection due to shearing of an **I** beam with square corners such as Fig. 109 may be found by integration in two ranges over which the breadth is constant (see example below), and this method might be used as an approximate for any **I** section by using mean values for the thickness of the flanges and web: an example is given below.

*Simple Approximation for* **I** *Sections.*—Owing to the limitations of the simple theory of bending, none of these calculations can be regarded

as correct, and perhaps the simplest approximation may also be the best, viz. to calculate the deflection due to shear as if the web carried the whole shearing force with uniform distribution, so that for a cantilever

$$\delta = \frac{Wl}{A\bar{N}}$$

and for a beam simply supported at its ends

$$\delta = \frac{Wl}{4A\bar{N}}$$

where A is the area *of the web* and $l$ is the length of the beam, all the linear units being, say, inches.

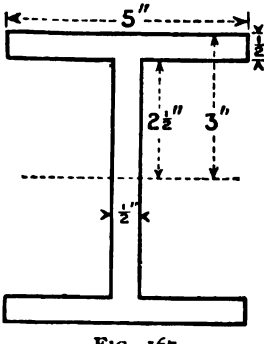



FIG. 167.

EXAMPLE.—Find the ratio of the deflections due to shearing and bending in a cantilever of I section, 6 inches deep and 5 inches wide, the flanges and web each ½ inch thick, carrying an end load, $\frac{E}{\bar{N}}$ being taken as $\frac{5}{9}$ . I = 43·125 (inches)⁴ (see Fig. 167). In the flanges

$$q = \frac{W}{I}\int_y^3 y\,dy = \frac{W}{2I}(9 - y^2)$$

$$q^2 = \frac{W^2}{4I^2}(81 - 18y^2 \times y^4)$$

In the web

$$q = \frac{W}{\frac{1}{2}I}\left(\int_{2\frac{1}{2}}^3 5y\,.\,dy + \int_y^{2\frac{1}{2}} \tfrac{1}{2}\,.\,y\,dy\right) = \frac{2W}{I}\left(\tfrac{135}{16} - \frac{y^2}{4}\right) = \frac{W}{I}\left(\tfrac{135}{8} - \frac{y^2}{2}\right)$$

$$q^2 = \frac{W}{I^2}\left(\tfrac{18225}{64} - \tfrac{135}{8}y^2 + \frac{y^4}{4}\right)$$

Taking both sides of the neutral axis, the total shearing resilience is by (2)

$$\tfrac{1}{2}W\,.\,\delta = \frac{l}{2\bar{N}}\int_{-3}^{3} q^2 z\,dy = 2\,\frac{l}{2\bar{N}}\left\{\frac{5W^2}{4I^2}\int_{2\frac{1}{2}}^3 (81 - 18y^2 + y^4)\,dy + \frac{W^2}{2I^2}\int_0^{2\frac{1}{2}}\left(\tfrac{18225}{64} - \tfrac{135}{8}y^2 + \frac{y^4}{4}\right)dy\right\}$$

$$\delta = \frac{2lW}{\bar{N}I^2}(1{\cdot}65 + 314{\cdot}5) = \frac{632Wl}{I^2\bar{N}} = 0{\cdot}340\,\frac{Wl}{\bar{N}}$$

(This agrees closely with the result given for Fig. 166, being less in about the same proportion that the web thickness is greater, I being nearly the same in each.)

Ratio of deflections $\frac{\textit{shearing}}{\textit{bending}} = \frac{632Wl}{I^2\bar{N}} \times \frac{3EI}{Wl^3} = \frac{1896}{I}\,.\,\frac{N}{E}\,.\,\frac{I}{l^2}$, and

taking $I = 43{\cdot}125$ and $\frac{E}{N} = \frac{5}{2}$, this ratio is $\frac{110}{l^2}$ nearly. For a simply supported beam of span $l$ the ratio would be $\frac{440}{l^2}$, and if the span were 10 times the depth, or 60 inches, the ratio would be $\frac{440}{3600}$, or over 12 per cent.

## EXAMPLES VIII.

1. A beam is firmly built in at each end and carries a load of 12 tons uniformly distributed over a span of 20 feet. If the moment of inertia of the section is 220 inch units and the depth 12 inches, find the maximum intensity of bending stress and the deflection. (E = 13,000 tons per square inch.)

2. A built-in beam carries a distributed load which varies uniformly from nothing at one end to a maximum $w$ per unit length at the other. Find the bending moment and supporting forces at each end and the position where maximum deflection occurs.

3. A built-in beam of span $l$ carries two loads each W units placed $\frac{1}{3}l$ from either support. Find the bending moment at the supports and centre, the deflection at the centre and under the loads, and find the points of contraflexure.

4. A built-in beam of span $l$ carries a load W at a distance $\frac{1}{3}l$ from one end. Find the bending moment and reactions at the supports, the deflection at the centre and under the load, the position and amount of the maximum deflection, and the position of the points of contrary flexure.

5. A built-in beam of 20-feet span carries two loads, each 5 tons, placed 5 feet and 13 feet from the left-hand support. Find the bending moments at the supports.

6. A built-in beam of span $l$ carries a uniformly distributed load $w$ per unit of length over half the span. Find the bending moment at each support, the points of inflection, the position and magnitude of the maximum deflection.

7. The moment of inertia of cross-section of a beam built in at the ends varies uniformly from $I_0$ at the centre to $\frac{1}{2}I_0$ at each end. Find the bending moment at the end and middle, and the central deflection when a load W is supported at the middle of the span.

8. Solve the previous problem when the load W is uniformly distributed over the span.

9. A continuous beam rests on supports at its ends and two other supports on the same level as the ends. The supports divide the length into three equal spans each of length $l$. If the beam carries a uniformly spread load W per unit length, find the bending moments and reactions at the supports.

10. A continuous beam covers three consecutive spans of 30 feet, 40 feet, and 20 feet, and carries loads of 2, 1, and 3 tons per foot run respectively on the three spans. Find the bending moment and pressure at each support. Sketch the diagrams of bending moment and shearing force.

11. A continuous beam ABCD 20 feet long rests on supports A, B, C, and D, all on the same level, AB = 8 feet, BC = 7 feet, CD = 5 feet. It carries loads of 7, 6, and 8 tons at distances 3, 11, and 18 feet respectively from A. Find the bending moment at B and C, and the reactions at

A, B, C, and D. Sketch the bending-moment diagram. (The results should be checked by using both methods given in Art. 90.)

12. Solve problem No. 9, (*a*) if one end of the beam is firmly built in, (*b*) if both ends are built in.

13. Solve problem No. 11, the end A being fixed horizontally.

14. Solve problem No. 11, if the support B sinks $\frac{1}{10}$ inch, I being 90 (inches)$^4$ and E = 13,000 tons per square inch.

15. If the limits of safe bending stress for steel and ash are in the ratio 8 to 1, and the direct moduli of elasticity for the two materials are in the ratio 20 to 1, compare the proof resilience per cubic inch of steel with that for ash and where both are bent in a similar manner. If steel weighs 480 lbs. per cubic foot, and ash 50 lbs. per cubic foot, compare the proof resilience of steel with that of an equal weight of ash.

16. A beam of I section is 20 inches deep and $7\frac{1}{2}$ inches broad, the thickness of web and flanges being 0·6 inch and 1 inch respectively. If the beam carries a load at the centre of a 20-feet span, find approximately what proportion of the total deflection is due to shearing if the ratio $\frac{E}{N} = 2{\cdot}5$.

# CHAPTER IX

## *DIRECT AND BENDING STRESSES*

**111. Combined Bending and Direct Stress.**—It often happens that the cross-section of a pillar or a tie-rod mainly subjected to a longitudinal thrust or pull has in addition bending stresses across it, the pillar or tie-rod suffering flexure in an axial plane; or that the cross-section of a beam resisting flexure has brought upon it further direct stress due to an end thrust or pull, the loads on the beam not being all transverse ones, such as were supposed in Chapters IV. and V., but such as make the beam also a strut or a tie. In either case the resultant longitudinal intensity of stress at any point in a cross-section will be the algebraic sum of the direct stress of tension or compression and the direct stresses due to bending. If $p$ is the intensity of stress anywhere on a section subjected to an end load—

$$p = p_0 + p_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $p_0$ is the total end load divided by the area of cross-section, and $p_b$ is the intensity of bending stress as calculated from the bending moments for purely transverse loading in Art. 63, and is of the same sign as $p_0$ in part of the section and of opposite sign in another part. The stress intensity $p$ will change sign somewhere in the section if the extreme values of $p_b$ are of greater magnitude than $p_0$, but the stress will not be zero at the centroid of the section as in the case of a beam bent only by transverse forces. The effect of the additional direct stress $p_0$ is to change the position of the neutral surface or to remove it entirely.

**112. Eccentric Longitudinal Loads.**—If the line of action of the direct load on a prismatic bar is parallel to the axis of the bar, and intersects an axis of symmetry of the cross-section at a distance $h$ from the centroid of the section, bending takes place in the plane of the axis of the bar and the line of action of the eccentric load. Thus, Fig. 168 represents the cross-section of a bar, the load P passing through the point C, and O is the centroid of the section. Let A be the area of cross-section, and $y_1$ the distance OD from the centroid O to the extreme edge D in the direction OC, and let I be the moment of inertia of the area of section about the central axis FG perpendicular to OC. Then, in addition to the direct tension or compression $\frac{P}{A}$ or $p_0$, there is a

bending moment $M = P.h$ on the section, the intensity of stress at any point distant $y$ from FG being—

$$p = p_0 + p_b = \frac{P}{A} + \frac{P.h.y}{I} \text{ (Art. 63)}$$

or since $I = Ak^2$, where $k$ is the radius of gyration about FG—

$$p = \frac{P}{A} + \frac{Phy}{Ak^2} = \frac{P}{A}\left(1 + \frac{h.y}{k^2}\right) \text{ or } p_0\left(1 + \frac{h.y}{k^2}\right) . \quad . \quad (1)$$

$y$ being positive for points on the same side of FG as C, and negative on the opposite side. The intensity varies uniformly with the dimension $y$, as shown in Figs. 168, 169.

The extreme stress intensities at the edges of the section will be—

$$p_0 + f_1 \text{ and } p_0 - f_1'$$

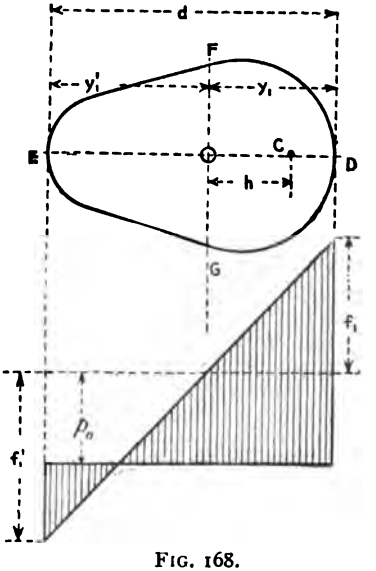



FIG. 168.

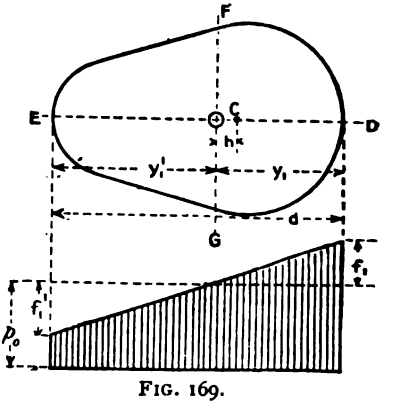



FIG. 169.

where $f_1$ and $f_1'$ are the opposite extreme values of $p_b$, or if $y_1$ and $y_1'$ are the distances of the extreme edges from the centroid O, the extreme stress intensities of stress are—

$$p = p_0\left(1 + \frac{hy_1}{k^2}\right) \text{ and } p = p_0\left(1 - \frac{hy_1'}{k^2}\right) . \quad . \quad . \quad (2)$$

on the extreme edges D and E, the former being on the same side of the centroid as C, and the latter on the opposite side. If the section is symmetrical about FG—

$$y_1 = y_1' = \frac{d}{2}$$

Evidently $p = 0$ for $y = -\frac{k^2}{h}$ if this distance is within the area of cross-section, *i.e.* if $\frac{k^2}{h}$ is less than $y_1'$ the distance from the centroid to

the edge E opposite to C. An axis parallel to FG and distant $\frac{k^2}{h}$ from it on the side opposite to C might be called the neutral axis of the section, for it is the intersection of the area of cross-section by a surface along which there is no direct longitudinal stress. The uniformly varying intensity of stress where $h$ is greater than $\frac{k^2}{y_1}$ is shown in Fig. 168. If $\frac{k^2}{h}$ is greater than $y_1'$, *i.e.* if $h$ is less than $\frac{k^2}{y_1'}$ the stress throughout the section is of the same kind as $p_0$; this uniformly varying distribution of stress is shown in Fig. 169. With loads of considerable eccentricity, it should be noted, such metals as cast iron, which are strong in compression, ultimately fail in tension under a compressive load.

*Rectangular Section.*—In the rectangular section of breadth $b$ and depth $d$, shown in Fig. 170, in order that the stress on the section shall be all of the same sign, the maximum deviation in the direction OE of the line of action of the resultant stress from the line GH through the centroid is—

$$h = k^2 \div y = \tfrac{1}{12}d^2 \div \frac{d}{2} = \frac{d}{6}$$

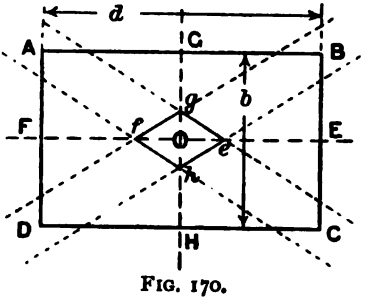


FIG. 170.

From this result springs the well-known rule for masonry, in which no tension is allowed—that across a rectangular joint (see Art. 213) the resultant thrust across the joint must fall within $\frac{1}{6}$ of the thickness from the centre line of the joint, or *within the middle third.* The limiting deviation in the direction OG under the same conditions is $\frac{1}{6}b$.

If the line of action of the stress is on neither of the centre lines of the section, the bending is unsymmetrical, and may conveniently be resolved in the planes of the two principal axes as in Art. 71. If the line of action of P fall in the quarter GOEB say, at a point the co-ordinates of which, referred to OE and OG as axes, are $x$ and $y$ reckoned positive toward E and G respectively, the bending moment about OE is $P.y$, and about OG is $P.x$, and the stress at any point in the section the co-ordinates of which are $x'$, $y'$, is—

$$p = \frac{P}{bd} + \frac{P.y.y'}{\frac{1}{12}bd^3} + \frac{P.x.x'}{\frac{1}{12}db^3} = \frac{12P}{bd}\left(\tfrac{1}{12} + \frac{y.y'}{b^2} + \frac{x.x'}{d^2}\right) \quad . \quad (3)$$

The least value of this is evidently always at D, where $x' = -\frac{d}{2}$ and $y' = -\frac{b}{2}$ when the least value of $p$ is—

$$\frac{6P}{bd}\left(\tfrac{1}{6} - \frac{y}{b} - \frac{x}{d}\right)$$

This just reaches zero when

$$\frac{y}{b}+\frac{x}{d}=\tfrac{1}{6},\text{ or } y=-\frac{b}{d}x+\frac{b}{6}$$

which is the equation to the straight line joining points $g$, $\frac{b}{6}$ from O along OG, and $e$, $\frac{d}{6}$ from O along OE. Similar limits will apply in other quarters of the rectangle, and the stress will be of the same sign in all parts of the section, provided the line of the resultant load falls within a rhombus *egfh*, the diagonals of which lie along EF and GH, and are of length $\frac{d}{3}$ and $\frac{b}{3}$ respectively. This rhombus is called the *core* of the section.

*Circular Section.*—In the case of a circular section of radius R, the deviation which just produces zero stress at one point of the perimeter of the section and double the average intensity diametrically opposite is—

$$h=k^2\div \mathrm{R}=\frac{\mathrm{R}^2}{4}\div \mathrm{R}=\tfrac{1}{4}\mathrm{R}$$

and for a hollow circular section of internal radius $r$ and external radius R the deviation would be—

$$h=\frac{\mathrm{R}^2+r^2}{4\mathrm{R}}$$

which approaches the limit $\frac{1}{2}$R in the case of a thin tube.

*Other Sections.*—A more general form of (3) is evidently—

$$p=\frac{\mathrm{P}}{\mathrm{A}}\left(1+\frac{yy'}{k_x^2}+\frac{xx'}{k_y^2}\right) \quad \ldots\ldots \quad (4)$$

where $k_x$ and $k_y$ are the radii of gyration of the area of section about the axes of $x$ and $y$ respectively, and for zero stress at a point the co-ordinates of which are $x'$, $y'$—

$$\frac{yy'}{k_x^2}+\frac{xx'}{k_y^2}=-1 \quad \ldots\ldots \quad (5)$$

For a symmetrical I section of breadth $b$ in the direction of $x$, and depth $d$ in the direction of $y$, the four corners will be limiting points of zero stress, and the limits of deviation of load from the centroid for no change in sign of the stress will be the bounding line—

$$y=-\frac{k_x^2}{k_y^2}\cdot\frac{b}{d}x-\frac{2k_x^2}{d} \quad \ldots\ldots \quad (6)$$

and three others forming a rhombus having the principal axes as diagonals. Similar bounding lines will fix the deviation limits or cores for various other sections the boundaries of which can be circumscribed by polygons.

For a symmetrical I section such as Fig. 62, if the axis OY takes in as the vertical principal axis of the section, for a corner—

$$x' = \frac{b}{2} \text{ and } y' = \frac{d}{2}$$

If $x$ and $y$ are the co-ordinates of the centre of the loading, the unit stress from (4) is—

$$p = \frac{P}{A}\left(\frac{yd}{2k_x^2} + \frac{xb}{2k_y^2} + 1\right) \text{ or } \frac{p}{p_0} - 1 = \frac{yd}{2k_x^2} + \frac{xb}{2k_y^2} \quad . \quad (7)$$

For various values of $\frac{p}{p_0}$ equation (7) would represent a series of straight lines on which the load centre would lie; the inclination of the lines to the axis OX would be at an angle $\theta$ such that—

$$\tan\theta = -\frac{k_x^2}{k_y^2} \cdot \frac{b}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and equation (6) is the particular line for $p = 0$. The minimum eccentricity of loading to give any ratio $\frac{p}{p_0}$ at the corner of the section would occur when a line joining the centroid to the load centre is perpendicular to the lines represented by (7), *i.e.* inclined to the axis OX at an angle the tangent of which is—

$$\frac{k_y^2}{k_x^2} \cdot \frac{d}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

*Common examples* of eccentric loads occur in tie-bars "cranked" to avoid an obstacle, frames of machines, such as reciprocating engines, members of steel structures, and columns or pillars of all kinds; but it is to be remembered that, particularly in the case of pillars, the deviation $h$ is a variable along the length if flexure takes place. Frequently, however, in columns which are short in proportion to their cross-sectional dimensions, and in which the deviation $h$ of resultant thrust from the axis is considerable, this variation in $h$ is negligible.

*Masonry Seating for Beam Ends.*—If we assume the forces exerted by the walls on a cantilever or a built-in beam to consist of a uniform upward pressure equal to the total vertical reaction R and equal upward and downward pressures varying in intensity uniformly along the length from zero at the centre of the seating to maxima at the ends, giving a resultant couple or fixing moment, formula (1) may be applied to calculate the maximum intensity of pressure on the masonry. If $b$ be the (constant) breadth of the beam and $d$ the length of the seating, $p_0 = \frac{R}{b \cdot d}$. The moment of the seating pressures about the centroid of the seating is nearly the same as the bending moment at the entrance to the wall if the seating is short, exceeding it by $R \times \frac{d}{2}$. Taking the case of a cantilever of length $l$ carrying an end load W (Fig. 75), the

moment is $W\left(l + \frac{d}{2}\right)$; writing this for P . $h$, and $b$ . $d$ for A, and $\frac{1}{6}bd^2$ for $\frac{Ak^2}{y_1}$ in (1) or (2), the extreme intensity of pressure at the entrance to the wall is—

$$p_{max.} = \frac{W}{bd} + \frac{6W\left(l + \frac{d}{2}\right)}{bd^2} = \frac{2W}{bd}\left(2 + \frac{3l}{d}\right)$$

which serves to calculate the maximum pressure intensity if $d$ is known, or to determine $d$ for a specified value (say about 500 pounds per square inch) of the working intensity of crushing stress on the seating.

EXAMPLE 1.—In a rectangular cross-section 2 inches wide and 1 inch thick the axis of a pull of 10 tons deviates from the centre of the section by $\frac{1}{10}$ inch in the direction of the thickness, and is in the centre of the width. Find the extreme stress intensities.

The extreme bending stresses are—

$$f = \frac{M}{Z} = \frac{\frac{1}{10} \times 10}{\frac{1}{6} \times 2 \times 1} = 3 \text{ tons per square inch}$$

tension and compression along the opposite long edges of the section. To these must be added algebraically a tension of—

$$\frac{10}{2} = 5 \text{ tons per square inch}$$

hence on the side on which the pull deviates from the centroid the extreme tension is—

$$5 + 3 = 8 \text{ tons per square inch}$$

and on the opposite side the tension is—

$$5 - 3 = 2 \text{ tons per square inch}$$

Here a deviation of the load a distance of $\frac{1}{10}$ of the thickness from the centroid increases the maximum intensity of stress to 60 per cent. over the mean value.

EXAMPLE 2.—A short cast-iron pillar is 8 inches external diameter, the metal being 1 inch thick, and carries a load of 20 tons. If the load deviates from the centre of the column by $1\frac{3}{4}$ inch, find the extreme intensities of stress. What deviation will just cause tension in the pillar?

The area of section is $\frac{\pi}{4}(64 - 36) = 22{\cdot}0$ square inches

The moment of resistance to bending is equal to—

$$20 \times 1\tfrac{3}{4} = 35 \text{ ton-inches}$$

hence the extreme intensities of bending stress are—

$$35 \div \frac{\pi}{32} \cdot \left(\frac{8^4 - 6^4}{8}\right) = \frac{35 \times 8 \times 32}{\pi \times 2800} = 1{\cdot}017 \text{ tons per square inch}$$

The additional compressive stress is—

$$\frac{20}{22} = 0{\cdot}909 \text{ ton per square inch}$$

hence the maximum compressive stress is 1·017 + 0·909 = 1·926 tons per square inch, and the minimum compression is 0·909 − 1·017 = − 0·108, *i.e.* 0·108 ton per square inch tension.

If there is just no stress on the side remote from the eccentric load the deviation would be—

$$1{\cdot}75 \times \frac{0{\cdot}909}{1{\cdot}017} = 1{\cdot}56 \text{ inch}$$

EXAMPLE 3.—A short stanchion of symmetrical I section withstands a thrust parallel to its axis such that the stress would be 2 tons per square inch if the thrust were truly axial. Determine the eccentricity which would be sufficient to produce a stress of 10 tons per square inch if the section is 9 inches deep, 7 inches wide, 17·06 square inches area, the principal moments of inertia being 229·5 (inches)$^4$ and 46·3 (inches)$^4$, the former being about an axis in the direction of the breadth.

$$\text{Taking } k_x^2 = \frac{I_x}{A} = \frac{229{\cdot}5}{17{\cdot}06} = 13{\cdot}45, \; k_y^2 = \frac{46{\cdot}3}{17{\cdot}06} = 2{\cdot}714$$

and in equation (7) $\frac{p}{p_0} = \frac{10}{2} = 5$; this gives—

$$5 - 1 = 4 = \frac{4{\cdot}5y}{13{\cdot}45} + \frac{3{\cdot}5x}{2{\cdot}714} \text{ or } y = -3{\cdot}854x - 11{\cdot}96$$

as the locus of the centre of pressure to produce the extreme stress at one corner. The inclination of this locus to the horizontal principal axis is—

$$\tan^{-1}(-3{\cdot}854) = 180 - 75{\cdot}55 = 104{\cdot}45^\circ$$

and for $x = 0$, $y = -11{\cdot}96$ inches.

Hence the distance of the line from the centroid is—

$$11{\cdot}96 \cos 75{\cdot}55^\circ = 3{\cdot}00 \text{ inches}$$

in a direction inclined 14·45° to the horizontal axis. If the centre of pressure were on the horizontal axis of the I section, the deviation to produce the same extreme stress would be—

$$\frac{11{\cdot}96}{3{\cdot}854} = 3{\cdot}1 \text{ inches}$$

EXAMPLE 4.—A cantilever 8 inches broad is at the wall subjected to a shearing force of 20 tons and a bending moment of 400 ton-inches. Assuming a uniformly varying pressure between the beam and its seating, find what length of the beam must be built into the wall in order that the pressure shall not exceed $\frac{1}{4}$ ton per square inch.

Taking the upward pressure to support the shearing force and the upward pressure constituting part of the fixing couple, if $d$ is the length required—

$$\frac{1}{4} = \frac{20}{8d} + \frac{400 + \left(20 \times \frac{d}{2}\right)}{\frac{1}{6} \times 8 \times d^2}$$

$$d^2 - 40d - 1200 = 0$$

Hence $d = 60$ inches, *i.e.* the beam must be built into the wall for a length of 5 feet.

**113. Pillars, Columns, Stanchions, and Struts.**—These terms are usually applied to prismatic and similar-shaped pieces of material under compressive stress. The effects of uniformly distributed compressive stress are dealt with in Chap. II. on the supposition that the length of the strut is not great. The uniformly varying stress resulting from combined bending and compression on a short prismatic piece of material is dealt with in Arts. 111 and 112. There remain the cases in which the strut is not short, in which the strut fails under bending or buckling due to a central or to an eccentric load. Theoretical calculation for such cases is of two kinds: first, exact calculation for ideal cases which cannot be even approximately realized in practice, and secondly, empirical calculation, which cannot be rigidly based on rational theories, but which can be shown to be reasonable theoretically, as well as in a fair measure of agreement with experiments. Calculations of each kind will be dealt with in the following articles, and the objections and uncertainties attaching to each will be pointed out, but the stresses and strains produced in struts by known loads cannot be estimated by any method with the same degree of approximation as in the case of beams or tie-rods, for reasons which will be indicated.

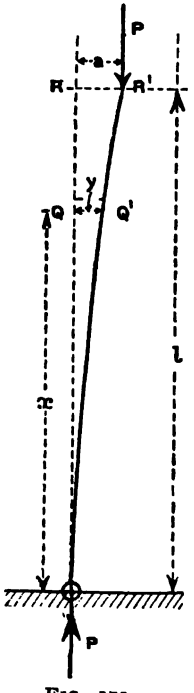


FIG. 171.

**114. Euler's Theory: Long Pillars.**—This refers to pillars which are very long in proportion to their cross-sectional dimensions, which are perfectly straight and homogeneous in quality, and in which the compressive loads are perfectly axially applied. Under such ideal conditions it is shown that the pillar would buckle and collapse under a load much smaller than would produce failure by crushing in a short piece of the same cross-section, and that until this critical load is reached it would remain straight. This evidently could not apply to any pillar so short that the elastic limit is reached before the buckling load.

The strength to resist buckling is greatly affected by the condition of the ends, whether fixed or free. A fixed end means one which is so supported or clamped as to constrain the direction of the strut at that point, as in the case of the ends of a built-in or encastré beam, while a free end means one which by being rounded or pivoted or hinged is free to take up any angular position due to bending of the strut. If the collapsing load for a strut with one kind of end support is found, the corresponding loads for other conditions may be deduced from it.

*Case I., Fig.* 171.—Notation as in the figure. One end O fixed, and the other end, initially at R, free to move laterally and to take up any angular position. Taking the fixed end

O as origin, measuring $x$ along the initial position of the strut OR, and bending deflections $y$ perpendicular to OR, the bending moment at Q' is $P(a-y)$ if the moment is reckoned positive for convexity towards the initial position OR; then, neglecting any effects of direct compression and using the relations for ordinary transverse bending, the curvature—

$$\frac{M}{EI} = \frac{P(a-y)}{EI} = \frac{d^2y}{dx^2} \text{ (approximately, as in Art. 93)}$$

where I is the least moment of inertia of the cross-section, which is assumed to be the same throughout the length—

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{P}{EI}.a \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The solution to this well-known differential equation is [1]—

$$y = a + B\cos\sqrt{\frac{P}{EI}}.x + C\sin\sqrt{\frac{P}{EI}}.x \quad . \quad . \quad (2)$$

where B and C are constants of integration which may be found from the end conditions. When $x = 0$, $y = 0$, hence—

$$0 = a + B + 0 \text{ or } B = -a$$

And when $x = 0$, $\frac{dy}{dx} = 0$, hence, differentiating (2)—

$$\frac{dy}{dx} = \sqrt{\frac{P}{EI}}\left(-B\sin\sqrt{\frac{P}{EI}}x + C\cos\sqrt{\frac{P}{EI}}x\right)$$

and $$0 = \sqrt{\frac{P}{EI}}(-0 + C) \qquad \text{hence } C = 0$$

and (2) becomes—

$$y = a\left(1 - \cos x\sqrt{\frac{P}{EI}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

This represents the deflection to a curve of cosines or sines, and holds for all values of $x$ to $x = l$. In particular, at the free end $x = l$ and $y = a$, hence—

$$a = a - a\cos l\sqrt{\frac{P}{EI}}$$

or,

$$-a\cos l\sqrt{\frac{P}{EI}} = 0$$

From this it follows that either $a = 0$ or the cosine is zero. In the former case evidently no bending takes place; in the latter case, if bending takes place—

$$\cos l\sqrt{\frac{P}{EI}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2}, \text{ etc.}$$

[1] See Lamb's "Infinitesimal Calculus," Art. 182.

Taking the first value $\frac{\pi}{2}$, which gives the least magnitude to P—

$$l^2\frac{P}{EI} = \frac{\pi^2}{4} \quad \text{or} \quad P = \frac{\pi^2 EI}{4l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This gives the collapsing load, and for a long column is much within

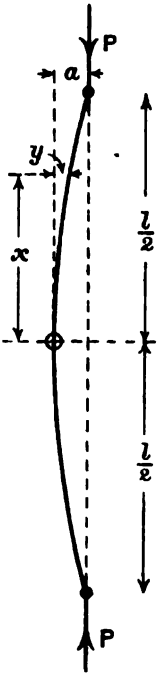



FIG. 172.

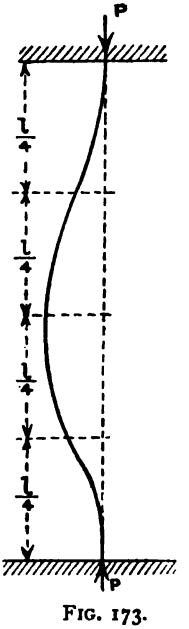



FIG. 173.

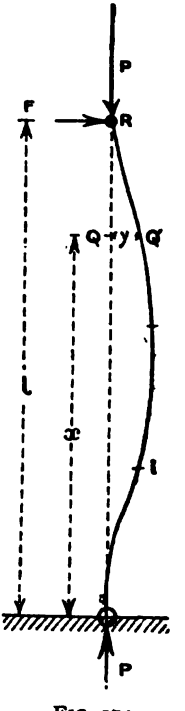



FIG. 174.

the elastic limit of compressive stress. Writing $A.k^2$ for I, where A is the constant area of cross-section and $k$ is the least radius of gyration

$$P = \frac{\pi^2 EA}{4}\left(\frac{k}{l}\right)^2$$

or the average intensity of compressive stress is—

$$p_0 = \frac{P}{A} = \frac{\pi^2 E}{4}\left(\frac{k}{l}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

*Case II., Fig.* 172.—Both ends on pivots on frictionless hinges or otherwise free to take up any angular position. If half the length of

the strut be considered, its ends and loading evidently satisfy the conditions of Case I.; hence the collapsing load

$$P = \frac{\pi^2 EI}{4\left(\frac{l}{2}\right)^2} = \frac{\pi^2 EI}{l^2} \quad \ldots\ldots\ldots \quad (6)$$

and

$$p_0 = \frac{P}{A} = \pi^2 E\left(\frac{k}{l}\right)^2 \quad \ldots\ldots\ldots \quad (7)$$

*Case III., Fig.* 173.—Both ends rigidly fixed in position and direction. If the length of the strut be divided into four equal parts, evidently each part is under the same end and loading conditions as in Case I., hence the collapsing load—

$$P = \frac{\pi^2 EI}{4\left(\frac{l}{4}\right)^2} = \frac{4\pi^2 EI}{l^2} \quad \ldots\ldots\ldots \quad (8)$$

and

$$p_0 = \frac{P}{A} = 4\pi^2 E\left(\frac{k}{l}\right)^2 \quad \ldots\ldots\ldots \quad (9)$$

Thus the ideal strut fixed at both ends is four times as strong as one freely hinged at both ends. These two are the most important cases.

*Case IV., Fig.* 174.—One end O rigidly fixed, and the other R hinged without friction, *i.e.* free to take any angular position, but not to move laterally. Evidently, if bending takes place, some horizontal force F at the hinge will be called into play, since lateral movement is prevented there. Take O as origin. The bending moment at Q′, reckoning positive those moments which tend to produce convexity towards OR, is $F(l - x) - P.y$, hence—

$$EI\frac{d^2y}{dx^2} = F(l - x) - Py$$

or,

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{F}{EI}(l - x)$$

the solution of which is—

$$y = B\cos x\sqrt{\frac{P}{EI}} + C\sin x\sqrt{\frac{P}{EI}} + \frac{F}{P}(l - x). \quad (10)$$

Finding the constants as before—

$y = 0$ for $x = 0$ gives $0 = B + 0 + \frac{F}{P}l$ and $B = -\frac{F}{P}l$

$\frac{dy}{dx} = 0$ for $x = 0$ gives $0 = 0 + C\sqrt{\frac{P}{EI}} - \frac{F}{P}$ and $C = \frac{F}{P}\sqrt{\frac{EI}{P}}$

and substituting these values in (10)—

$$y = \frac{F}{P}\left(-l\cos x\sqrt{\frac{P}{EI}} + \sqrt{\frac{EI}{P}}\sin x\sqrt{\frac{P}{EI}} + l - x\right)$$

for all values of $x$. And putting $y = 0$ for $x = l$—

$$0 = \frac{F}{P}\left(-l\cos l\sqrt{\frac{P}{EI}} + \sqrt{\frac{EI}{P}}\sin l\sqrt{\frac{P}{EI}}\right)$$

hence either $F = 0$, in which case there is no bending, or—

$$\tan l\sqrt{\frac{P}{EI}} = l\sqrt{\frac{P}{EI}}$$

an equation in $l\sqrt{\frac{P}{EI}}$, which may be easily solved by a table giving the values of tangents and of angles in radians. The solution for which P is least (other than $P = 0$) is approximately—

$$l\sqrt{\frac{P}{EI}} = 4{\cdot}5 \text{ radians}$$

from which

$$P = 20\tfrac{1}{4}\frac{EI}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$p_0 = \frac{P}{A} = 20\tfrac{1}{4}E\left(\frac{k}{l}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

By substituting the known values of $y$ in the original equation, and equating $\frac{dy^2}{dx^2}$ to zero, we find approximately $4{\cdot}5 = \tan\frac{4{\cdot}5x}{l}$, which is satisfied by $x = l$ or $x = 0{\cdot}30l$, *i.e.* the point of inflection I (Fig. 174) is $0{\cdot}30l$ from 0 and $0{\cdot}70l$ (approximately) from R, 0·35 of the length being under conditions similar to Case I.

The ultimate strength of the strut in each case is inversely proportional to the square of its length, and comparison between the four cases above shows that the strengths are inversely proportional in Figs. 171, 172, 173, and 174 to the square of the numbers 1, $\frac{1}{2}$, $\frac{1}{4}$, and 0·35 (approx.), the fraction of the lengths between a point of inflection and a point of maximum curvature. The strengths in the same order are therefore proportional to the numbers 1, 4, 16, and 8 (approx.).

**115. Use of Euler's Formulæ.**—Since actual struts deviate from many of the conditions of the ideal cases of Art. 114, the use of the formulæ there derived must be accompanied by a judicious factor to take account of such deviations beyond the ordinary margin of a factor of safety, the effect of very small deviations from the ideal conditions being very great (see Art. 118).

*"Fixed" and "Free" Ends.*—Most actual struts will not exactly fulfil the condition of being absolutely fixed or perfectly free at the ends, and, in applying Euler's rules, allowance must be made for this. An end consisting of a broad flat flange bolted to a fairly rigid foundation will approximate to the condition of a perfectly "fixed" end, and an end which is attached to part of a structure by some form of pin-joint will approximate to the "free" condition; in other cases the ends may be so fastened as to make the strength conditions of the strut intermediate between two of the ideal cases of Art. 114, and sometimes to make the conditions different for different planes of bending.

*Elastic Failure.*—Euler's rules have evidently no application to struts so short that they fail by reaching the yield point of crushing or compressive stress before they reach the values given in Art. 114. For example, considering, say, a mild steel strut freely hinged at both ends (Case II., Art. 114), and taking E = 13,000 tons per square inch, and the yield point 21 tons per square inch, the shortest length to which formula (7) could *possibly* apply would be such that

$$p_0 = 21 = \pi^2 . 13{,}000 . \left(\frac{k}{l}\right)^2$$

$l$ being about 80 times $k$, which would be about 20 diameters for a solid circular section, and 28 diameters for a thin tube. Since these rules only contemplate very long struts, it is to be expected that they would not give very accurate values of the collapsing load until lengths considerably greater than those above mentioned have been reached. For shorter struts than these Euler's rules are not applicable, and will, if used, evidently give *much* too high a value of the collapsing load; such shorter or medium-length struts are, however, of very common occurrence in structures and machines. The values of $p_0$ for columns of mild steel and cast iron with freely hinged ends, as calculated by (7), Art. 114, are shown in Fig. 175.

**116. Rankine's and Other Empirical Formulæ.**

*Rankine.*—For a strut so very short that buckling is practically impossible the ultimate compressive load is—

$$P_c = f_c \times A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A is the area of cross-section and $f_c$ is the ultimate intensity of compressive stress, a quantity difficult to find experimentally, because in short specimens frictional resistance to lateral expansion augments longitudinal resistance to compression, and in longer specimens failure takes place by buckling; $f_c$ may well be taken as the intensity of stress at the yield point in compression.

The ultimate load for a *very* long strut is given fairly accurately by Euler's rules (see Art. 114). Let this load be denoted by $P_e$; then, taking the case of a strut free at both ends (Case II., Art. 114)—

$$P_e = \frac{\pi^2 EI}{l^2} = \pi^2 EA\left(\frac{k}{l}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If P is the crippling load of a strut of *any* length $l$ and cross-section A, the equation

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

evidently gives a value of P which holds well for a very short strut, for $\frac{1}{P}$ then becomes negligible, or $P = P_c$ very nearly, and also holds for a very long strut, for $\frac{1}{P_c}$ then becomes negligible in comparison with $\frac{1}{P_e}$ and $P = P_e$ very nearly. Further, since the change in P is caused

by increasing $l$, for a constant value of A must be a continuous change, it is reasonable to take (3) as giving the value of P for any length of strut.

For a strut with both ends freely hinged, the equation (3) may be written—

$$P = \frac{1}{\frac{1}{f_c \cdot A} + \frac{l^2}{\pi^2 E I}} = \frac{f_c A}{1 + \frac{f_c \cdot l^2}{\pi^2 E k^2}} = \frac{f_c \cdot A}{1 + a\left(\frac{l}{k}\right)^2} \quad . \quad . \quad (4)$$

where $a = \frac{f_c}{\pi^2 E}$, a constant for a given material, or if $p_0$ is the mean intensity of compressive stress on the cross-section—

$$p_0 = \frac{P}{A} = \frac{f_c}{1 + a\left(\frac{l}{k}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In the case of a strut "fixed" at both ends the constant is $\frac{a}{4}$, or half the length may be used for $l$ in (5), and for a strut fixed at one end with angular freedom at the other the constant is $\frac{a}{2}$[1] (approximately), or $\frac{l}{\sqrt{2}}$ may be used for $l$ in (5), and for a strut fixed at one end and free to move in direction and position at the other it is $4a$ (see Cases III., IV., and I., Art. 114). The above are Rankine's rules for struts; they are really empirical, and give the closest agreement with experiments on a series of struts of different ratios $\frac{l}{k}$ when the constants are determined from such experiments rather than from the values of E and $f_c$ for a short length. The values $f$ and $\frac{f_c}{\pi^2 E}$ of the constants in (4) may be called the "theoretical" constants; the value of $a$ would evidently be less than $\frac{f_c}{\pi_2 E}$ for ends with hinges which are not frictionless, and which consequently help to resist bending.

*Gordon's Rule.*—Rankine's rule is a modification of an older rule of Gordon's, viz.—

$$P = \frac{f_c \cdot A}{1 + c\left(\frac{l}{d}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $d$ is the least breadth or diameter of the cross-section in the direction of the least radius of gyration, and $c$ is a constant which will differ not only for different materials and end fixings, but with

[1] $\frac{a}{2}$ is simpler and more correct than the value $\frac{4}{9}a$ often given (see Case IV., Art. 114).

the shape of cross-section, its relation to Rankine's constant $a$ being—

$$\frac{c}{d^2} = \frac{a}{k^2} \quad \text{or} \quad c = a\left(\frac{d}{k}\right)^2$$

*e.g.* in a solid circular section of radius R, $d = 2R$, $k = \frac{R}{2}$, and $c = 16a$.

*Rankine's Constants.*—The usually accepted values of $f_c$ and $a$ in Rankine's formula are about as follow :—

| Material. | $f_c$ tons per square inch. | $a$ |
|---|---|---|
| Mild steel . . . . . | 21 | $\frac{1}{7500}$ |
| Wrought iron . . . . | 16 | $\frac{1}{9000}$ |
| Cast iron . . . . . | 36 | $\frac{1}{1600}$ |

The above constants for wrought and cast-iron are those given as average values by Rankine and widely adopted. The value of $f_c$ for mild steel taken as the yield point may be rather lower than that given above, and rather higher for many kinds of machinery steel, the value of $a$ being altered in about the same proportion. The values of $p$ obtained from Rankine's formula (5) with the above constants will generally be rather above the values of Euler's "ideal" strut, and therefore obviously too high for very long columns with absolutely free ends, because the values of $a$ (generally deduced from experiments in which the ends are not absolutely free) are smaller than the "theoretical" value $\frac{f_c}{\pi^2 E}$. The average intensities of stress, or load per unit area of cross-section occurring at the ultimate load for mild steel and cast-iron struts of various strength with free ends, as calculated by Rankine's formula, and the above constants, are shown in Fig. 175.

*Choice of a Formula.*—If the ratio $\frac{l}{k}$ exceeds about 150, which it rarely if ever does, Euler's values may be used to give the breaking loads, and factors of safety on the average intensity of stress of 5 for steel and wrought iron, 6 for cast iron, and 10 for timber may be used to give the working loads. For shorter struts Rankine's formula may be used with factors of safety of about 3 or 4 for steel.

It may be noted that the specifications of the American Bridge Co. for dead loads give the permissible loads in pounds per square inch of cross-section, as

$$p = \frac{15{,}000}{1 + \frac{1}{13{,}500}\left(\frac{l}{k}\right)^2} \quad \text{(for soft steel)}$$

and

$$p = \frac{17{,}000}{1 + \frac{1}{11{,}000}\left(\frac{l}{k}\right)^2} \quad \text{(for medium steel)}$$

where $l$ is the length of a structural strut centre to centre of the pins at its ends.

Various other values of the constants in Rankine's formula are in use depending upon the quality of the material, the type of end connections and factor of safety used.

Euler's formula, for cases in which it may reasonably be used, has the advantage of directness; the necessary area of cross-section may be found for a given load from (4), (6), (8), or (11), Art. 114.

Rankine's formula, like all others except Euler's, while quite convenient for finding the working or the ultimate load for a given area

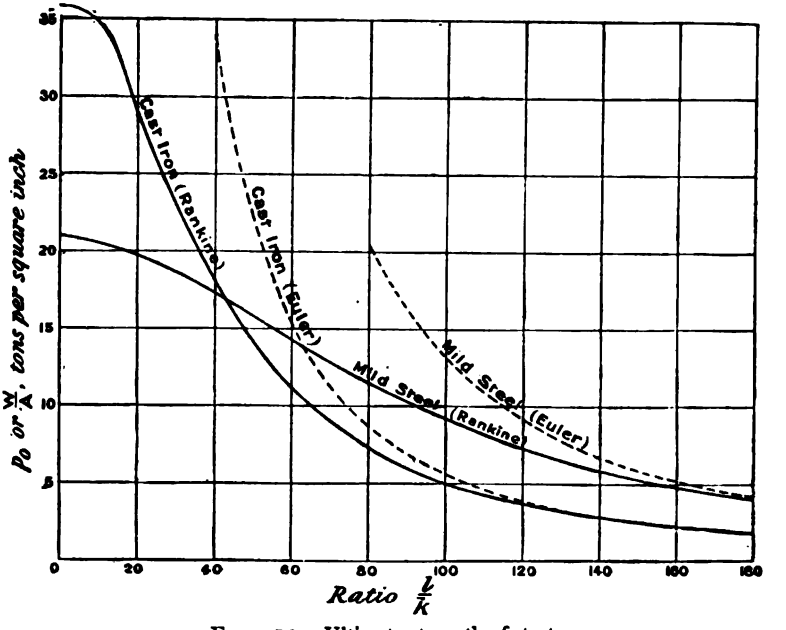



FIG. 175.—Ultimate strength of struts.

and shape of cross-section, is not very direct for finding the dimensions of cross-section in order to carry a given load; it leads to a quadratic equation in the square of some dimension. For practical purposes, however, with standard forms of section the area required is easily found by trial.

*Johnson's Parabolic Formula.*—Prof. J. B. Johnson adopted an empirical formula

$$p_0 = f_c - b\left(\frac{l}{k}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which, when plotted on a base-line giving values of $\frac{l}{k}$, is a parabola, $f_c$ is the yield point in compression, and $b$ is a constant determined so

as to make the parabola meet the curve plotted with Euler's values of $p_0$ tangentially. For a strut absolutely freely hinged at the ends this condition makes $b = \frac{f_c^2}{4\pi^2 E}$, and, owing to friction, Johnson adopted the smaller values of about $\frac{f_c^2}{64E}$ for pin ends and $\frac{f_c^2}{100E}$ for flat ends. For values of $\frac{l}{k}$ beyond the point of tangency with Euler's curve, Euler's values of $p_0$ must be adopted, and to allow for the frictional resistance to bending offered by pin or flat ends, (7) of Art. 114 is modified to $16E\left(\frac{k}{l}\right)^2$ and $25E\left(\frac{k}{l}\right)^2$ respectively, these values of $p_0$ being based on experimental results. The form of Johnson's formula is a trifle more convenient than that of Rankine's.

**Straight Line Formula.**—A great many experimental determinations of the ultimate strength of struts have been made under various conditions,[1] and various empirical formulæ have been devised to suit the various results. The results have been most consistent, and in agreement with empirical algebraic formulæ, as might be expected, when the conditions of loading and fixing have approached most nearly to the ideal, but, on the other hand, such conditions do not correspond to those for the practical strut, as used in machines and structures, which deviate from the ideal in want of straightness and homogeneity of material, more or less eccentricity of the thrust, and in the conditions of freedom or fixture at the ends. The results of tests obtained for struts under more or less working conditions show great variations, and no formula, empirical or otherwise, can more than roughly predict the load at which failure will take place in a given case. This being so, for design purposes one empirical formula is generally about as accurate as another, and the simplest is the best form to use, the constants in any case being deduced from a (short) range of values of $\frac{l}{k}$, within limits for which experimental information is available; for example, straight-line formulæ of the type

$$p_0 = f - \left(\text{constant} \times \frac{l}{k}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $p_0$ is the load per unit area of cross-section and $f$ is a constant, may be used to give the working or the breaking-stress intensities over short ranges of $\frac{l}{k}$.

[1] See "Experimental Researches on Cast Iron Pillars," Hodgkinson, *Phil. Trans. Roy. Soc.*, 1840; "Iron Bridges," by T. C. Clark, *Proc. Inst. C.E.*, vol. liv.; "Experiments on Strength of Wrought Iron Struts," J. Christie, *Trans. Am. Soc. Civ. Eng.*, 1884, vol. xiii.; also extract *Proc. Inst. C.E.*, vol. lxxvii. p. 396.

An extensive analysis of experimental results has been made by J. M. Moncrieff, *Trans. Am. Soc. Civ. Eng.*, vol. xlv., and *Engineering*, June 6, 1902. Ample synopses of results may be found in Warren's "Engineering Construction" (Longmans); H. Fidler's "Notes on Construction in Mild Steel" (Longmans); and T. C. Fidler's "Treatise on Bridge Construction" (Griffin).

For example, an American rule for the safe load on a built-up steel column with square (or flat) ends per square inch of section is 12,000 lbs. for values of $\frac{l}{k}$ less than 90 and above this length

$$p_0 = 17{,}100 - 57\frac{l}{k} \text{ pounds per square inch} \quad . \quad . \quad . \quad (9)$$

which is equivalent to about

$$p_0 = 7{\cdot}5 - 0{\cdot}025\frac{l}{k} \text{ (British) tons per square inch} \quad . \quad . \quad (10)$$

these giving about $\frac{1}{4}$ of the ultimate load per square inch.

Consideration of the ideal strut would suggest doubling the coefficient (57 or 0·025) for struts freely hinged at both ends, but flat-ended struts fall short of absolute fixture, and round ends or hinges of struts offer more resistance to turning than ideally freely hinged ends, and to apply an empirical experimental straight-line formula such as (9) to struts not fixed at the ends the coefficient (or $l$) should be multiplied by about 1·25 only. The formula must not, of course, be used for values of $\frac{l}{k}$ below that stated, or it would give too high a working stress.

Using constants for the average stress per square inch at the crippling load corresponding closely to Rankine's values on Fig. 175 for a strut hinged at both ends

$$p_0 = 22 - 0{\cdot}125\frac{l}{k} \text{ tons per square inch} \quad . \quad . \quad . \quad (11)$$

which may be plotted on Fig. 175 and used with a factor of safety of 3 or 4. Beyond $\frac{l}{k} = 120$ it gives low values, but such lengths are seldom or never used.

The foregoing is not an exhaustive account of all the various strut formulæ in use, but the reader can compare any one with others by a diagram such as Fig. 175. A point of great uncertainty in the design of struts, and particularly of stanchions, is the condition of the ends. Whether a base and its foundation is so rigid as to be taken as "fixed," and whether a top end or cap is to be taken as "fixed," "hinged," or absolutely free, makes much difference in estimated strength, but must generally be a matter of individual judgment (see Art. 185).

Experiments always show that flexure of struts intended to be axially loaded begins at loads much below the maximum ultimately borne, this being due to eccentricity and other variations from the premises upon which Euler's and Rankine's rules depend. This leads us to consider in Art. 118 the effect of eccentric loading on a long column where the flexure is not negligible (as it is in a very short one), and where the greatest bending moment may be mainly from the increased eccentricity which results from flexure.

EXAMPLE 1.—A mild-steel strut hinged at both ends has a T section 6″ × 4″ × $\frac{3}{8}$″ (see B.S.T. 21, Table VI. Appendix), the area being 3·634

square inches, and the least moment of inertia is 4·70 (inches)⁴. Find, by Rankine's formula, the crippling load of the strut, which is 6 feet long, if the ultimate crushing strength is taken at 21 tons per square inch.

The square of the least radius of gyration is $\frac{4\cdot7}{3\cdot634} = 1\cdot293$ (inches)²

$$\left(\frac{l}{k}\right)^2 = \frac{72 \times 72}{1\cdot293} = 4000$$

Using the constant given in the text, viz. $\frac{1}{7500}$ for this case

$$P = \frac{3\cdot634 \times 21}{1 + \frac{4000}{7500}} = \tfrac{15}{23} \times 3\cdot634 \times 21 = 49\cdot7 \text{ tons}$$

EXAMPLE 2.—A steel stanchion of the form shown in Fig. 176 has a cross-sectional area of 39·88 square inches, and its least radius of gyration is 3·84 inches. Both ends being fixed, and the length being 40 feet, find its crippling load, (1) by Euler's formula, (2) by Rankine's formula, (3) by the straight-line formula. (E = 13,000 tons per square inch.)

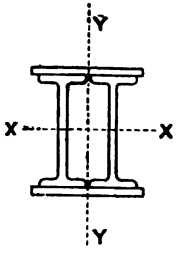


FIG. 176.

By Euler's formula

$$P = \frac{4\pi^2 \times 13{,}000 \times 39\cdot88 \times (3\cdot84)^2}{480 \times 480} = 1307 \text{ tons}$$

By Rankine's formula, and the constants given

$$P = \frac{21 \times 39\cdot88}{1 + \frac{480 \times 480}{3\cdot84 \times 3\cdot84 \times 30{,}000}} = \frac{21 \times 39\cdot88}{1\cdot520} = 551 \text{ tons}$$

Formula (10) gives $p_0 = 7\cdot5 - \frac{1}{40} \times \frac{480}{3\cdot84} = 4\cdot38$ tons per square inch, which corresponds to a working load of $39\cdot88 \times 4\cdot38 = 175$ tons, and to a crippling load of $4 \times 175 = 700$ tons.

Taking the equivalent length for hinged ends as 20 feet, $\frac{l}{k} = \frac{240}{3\cdot84} = 62\cdot5$, and formula (11) gives $p_0 = 14\cdot1875$ tons per square inch, or a crippling load of $14\cdot1875 \times 39\cdot88 = 565$ tons, agreeing closely with Rankine's formula.

EXAMPLE 3.—Find the necessary thickness of metal in a cast-iron column of hollow circular section, 20 feet long, fixed at both ends, the outside diameter being 8 inches, if the axial load is to be 80 tons, and the crushing load is to be 6 times this amount.

Let $d$ be the necessary internal diameter in inches.

The sectional area is $\frac{\pi}{4}(8^2 - d^2)$, and $I = \frac{\pi}{64}(8^4 - d^4)$, hence $k^2 = \frac{1}{16}(8^2 + d^2)$.

The breaking load being 480 tons, Rankine's formula, with the constants given in Art. 102, becomes

$$480 = \frac{36 \times \frac{\pi}{4}(8^2 - d^2)}{1 + \frac{240 \times 240 \times 16}{6400(8^2 + d^2)}} = \frac{9\pi(8^4 - d^4)}{208 + d^2}$$

$$d^4 + 17d^2 - 560 = 0$$

$$d^2 = 16{\cdot}65 \qquad d = 4{\cdot}08''$$

Thickness of metal $= \frac{8 - 4{\cdot}08}{2} = 1{\cdot}96$, or nearly 2 inches.

**117. Forms of Section for Stanchions and Built-up Struts.**—The theory of bending or the theory of buckling of struts (see (7) Art. 114 or (5) Art. 116) shows that for economy of material the section of a stanchion, strut, or column must have a radius of gyration large in proportion to its area. This involves a spread-out form of section, and for cast-iron columns hollow circular sections with comparatively thin walls are usual. For steel stanchions the commonest forms of cross-section are illustrated in Fig. 177; these consist of

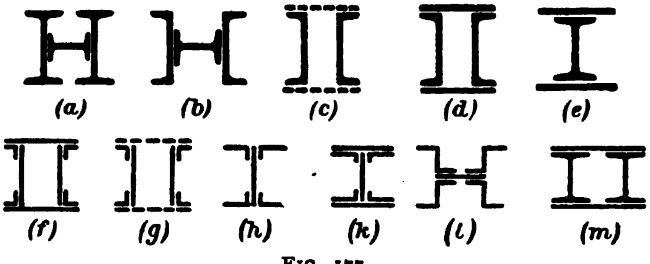



FIG. 177.

sections built up of I, angle, channel, Z and plate sections, and for comparatively small members single I, T, channel or angle bars are also used. (For caps and bases see Art. 185.) The moments of inertia, etc., of the built-up stanchions may be found by the rules given in Chapter III. In sections such as (*c*), (*d*), (*f*), (*g*) it is easy to so space the plates or channels that the moments of inertia about both principal axes of the compound section are equal.

*Latticed Stanchions and Struts.*—Built-up stanchions often consist partly of open lattice work, as shown in Figs. 178 and 179, and indicated by dotted lines in the sections (*c*) and (*g*), Fig. 177. In estimating the moment of inertia of a latticed stanchion section the lattice bars are neglected. Thus in section (*c*), Fig. 177, if $I_1$ is the moment of inertia of each of the channel sections about an axis parallel to its base, A its area, $d$ the distance apart of the centroids of the two channels, the moment of inertia about a central axis parallel to the channel basis is by Theorem 1, Art. 52.

$$2I_1 + 2A\left(\frac{d}{2}\right)^2 = 2I_1 + A \cdot \frac{d^2}{2} \quad \ldots \quad (1)$$

And if $I_2$ is the moment of inertia of each channel section about an axis perpendicular to its base the moment of inertia of the built-up section about the principal axis perpendicular to the channel bases is—

$$2I_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The lattice bars are usually designed to withstand any shearing force to which the stanchion may be subjected. If F is the shearing force at

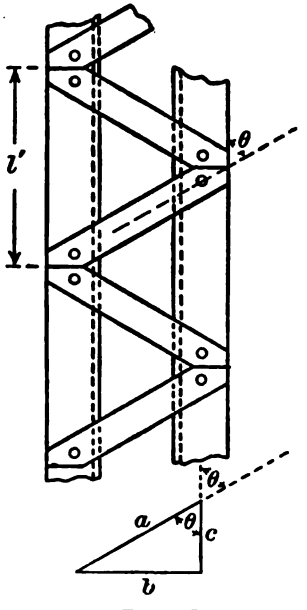



FIG. 178.

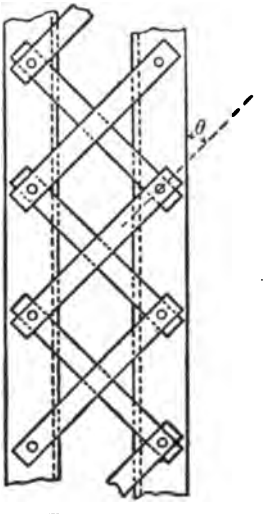



FIG. 179.

any cross-section and $\theta$ is the angle (Fig. 178) which the lattice bars make with the axis of the column, the pull or thrust in a lattice bar with single lacing on one side only is

$$F \text{ cosec } \theta$$

and the bars must be sufficient to withstand this as a tie or a strut. With single lacing on two sides the force will be halved, and with double lacing it will be again halved. With single lacing the angle $\theta$ (Fig. 178) is usually not less than 60°, and with double lacing (Fig. 179) not less than 45°. With regard to resistance of axial loads only it is evident that a single channel of length $l'$ between lattice bar ends must be capable of carrying at least half the total load as a strut. Let L be the equivalent length of a column hinged at both ends and of the full latticed section, and let K be its least radius of gyration; let $k$ be the least radius of gyration of one channel or other component section, then it is

evident from Euler's rules that since one of the two channels carries at least half the load—

$$A\left(\frac{k}{l'}\right)^2 \text{ must be at least equal to } \frac{1}{2}\cdot\frac{2AK^2}{L^2}$$

$$\text{or that } \frac{k}{l'} \text{ must be at least equal to } \frac{K}{L}$$

$$\text{that is, } l' \text{ must not exceed } \frac{k}{K}\cdot L \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

*Proportions of Lattice Bars.*—The usual thickness of the lattice bars for single lacing is about $\frac{1}{40}$th of their length, and for double lacing about $\frac{1}{60}$th of their length, while their width usually varies from $2\frac{1}{2}$ to $1\frac{3}{4}$ ins. according to the size of channel used, being not less than three times the diameter of the rivet passing through them nor less than $\frac{1}{3}$th of their length.

The use of lattice bars instead of solid plate renders all parts of a column accessible for painting.

The above rules would make the lattice bar dimensions nearly the same for any size of channels, *i.e.* for any proportions of $k$ to $l$, or would make a quite arbitrary connection between the cross-section of the column and the cross-section of lattice bars used. Actually the latticing should be heavier in short than in long columns to develop the full strength of the former. A better empirical rule might perhaps be framed as follows, although an entirely rational treatment of so complex a built-up structure is quite impracticable as the distribution of stress is indefinitely known and depends upon the method of manufacture. Let $f$ be the working unit stress for a very short column and $p_0$ be the working unit stress for the actual column. Then

$$f - p_0$$

may be looked upon as the allowance for flexural stress in the channels. The moment of resistance to bending may be taken as

$$M = (f - p_0)Z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in a plane parallel to the lattices where Z is the modulus of section about a central axis of the section perpendicular to the lattice planes. Then for equivalent shear due to transverse loading at the ends over a length $\frac{L}{2}$ from the centre to the end of a double pinned column; since $\frac{dM}{dx} = F = \text{constant}$—

$$F = M \div \frac{L}{2} = \frac{2(f - p_0)Z}{L} \quad . \quad . \quad . \quad . \quad (5)$$

and the stress in a bar of a single lattice (with lacing on both sides of the stanchion) will be as before—

$$\frac{F}{2}\operatorname{cosec}\theta = \frac{(f - p_0)Z}{L}\operatorname{cosec}\theta \quad . \quad . \quad . \quad . \quad (6)$$

For double lacing this must be halved, and for stanchions fixed at one end and entirely free to move at the other L must stand for twice the length, while for columns fixed at both ends it must stand for half the length of the column; in any case for the length of the equivalent doubly pinned strut.

For Z the product A × $d$ may be approximately substituted where $d$ is the distance between the centroids of the two channels and A the area of one channel section, giving the stress in (6) as—

$$\frac{(f-p_0)\mathrm{A}\,.\,d\,.\,\mathrm{cosec}\,\theta}{\mathrm{L}} \text{ or } \frac{(f-p_0)\mathrm{A}l}{\mathrm{L}} \quad . \quad . \quad . \quad . \quad (7)$$

where $l$ is the length of a lattice bar.

The lattice bar should then be of such a section as to carry this amount of thrust as a strut with pin ends, and low unit stress as there may be considerable eccentricity of loading.

*Secondary Stress.*—Stresses in the lattice bars may arise due to the strain of the column. Thus if the column is shortened and the width remains unchanged the diagonal lattice bars are also shortened, inducing a secondary stress. Thus if $c$, Fig. 178, is shortened by an amount $\delta c$, since $a^2 = b^2 + c^2$, differentiating, $2a\frac{da}{dc} = 2c$, and hence the proportional strain $\frac{\delta a}{a} = \frac{c^2}{a^2}\cdot\frac{\delta c}{c} = \cos^2\theta\cdot\frac{\delta c}{c}$, and if all parts are of steel, $\frac{1}{\mathrm{E}} \times$ secondary stress $= \frac{p_0}{\mathrm{E}} \times \cos^2\theta\,. = \frac{p_0}{\mathrm{E}}\cos^2\theta$, and

$$\text{secondary stress in lattice bars} = p_0 \times \cos^2\theta \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since $\cos^2 60° = \frac{1}{4} = \frac{1}{2}\cos^2 45°$, both types of lattice would get an equal secondary unit stress.

Comparatively little is known as to the real distribution of stress in a latticed strut, but investigations made in America[1] on large latticed columns show (by means of strain measurements) great variations such as 40 to 50 per cent. of extreme stress from the average over the section as well as great changes of stress for small axial changes of distances which would indicate local flexure. The experiments of Talbot and Moore showed small strains of the lattice bars, but quite irregular variations in different parts of the column length. The average stress on cross-sections of the lattice bars was such as would be produced by a transverse shear on the column of from 1 to 3 per cent. of the compression load. Individual compression tests of lattice bars showed very low ultimate strengths, these being below half the yield point of the material. The tests of Howard and Buchanan showed marked elastic failure at loads below 9 tons per square inch of column section with

[1] See "An Investigation of Built-up Columns under Load," by Talbot and Moore, *Engineering Bulletin*, No. 44, of Univ. of Illinois; also "Some Tests of Large Steel Columns," by J. E. Howard, in *Proc. American Soc. of Civil Engineers*, Feb., 1911; or an extract from both these papers in *Engineering News*, Vol. 65, No. 11, March 16, 1911.

complete failure below 14 tons per square inch, on struts having a ratio $\frac{l}{k}$ less than 50. Such results point to the desirability of a conservative allowance of unit stresses in built-up columns. It may be recalled that the Quebec bridge disaster resulted from the failure of a latticed member of a compression chord.

EXAMPLE 1.—A stanchion consists of two British standard channels 12 inches × 3½ inches × 32·88lbs. per foot (see B.S.C. 25, Table II., Appendix) placed back to back 6½ inches apart and connected by ½-inch plates 14 inches wide. Find the working load which is to be ¼ that given by Rankine's rule if one end of the column is fixed and the other end hinged, the length being 30 feet.

Using the values from Table II. in the Appendix without the plates, from (2), the moment of inertia about the central axis parallel to the plates is

$$(2 \times 190{\cdot}7) = 381{\cdot}4 \text{ (inches)}^4$$

and for the two plates add $\frac{1}{12} \times 14(13^3 - 12^3) = 547{\cdot}2$ (inches)$^4$

$$\text{Total} \quad . \quad . \quad = 928{\cdot}6 \text{ (inches)}^4$$

About the central axis parallel to the channel, using (1) the moment of inertia, is

For the channels

$$(2 \times 8{\cdot}922) + \frac{9{\cdot}671}{2} \times (6{\cdot}5 + 2 \times 0{\cdot}867)^2 = 345{\cdot}7 \text{ (inches)}^4$$

$$\text{For the plates } \tfrac{1}{12} \times 14^3 = 228{\cdot}7 \quad \text{,,}$$

$$\text{Total} \quad . \quad . \quad = 574{\cdot}4 \text{ (inches)}^4$$

The total area of section is

$$2 \times 9{\cdot}671 + 14 = 33{\cdot}34 \text{ square inches.}$$

Hence the least radius of gyration is

$$\sqrt{\frac{574{\cdot}4}{33{\cdot}34}} = 4{\cdot}15 \text{ inches.}$$

The equivalent length of strut with ends freely hinged is $\frac{30}{\sqrt{2}}$ feet $= \frac{360}{\sqrt{2}}$ inches; hence $\left(\frac{l}{k}\right)^2$ in (5), Art. 116, is $\frac{360 \times 360}{2 \times (4{\cdot}15)^2} = 3764$; hence from (5), Art. 116, the allowable stress is

$$\frac{\frac{1}{4} \times 21}{1 + \frac{3764}{7500}} = \frac{5{\cdot}25 \times 7500}{11{,}264} = 3{\cdot}49 \text{ tons per square inch,}$$

and the working load is $3{\cdot}49 \times 33{\cdot}34 = 116{\cdot}5$ tons.

EXAMPLE 2.—How far apart should two 15-inches × 4-inches British Standard (see B.S.C. 27, Table II., Appendix) channel-shaped sections be placed back to back in a latticed stanchion in order that the resistance to buckling may be approximately equal in all directions.

To satisfy this condition the moments of inertia about the two

principal axes of the compound section must be equal, hence from (1) and (2), using the table.

$$(2 \times 14{\cdot}55) + \frac{12{\cdot}334}{2} \times d^2 = 2 \times 377$$

hence $$d = 10{\cdot}83''$$

Distance apart of channels $= 10{\cdot}83 - 2 \times 0{\cdot}935 = 8{\cdot}96$ inches.

These channels are often spaced $9\frac{1}{2}$ inches apart, and then the value about the axis parallel to the bases is somewhat greater than $(2 \times 377)$ about the other principal axis.

EXAMPLE 3.—What is the maximum distance apart of the lacing bar ends in the previous example, if each channel between these points of support is to be of resistance at least equal to that of the whole stanchion, 30 feet long, fixed at one end and hinged at the other. Equivalent length of stanchion with hinged ends is

$$\frac{30}{\sqrt{2}} \text{ feet} = 21{\cdot}21 \text{ feet.}$$

Hence, referring to Table II., from (3) $l'$ must not exceed $\frac{1{\cdot}09}{5{\cdot}53} \times 21{\cdot}21$ feet $= 4{\cdot}18$ feet (actually it would be less than one-third of this length).

EXAMPLE 4.—Estimate a suitable width for single lattice bars (both sides of the stanchion) for the data in Example 2, if the stanchion is 30 feet long, fixed at the base and hinged at the top, using Rankine's formula, with a factor of safety of 4. The distance of the rivet-hole centres from the outside edge of the channels is 1·7375 inch.

$$k^2 = \frac{377}{12{\cdot}334} = 30{\cdot}5 \text{ (inches)}^2 \qquad 30 \text{ feet} = 360 \text{ inches}$$

hence from (5), Art. 116

the working unit stress $$p_0 = \frac{\frac{1}{4} \times 21}{1 + \frac{360 \times 360}{15{,}000 \times 30{\cdot}5}} = \frac{5{\cdot}25}{1{\cdot}283}$$

$$= 4{\cdot}09 \text{ tons per square inch}$$

$$f - p_0 = 5{\cdot}25 - 4{\cdot}09 = 1{\cdot}16$$

The horizontal distance apart of the lines of rivet centres is $8{\cdot}96 + 2 \times 4 - 2 \times 1{\cdot}7375 = 13{\cdot}485$ inches, and with 60° lacing $l = 13{\cdot}485 \times \frac{2}{\sqrt{3}} = 15{\cdot}57$ inches; and $L = \frac{360}{\sqrt{2}} = 254{\cdot}5$ inches.

Hence from (7) the thrust in a bar may be

$$\frac{1{\cdot}16 \times 12{\cdot}334 \times 15{\cdot}57}{254{\cdot}5} = 0{\cdot}8754 \text{ ton.}$$

If thickness $= \frac{1}{40}$ of length, for rectangular section $\left(\frac{l}{k}\right)^2 = 40^2 \times 12$ $= 19{,}200$; allowable unit stress by (5), Art. 116 $= \frac{5{\cdot}25}{1 + \frac{19200}{7500}} = 1{\cdot}474$ tons per square inch.

$$\text{Area required} = \frac{0{\cdot}8754}{1{\cdot}474} = 0{\cdot}594, \text{ or say } 0{\cdot}6 \text{ square inch.}$$

$$\text{Thickness} = \tfrac{1}{40} \text{ of } 15{\cdot}57 = \tfrac{3}{8} \text{ inch say;}$$

hence $$\text{width} = 0{\cdot}6 \div \tfrac{3}{8} = 1{\cdot}6 \text{ inches.}$$

If we allow for secondary stress, by (8) the amount is $p_0 \times \cos^2 60° = 4{\cdot}09 \times \frac{1}{4} = 1{\cdot}02$ tons per square inch. This, if reckoned additional to the stress due to shearing, would require a larger section, viz. $\frac{0{\cdot}8754}{1{\cdot}474 - 1{\cdot}02} = 1{\cdot}92$ square inches. For a reasonable width of bar on this basis double lacing would be required, but probably the assumption of pin ends is too severe, and a higher stress than 1·474 tons per square inch may be allowed.

**118. Long Columns under slightly Eccentric Load.**—As Euler's formulæ are only strictly applicable to struts absolutely axially loaded, it is interesting to find what modifications follow if there is a small eccentricity $h$ at the points of application of the load. Variation of elasticity of the material and initial curvature of the strut must give a similar effect, and may be looked upon as an increased value of $h$. Taking Case I., Art. 114, if P is applied at a distance $h$ from the centre at R′, Fig. 171 (and on the principal axis perpendicular to that about which the minimum value of I is taken), the bending moment at Q will be $P(a + h - y)$, and (1), Art. 114, becomes

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{P}{EI}(a + h) \quad \ldots\ldots \quad (1)$$

and the solution (2a) of Art. 114 becomes

$$y = (a + h)\left(1 - \cos x\sqrt{\frac{P}{EI}}\right) \quad \ldots\ldots \quad (2)$$

and at $x = l$ this becomes—

$$y = a = (a + h)\left(1 - \cos l\sqrt{\frac{P}{EI}}\right)$$

$$a \cos l\sqrt{\frac{P}{EI}} = h\left(1 - \cos l\sqrt{\frac{P}{EI}}\right)$$

$$a = h\left(\sec l\sqrt{\frac{P}{EI}} - 1\right) \quad \ldots\ldots \quad (3)$$

The eccentricity of loading at the origin O is—

$$a + h = h \sec l\sqrt{\frac{P}{EI}} \quad \ldots\ldots \quad (4)$$

the bending moment there being increased $\sec l\sqrt{\frac{P}{EI}}$ times due to flexure. The bending moment at O is $P(a + h) = Ph \sec l\sqrt{\frac{P}{EI}}$, which, so long as the intensity of stress is proportional to the strain,

causes in a symmetrical section equal and opposite bending stresses of intensity—

$$p_b = \frac{Ph}{Z} \sec l\sqrt{\frac{P}{EI}} = \frac{Phd}{2I} \sec l\sqrt{\frac{P}{EI}}$$

where $d$ is the depth of section in the plane of bending, *i.e.* in the direction of the least radius of gyration; if the section is unsymmetrical, $y_c$ and $y_t$ must be used instead of $\frac{d}{2}$ (see Art. 63); hence the greatest compressive stress $p$, by (1), Art. 111, is—

$$p = \frac{P}{A} + \frac{Phd}{2k^2A} \sec l\sqrt{\frac{P}{EI}} = \frac{P}{A}\left(1 + \frac{hd}{2k^2} \sec l\sqrt{\frac{P}{EI}}\right) \quad (5)$$

which becomes infinite, as in Art. 114, when—

$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2} \quad \text{or} \quad P = \frac{\pi^2 EI}{4l^2}$$

Also

$$\frac{P}{A} = \frac{p}{1 + \frac{hd}{2k^2} \sec l\sqrt{\frac{P}{EI}}} \quad . \quad . \quad . \quad (6)$$

and if $f_c$ is the crushing strength of the material, *i.e.* say the stress intensity at the yield point in compression, at failure by buckling—

$$p_0 = \frac{P}{A} = \frac{f_c}{1 + \frac{hd}{2k^2} \sec l\sqrt{\frac{P}{EI}}} \quad . \quad . \quad . \quad (7)$$

In the case of a column free at both ends (Case II., Art. 114, and Fig. 172), with an eccentricity $h$ of the thrust at the ends, by writing $\frac{l}{2}$ instead of $l$, (4) becomes

$$(a + h) = h \sec \frac{l}{2}\sqrt{\frac{P}{EI}} \quad . \quad . \quad . \quad . \quad (8)$$

and (5) becomes—

$$p = \frac{P}{A}\left(1 + \frac{hd}{2k^2} \sec \frac{l}{2}\sqrt{\frac{P}{EI}}\right) \quad . \quad . \quad . \quad (9)$$

and at failure by compressive yielding (7) becomes—

$$p_0 = \frac{P}{A} = \frac{f_c}{1 + \frac{hd}{2k^2} \sec \frac{l}{2}\sqrt{\frac{P}{EI}}} = \frac{f_c}{1 + \frac{hd}{2k^2} \sec \frac{l}{2}\sqrt{\frac{p_0}{k^2E}}} \quad (10)$$

Allowing for a slight difference of notation, when $l = 0$, (5) and (9) reduce to the form (1) of Art. 112, the increase of bending stress due to flexure being only important when the length is considerable.

If failure occurs by tension, as is usual in cast iron, the greatest intensity of tensile stress corresponding to (9) is—

$$p = \frac{P}{A}\left(\frac{hd}{2k^2} \sec \frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right) \quad . \quad . \quad . \quad . \quad (11)$$

and if $f_t$ is the limit of tensile-stress intensity at fracture, instead of (10) at failure by *tension* the average *compressive* stress is—

$$p_0 = \frac{P}{A} = \frac{f_t}{\frac{hd}{2k^2}\sec\frac{l}{2}\sqrt{\frac{P}{EI}} - 1} = \frac{f_t}{\frac{hd}{2k^2}\sec\frac{l}{2}\sqrt{\frac{p_0}{k^2E}} - 1} \quad (12)$$

From equations (9) and (11) the extreme intensities of compressive and tensile stress may be found for a strut with given dimensions, load, and eccentricity, or the eccentricity which will cause any assigned intensity of stress may be found.

It is evident that $p$ becomes infinite for $P = \frac{\pi^2 EI}{l^2}$, just as in Euler's theory, where the eccentricity $h = 0$; but these equations show that where $h$ is not zero, $p$ approaches the ultimate compressive or tensile strengths for values of P much below Euler's critical values. The reader will find it instructive to plot the values of P and $p$ for any given section, and for several different magnitudes of the eccentricity $h$, and to observe how $p$ increases with P in each case.

For a strut of given dimensions with given eccentricity $h$, the ultimate load P (or $p_0$) to satisfy equations (10) or (12) for a given ultimate stress intensity $f_c$ or $f_t$ may be found by trial or by plotting as ordinates the difference of the two sides of either equation, on a base-line of values of P, and finding for what value of P the ordinate is zero. It is convenient to write $\frac{l}{2}\sqrt{\frac{P}{EI}} = \frac{\pi}{2}\sqrt{\frac{P}{P_e}}$ where $P_e = \frac{\pi^2 EI}{l^2}$ when solving for P by trial, the angle in degrees being $90\sqrt{\frac{P}{P_e}}$.

Fig. 180 shows the ultimate values of $p_0$ for mild-steel struts of circular section and various lengths, taking $f_c = 21$ tons per square inch with various degrees of eccentricity. It shows that for struts about 20 diameters in length, for example, an eccentricity of $\frac{1}{100}$ of the diameter greatly decreases the load which the ideal strut would support. Also that when there is an eccentricity of $\frac{1}{10}$ of the diameter an additional eccentricity of $\frac{1}{100}$ of the diameter does not greatly reduce the strength.

To find the dimensions of cross-section, for a strut of given length, load, and eccentricity, and shape of cross-section, in order not to exceed a fixed intensity of stress $f_c$ or $f_t$, the above equations may be solved by trial or plotting if A and $k$ (or I) are put in terms of $d$, viz. $A = c_1 \times d^2$, $k^2 = c_2 \times d^2$ (or $I = c_3 \times d^4$), where $c_1$ and $c_2$ (or $c_3$) are constants depending on the shape of cross-section. In solving by trial a first approximation to the unknown quantity may be found by taking the secant as unity, as in Art. 112; the further adjustment of the result is then simple. Prof. R. H. Smith[1] has shown how, where a large number of such problems are to be solved, the calculation may be facilitated by drawing

[1] See the *Engineer*, October 14 and 28, and November 25, 1887.

a series of curves corresponding to various degrees of eccentricity and adaptable to any shape of section.

It may be noticed from equations (9) and (10) that with increase of load P the maximum intensity of stress is increased more than proportionally, because the part due to bending increases with the increased eccentricity due to flexure as well as with the increased load. Hence the ratio of the ultimate or crippling loads to any working load

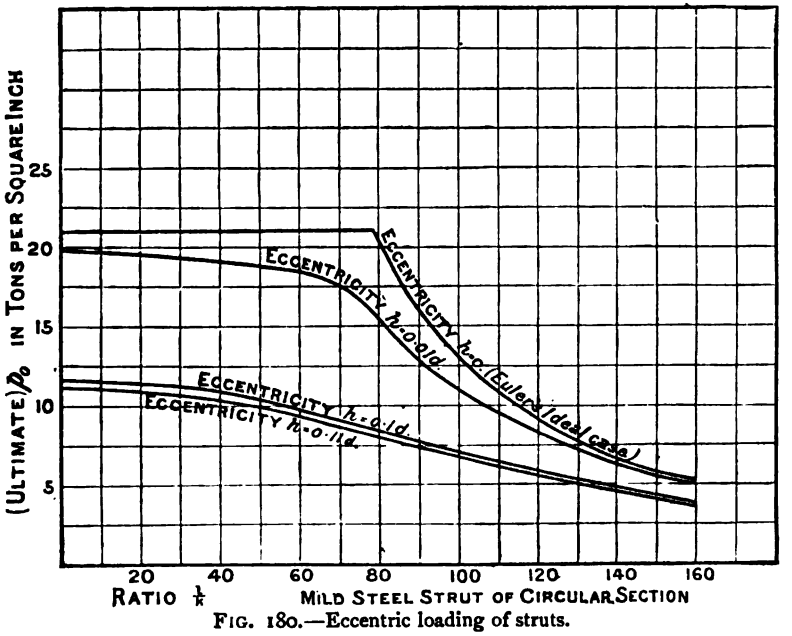



FIG. 180.—Eccentric loading of struts.

will be less than the factor of safety, as understood by the ratio of the maximum intensity of stress to the ultimate intensity of crushing stress (at the yield point, say). This point is illustrated in Examples Nos. 3 and 4 at the end of the present article.

In the case of a long tie-rod with an eccentric load the greatest intensities of stress are at the end sections, where the eccentricity is $h$; in the centre it is only $h$ sech $\frac{l}{2}\sqrt{\frac{P}{EI}}$.

*Approximate Method.*[1]—If the secant in (9) be expanded thus—

$$\sec\theta = 1 + \frac{\theta^2}{2!} + \frac{5\theta^4}{4!} + \frac{61\theta^6}{6!} + \frac{1385\theta^8}{8!} + \text{etc.}$$

it gives $$\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}} = 1 + \frac{\pi^2}{8}\cdot\frac{P}{P_e} + \frac{5\pi^4}{384}\left(\frac{P}{P_e}\right)^2 + \frac{61\pi^6}{41280}\left(\frac{P}{P_e}\right)^3.$$

[1] Professor Perry makes a somewhat similar approximation in the *Engineer*, December 10 and 24, 1886, but the form here given is more accurate over larger ranges of load.

And for working values of $\frac{P}{P_e}$ say less than $\frac{1}{4}$, this may be very closely represented (with error on the safe side) by

$$\frac{1}{1 - 1{\cdot}2\frac{P}{P_e}} \text{ or } \frac{P_e}{P_e - 1{\cdot}2P} \text{ or } \frac{1}{1 - 0{\cdot}12\frac{Pl^2}{EI}} \text{ or } \frac{1}{1 - \frac{Pl^2}{8EI}} \quad (13)$$

Making this substitution (9) becomes

$$p = \frac{P}{A}\left\{1 + \frac{hd}{2k^2\left(\frac{0{\cdot}12Pl^2}{EI}\right)}\right\} \text{ or } \frac{P}{A} + \frac{M_1 y_1}{I - \frac{0{\cdot}12Pl^2}{E}} \quad . \quad (14)$$

where $y = \frac{d}{2}$ and $M_1 = Ph$, the bending moment neglecting the flexure which reduces to the form in Art. 112 when $l = 0$. (14) may also be written,

$$\left(\frac{pA}{P} - 1\right)\left(1 - \frac{1{\cdot}2P}{P_e}\right) \text{ or } \left(\frac{p}{p_0} - 1\right)\left(1 - \frac{0{\cdot}12p_0l^2}{Ek^2}\right) = \frac{hd}{2k^2} \quad (15)$$

And for failure in tension (11) becomes for the maximum tension—

$$p = \frac{P}{A}\left\{\frac{hd}{2k^2\left(1 - \frac{0{\cdot}12Pl^2}{EI}\right)} - 1\right\} \text{ or } \frac{M_1 y_1}{I - \frac{0{\cdot}12Pl^2}{E}} - \frac{P}{A} \quad . \quad (16)$$

which may be written by transposing—

$$\left(\frac{p}{p_0} + 1\right)\left(1 - \frac{0{\cdot}12Pl^2}{Ek^2}\right) = \frac{hd}{2k^2} \quad . \quad . \quad . \quad . \quad (17)$$

*Another Method.*—The approximate results may also be obtained as follows. Neglecting the variation of bending moment due to flexure the strut has $M = Ph$ throughout. This *alone* would cause a central deflection $\frac{Phl^2}{8EI}$ (see (1), Art. 92). Hence the maximum bending moment exceeds $P\left(h + \frac{Phl^2}{8EI}\right)$ or $Ph\left(1 + \frac{Pl^2}{8EI}\right)$, and for moderate values of P may be taken as $\frac{Ph}{1 - \frac{Ph}{8EI}}$, hence

$$p = \frac{P}{A} + \frac{Phd}{2I\left(1 - \frac{Ph}{8EI}\right)} \text{ or } \frac{P}{A}\left\{1 + \frac{hd}{2k^2\left(1 - \frac{Ph}{8EI}\right)}\right\} \text{ as in (14)}$$

As before, from (15) and (17) the extreme intensities of stress may be found for a strut of known dimensions carrying a known load with any assigned eccentricity; or the allowable eccentricity may be calculated for a given limit of the tensile or compressive-stress intensity. Also for a strut of given dimensions, and maximum safe intensity of stress with a given eccentricity, the load P may be calculated directly

as the root of the quadratic equation (15) or (17), according as the specified stress limit is compressive ($p = f_c$) or tensile ($p = f_t$).

The dimensions of cross-section for a strut of given length and shape to carry a given load, with given eccentricity and a given stress limit, may be found by taking, as before, $A = c_1 . d^2$, $k^2 = c_2 . d^2$, $I = c_3 . d^4 = c_1 . c_2 . d_4$, where $c_1$ and $c_2$ are constants, in (15) or (17). Since $P_e$ is proportional to $d^4$, these equations evidently become sextic (or sixth-power) equations in $d$, and (15) or (17) being used according as the specified limit of stress intensity is compressive or tensile, $d$ may be found by trial or plotting. For a solution by trial a first approximation may be obtained by taking $l = 0$ when equation (15) reduces to the form of (1), Art. 112. If $h$ should be specified as a fraction of $d$, the equation will reduce to a cubic in $d^2$.

The approximate solution may be tested by the more exact rules (10) and (12), and adjusted to satisfy them.

Assuming any initial curvature of a strut to be of the form of a curve of cosines, Prof. Perry, in the form referred to above, shows that initial curvature is equivalent to eccentricity not greatly different from the maximum deflection of the strut at the centre from its proper position of straightness. This may be verified by substituting $h_1 \cos \frac{\pi x}{2 l}$ for $h$ in (1), the conditions being $y = 0$ and $\frac{dy}{dx} = 0$ for $x = 0$ and $y = a$ for $x = l$; the maximum bending moment is then $P(a + h_1)$, which is equal to—

$$\frac{P h_1}{1 - \frac{P}{P_e}}$$

where $P_e = \frac{\pi^2 EI}{4 l^2}$. A similar value holds for other cases when the value of $P_e$ is modified as in Art. 114.

EXAMPLE 1.—A cast-iron pillar is 8 inches external diameter, the metal being 1 inch thick, and carries a load of 20 tons. If the column is 40 feet long and rigidly fixed at both ends, find the extreme intensities of stress in the material if the centre of the load is $1\frac{3}{4}$ inch from the centre of the column. What eccentricity would be just sufficient to cause tension in the pillar? (E = 5000 tons per square inch.) The corresponding problem for a very short column has been worked in Ex. 2, Art. 112, and these results may be used—

$$p_e = 0{\cdot}909 \text{ ton per square inch} \qquad k^2 = \tfrac{1}{16}(8^2 + 6^2) = \tfrac{25}{4}$$

The bending stress is increased in the ratio $\sec \frac{l}{4}\sqrt{\frac{P}{EI}}$ or

$$\sec \frac{l}{4}\sqrt{\frac{p_e}{E k^2}} = \sec \tfrac{480}{4}\sqrt{\frac{0{\cdot}909 \times 4}{5000 \times 25}} = \sec 0{\cdot}646 = \sec 37^\circ = 1{\cdot}25.$$

Hence the bending-stress intensity is—

$$1{\cdot}017 \times 1{\cdot}25 = 1{\cdot}27 \text{ ton per sq. in.}$$

The maximum compressive stress = 1·27 + 0·909 = 2·18 tons per sq. in.
The maximum tensile stress = 1·27 − 0·909 = 0·36 ton per sq. in.

or more than treble that when there is no flexure increasing the eccentricity.

If the eccentricity is just sufficient to cause tension in the pillar, its amount is—

$$1{\cdot}75 \times \frac{0{\cdot}909}{1{\cdot}27} = 1{\cdot}25 \text{ inch}$$

EXAMPLE 2.—A compound stanchion has the section shown in Fig. 176; its radius of gyration about YY is 3·84 inches, and its breadth parallel to XX is 14 inches. The stanchion, which is to be taken as freely hinged at both ends, is 32 feet long. If the load per square inch of section is 4 tons, how much may the line in which the resultant force acts at the ends deviate from the axis YY without producing a greater compressive stress than 6 tons per square inch, the resultant thrust being in the line XX? How much would it be in a very short pillar? (E = 13,000 tons per square inch.)

Evidently from (9) the bending-stress intensity must be 6 − 4 = 2 tons per square inch; hence, if $h$ is the eccentricity—

$$4\frac{hd}{2k^2}\sec\frac{l}{2}\sqrt{\frac{p_0}{Ek^2}} = 2$$

$$\frac{4 \cdot h \cdot 14}{2 \times (3{\cdot}84)^2}\sec\frac{192}{3{\cdot}84}\sqrt{\frac{4}{13{,}000}} = 2$$

$$h(1{\cdot}897 \sec 50{\cdot}3^\circ) = 2{\cdot}97h = 2$$

$$h = 0{\cdot}675 \text{ inch}$$

For a very short pillar where the flexure is negligible this would evidently be—

$$h \times 1{\cdot}897 = 2 \qquad h = 1{\cdot}055 \text{ inch}$$

the equation reducing to the form (1), Art. 112, since the secant is practically unity.

It is interesting to compare the solution by (15)—

$$(\tfrac{6}{4} - 1)\left(1 - \frac{0{\cdot}12 \times 10{,}000 \times 4}{13{,}000}\right) = \frac{14}{14{\cdot}75} \times \frac{h}{2}$$

$$h = 0{\cdot}665 \text{ inch}$$

which agrees well with the previous result, and is slightly on the safe side.

EXAMPLE 3.—Find the load per square inch of section which a column of the cross-section given in Ex. 2 will carry with an eccentricity of 1½ inch from XX, the column being 28 feet long and free at both ends, the maximum compressive stress not exceeding 6 tons per square inch. Find also the ultimate load per square inch of section if the ultimate compressive strength is 21 tons per square inch. (E = 13,000 tons per square inch.)

Using first the approximate method, (15) gives—

$$\left(\frac{6}{p_0}-1\right)\left\{1-\frac{0\cdot12\times p_0\left(\frac{28\times12}{3\cdot84}\right)^2}{13{,}000}\right\}=\tfrac{1}{2}\times\tfrac{3}{2}\cdot\frac{14}{(3\cdot84)^2}$$

or, $$p_0^2-30\cdot2p_0+85=0$$

hence $$p_0=3\cdot15 \text{ tons per square inch}$$

Testing this value in (9)—

$$3\cdot15\left(1+\tfrac{3}{2}\times\frac{14}{2\times14\cdot75}\sec\frac{168}{3\cdot84}\sqrt{\frac{3\cdot15}{13{,}000}}\right)$$

$$=3\cdot15(1+0\cdot715\sec39^\circ)=6\cdot05 \text{ tons per square inch}$$

instead of 6, hence 3·15 is slightly too high. Trial shows that

$$p_0=3\cdot12 \text{ tons per square inch}$$

satisfies (9), and is the allowable load per square inch of section. Substituting 21 tons per square inch for 6 in the above work gives 8·2 tons per square inch of section as the crippling load. Note that while the factor of safety reckoned on the stress is $\frac{21}{6}=3\frac{1}{2}$, the ratio of ultimate to working load is $\dfrac{8\cdot2}{3\cdot12}=2\cdot63$.

EXAMPLE 4.—A steel strut is to be of circular section, 50 inches long and hinged at both ends. Find the necessary diameter in order that, if the thrust of 15 tons deviated at the ends by $\frac{1}{10}$ of the diameter from the axis of the strut, the greatest compressive stress shall not exceed 5 tons per square inch. If the yield point of the steel in compression is 20 tons per square inch, find the crippling load of the strut. (E = 13,000 tons per square inch.)

$$k=\frac{d}{4}\qquad A=\frac{\pi d^2}{4}\qquad h=\frac{d}{10}$$

Using the approximate equation of (15)—

$$\left(\frac{5\pi d^2}{4\times15}-1\right)\left(1-\frac{0\cdot12\times15\times64\times2500}{13{,}000\times\pi d^4}\right)=\tfrac{1}{2}\times\frac{d}{10}\times\frac{d\times16}{d^2}=0\cdot80$$

$$(0\cdot2616d^2-1)\left(1-\frac{7\cdot056}{d^4}\right)=0\cdot80$$

$$d^6-6\cdot88d^4-7\cdot065d^2+27=0$$

a cubic equation in $d^2$, which by trial gives—

$$d^2=7\cdot3$$
$$d=2\cdot70 \text{ inches}$$

Testing this result by equation (9)—

$$\frac{15\times4}{\pi\times7\cdot3}(1+\tfrac{16}{20}\sec0\cdot522)=4\cdot84$$

instead of 5 tons per square inch.

By trial $d=2\cdot7$ inches very nearly, as in the approximate method.

Taking this value for failure when $p = 20$ tons per square inch, (15) gives—

$$\left(\frac{20}{p_e} - 1\right)\left(1 - \frac{657p_e}{13{,}000}\right) = 0{\cdot}80$$

$$p_e = 8{\cdot}4 \text{ tons per square inch}$$

and by trial, from (9)—

$$p_e = 8{\cdot}43 \text{ tons per square inch}$$

the whole load on the strut being—

$$8{\cdot}43 \times \frac{\pi}{4} \times (2{\cdot}7)^2 = 48{\cdot}4 \text{ tons}$$

Thus the factor of safety reckoned on the greatest intensity of stress is $\frac{20}{5} = 4$, but the ratio of crippling load to working load is $\frac{48{\cdot}4}{15} = 3{\cdot}22$.

**119. Struts and Tie-rods with Lateral Loads.**—When a prismatic piece of material is subject to axial and lateral forces it may be looked upon as a beam with an axial thrust or pull, or as a strut or tie-rod with lateral bending forces. A good example occurs in the case of a sloping beam acted upon by vertical forces as in the main rafters of a roof. The stress intensity at any cross-section is, as indicated by (1), Art. 111, the algebraic sum of the bending stress, and the direct stress which the axial thrust would cause if there were no lateral forces.

In a beam which is only allowed a very limited deflection, *i.e.* which is not very long in proportion to its dimensions of cross-section, the bending stress may usually be taken as that resulting from the transverse loads only. If, however, the beam is somewhat longer in proportion to its cross-section, the longitudinal force, which may be truly axial only at the ends, will cause a considerable bending stress due to its eccentricity elsewhere, and will play an appreciable part in increasing or decreasing the deflection produced by the lateral load, according as it is a thrust or a pull. In this case, the bending stresses at any section are the algebraic sum of those produced by the transverse loads, and those produced by the eccentricity of the longitudinal forces. Unless the bar is very long, or the longitudinal force is very great, a fairly close approximation to the bending moment may be found by taking the algebraic sum of that resulting from the transverse forces and that resulting from the eccentricity of the longitudinal force, on the assumption that the deflection or eccentricity is that due to the transverse loads only. The solution of a problem under these approximations has already been dealt with, the bending stress due to transverse loads being as calculated in Chapter V., the deflection being as calculated in Chapter VII., and the stresses resulting from the eccentric longitudinal force being calculated as in Art. 112. It remains to deal with those cases where the end thrust or pull materially affects the deflection, and where consequently the above approximation is not valid; this is the work of the two following articles, which give the stress intensities for members of any proportion, and indicate the circumstances under which the simpler solution of the problem will be approximately correct.

*Strut with Lateral Load.*—Let $l$ be the length of a uniform strut freely hinged at each end and carrying a load $w$ per unit length. Let the end thrust which passes through the centroid of the cross-section at each end be P. Take the origin O (Fig. 181) midway between the ends, the line joining the centroids of the ends being the

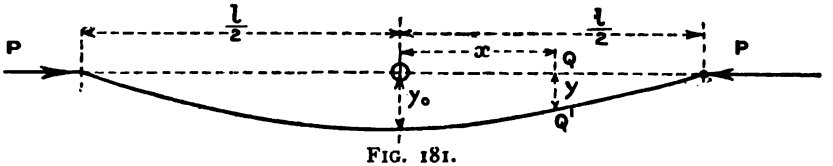



FIG. 181.

axis of $x$. The bending moment at Q' is $-\frac{w}{2}\left(\frac{l^2}{4}-x^2\right)$ due to the lateral load and $-\mathrm{P}.y$ due to the end thrust P. The sum is equal to $\mathrm{EI}\frac{d^2y}{dx^2}$, where I is the (constant) moment of inertia of the cross-section about an axis through its centroid and perpendicular to the plane of flexure, or—

$$\mathrm{EI}\frac{d^2y}{dx^2} = -\frac{w}{2}\left(\frac{l^2}{4}-x^2\right) - \mathrm{P}.y \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{d^2y}{dx^2} + \frac{\mathrm{P}}{\mathrm{EI}}.y = -\frac{w}{2\mathrm{EI}}\left(\frac{l^2}{4}-x^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The solution of this equation under the conditions $\frac{dy}{dx} = 0$ for $x=0$, and $y = 0$ for $x = \frac{l}{2}$ is—

$$y = \frac{w}{2\mathrm{P}}x^2 - \frac{wl^2}{8\mathrm{P}} - \frac{w\mathrm{EI}}{\mathrm{P}^2}\left(1 - \sec\frac{l}{2}\sqrt{\frac{\mathrm{P}}{\mathrm{EI}}}\cos\sqrt{\frac{\mathrm{P}}{\mathrm{EI}}}.x\right) \quad (3)$$

and the maximum bending moment at O is—

$$-\mathrm{M}_0 = \mathrm{P}.y_0 + \tfrac{1}{8}wl^2 = \frac{w\mathrm{EI}}{\mathrm{P}}\left(\sec\frac{l}{2}\sqrt{\frac{\mathrm{P}}{\mathrm{EI}}} - 1\right) \quad . \quad . \quad (4)$$

or,
$$-\mathrm{M}_0 = \frac{w\mathrm{EI}}{\mathrm{P}}\left(\sec\frac{\pi}{2}\sqrt{\frac{\mathrm{P}}{\mathrm{P}_e}} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $\mathrm{P}_e = \frac{\pi^2\mathrm{EI}}{l^2}$, Euler's limiting value for the ideal strut (Case II., Art. 114). If $\mathrm{P} = \mathrm{P}_e$, $\mathrm{M}_0$ and $y_0$ become infinite. The expansion—

$$\sec\theta - 1 = \frac{\theta^2}{2!} + \frac{5\theta^4}{4!} + \frac{61\theta^6}{6!} + \frac{1385\theta^8}{8!} +, \text{ etc.} \quad . \quad . \quad (6)$$

may be applied to (4), which then reduces to—

$$-\mathrm{M}_0 = \frac{wl^2}{8}\left\{1 + \frac{5\pi^2}{48}\left(\frac{\mathrm{P}}{\mathrm{P}_e}\right) + \frac{61\pi^4}{5760}\left(\frac{\mathrm{P}}{\mathrm{P}_e}\right)^2 + \frac{277\pi^6}{258{,}048}\left(\frac{\mathrm{P}}{\mathrm{P}_e}\right)^3 +, \text{ etc.}\right\} \quad (7)$$

or

$$-\mathrm{M}_0 = \frac{wl^2}{8} + \frac{5}{384}\cdot\frac{wl^4}{\mathrm{EI}}.\mathrm{P}\left\{1 + \frac{61\pi^2}{600}\cdot\frac{\mathrm{P}}{\mathrm{P}_e} + \frac{277\pi^4}{26{,}880}\left(\frac{\mathrm{P}}{\mathrm{P}_e}\right)^2 +, \text{ etc.}\right\} \quad (8)$$

These two forms (7) and (8) show the relation of the approximate methods mentioned at the beginning of this article to the more exact method of calculating bending moment. The first term in each is the bending moment due to the lateral loads alone; the second term in (8) is the product of the axial thrust P and the deflection $\frac{5}{384}\cdot\frac{wl^4}{EI}$ (see (11), Art. 94) due to the transverse load alone. Even in the longest struts $\frac{P}{P_e}$ will not exceed about $\frac{1}{3}$, and in shorter ones will be much less. The errors involved in the approximate method of calculation, which gives the first two terms in (9), are evidently then not great.

If the strut carried a lateral load W at the centre instead of the uniformly distributed load, equation (2) becomes—

$$\frac{d^2y}{dx^2}+\frac{P}{EI}\cdot y=-\frac{W}{2EI}\left(\frac{l}{2}-x\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$y_0=\frac{W}{2P}\sqrt{\frac{EI}{P}}\tan\frac{l}{2}\sqrt{\frac{P}{EI}}-\frac{Wl}{4P} \quad . \quad . \quad (10)$$

$$-M_0=\frac{2}{W}\sqrt{\frac{P}{EI}}\tan\frac{l}{2}\sqrt{\frac{P}{EI}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Using the expansion $\tan\theta=\theta+\frac{1}{3}\theta^3+\frac{2}{15}\theta^5+\frac{17}{315}\theta^7+\quad .\quad (12)$

$$-M_0=\frac{Wl}{4}\left\{1+\frac{\pi^2}{12}\cdot\frac{P}{P_e}+\frac{\pi^4}{120}\left(\frac{P}{P_e}\right)^2+\frac{17\pi^6}{20160}\left(\frac{P}{P_e}\right)^3+\text{etc.}\right\} \quad (13)$$

or—

$$-M_0=\frac{Wl}{4}+\frac{Wl^3}{48ET}P\left\{1+\frac{\pi^2}{10}\cdot\frac{P}{P_e}+\frac{17\pi^4}{1680}\left(\frac{P}{P_e}\right)^2+\text{etc.}\right\} \quad . \quad (14)$$

which illustrates again the same points.

Other cases may be found in a paper in the *Philosophical Magazine*, June, 1908.

The expression in brackets in (7) and (13) approximates to [1]—

$$1+\frac{P}{P_e}+\left(\frac{P}{P_e}\right)^2+\left(\frac{P}{P_e}\right)^3+\left(\frac{P}{P_e}\right)^4+\text{etc.}=\frac{1}{1-\frac{P}{P_e}}\text{ or }\frac{1}{1-\frac{Pl^2}{10EI}}$$

$$\text{or }\frac{P_e-P}{P_e}\text{ nearly (15)}$$

$\frac{P}{P_e}$ being a fraction less than $\frac{1}{4}$ say, hence to find the bending stress approximately in any case for a strut hinged at both ends we simply use the maximum bending moment M, say, due to the lateral loads alone and increase it in the ratio given by (15), so that—

$$M_0=\frac{M_1}{1-\frac{Pl^2}{10EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

[1] Prof. Perry obtains this result and the succeeding one (27) for tension in a different way by substituting an approximation for the right-hand side of equations (2) and (22).

Whether the bending moment is calculated by the approximate methods of the previous article applicable to short struts, or by (5) or by (16), the maximum intensity of bending stress $p_b$ disregarding sign, by Art. 63, is—

$$p_b = \frac{M_0 y_1}{I} = \frac{M_0}{Z} = \frac{M_0 d}{2I} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $y_1$ is the half-depth $\frac{d}{2}$ in a symmetrical section, and Z is the modulus of section. Hence, by Art. 111 (1) the maximum intensity of compressive stress—

$$f_c = \frac{M_0}{Z} + p_0 \text{ or } \frac{M_0 d}{2I} + p_0 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where $p_0$ is the mean intensity of compressive stress on the section, viz. $\frac{P}{A}$, where A is the area of cross-section, and the bending moment is taken as positive.

And the maximum intensity of tensile stress is—

$$f_t = \frac{M_0}{Z} - p_0 \text{ or } \frac{M_0 d}{2I} - p_0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which, if negative, gives the minimum intensity of compressive stress. If the section is not symmetrical, the value of the unequal tensile and compressive bending stress intensities must be found as in Art. 63 (6).

The formula (18) affords an indirect means of calculating the dimensions of cross-section for a strut of given shape, in order that, under given axial and lateral loads, the greatest intensity of stress shall not exceed some specified amount. As the method is indirect, involving trial, the value $M_1$ may be used to give directly a first approximation to the dimensions, which may then be adjusted by testing the values of $f_c$ by the more accurate expression (18), where $M_0$ satisfies (5) or (11) or (16).

Using the approximate values (18) becomes—

$$f_c = \frac{M_1 y_1}{I - \frac{Pl^2}{10E}} + \frac{P}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and (19) becomes—

$$f_t = \frac{M_1 y_1}{I - \frac{Pl^2}{10E}} - \frac{P}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

*Tie Rod with Lateral Load.*—If the axial load P is tensile the sign of P in (2) is reversed and the equation becomes

$$\frac{d^2y}{dx^2} - \frac{P}{EI} \cdot y = -\frac{w}{2EI}\left(\frac{l^2}{4} - x^2\right) \quad . \quad . \quad . \quad . \quad (22)$$

and the conditions of fixing being the same, the solution is

$$y = -\frac{w}{2P}x^2 + \frac{wl^2}{8P} - \frac{wEI}{P^2}\left(1 - \operatorname{sech}\sqrt{\frac{P}{EI}}\frac{l}{2}\cosh\sqrt{\frac{P}{EI}}x\right) \quad (23)$$

and

$$-M_0 = \frac{wEI}{P}\left(1 - \operatorname{sech}\frac{l}{2}\sqrt{\frac{P}{EI}}\right) = \frac{wEI}{P}\left(1 - \operatorname{sech}\frac{\pi}{2}\sqrt{\frac{P}{P_e}}\right) \quad (24)$$

This when expanded gives a series identical with (7) and (8), except that the signs of successive terms are alternately positive and negative. The second term, viz. $-P \cdot \frac{5}{384} \cdot \frac{wl^4}{EI}$ gives the reduction in bending moment resulting from the eccentricity of the tension on the assumption that the deflection is that due to the transverse load only.

If the tie-rod carries only a lateral load W at the centre, (11) becomes

$$\frac{d^2y}{dx^2} - \frac{P}{EI} \cdot y = -\frac{W}{2EI}\left(\frac{l}{2} - x\right) \quad . \quad . \quad . \quad . \quad (25)$$

and

$$y_0 = \frac{W}{2P}\sqrt{\frac{EI}{P}} \tanh \frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$-M_0 = \frac{W}{2}\sqrt{\frac{EI}{P}} \tanh \frac{l}{2}\sqrt{\frac{P}{EI}} \quad . \quad . \quad . \quad . \quad (26)$$

Other cases may be found in a paper in the *Philosophical Magazine*, June, 1908.

The expansion of (26), which is similar to that of (11), further illustrates the same points as the previous cases.

Proceeding as for struts, the approximation for either (24) or (26) is

$$M_0 = M_1\left\{1 - \frac{P}{P_e} + \left(\frac{P}{P_e}\right)^2 - \left(\frac{P}{P_e}\right)^3 + \text{etc.}\right\} = \frac{M_1}{1 + \frac{P}{P_e}} \text{ or } \frac{M_1}{1 + \frac{Pl^2}{10EI}} \text{ nearly}$$

$$. \quad . \quad (27)$$

and corresponding to (18) or (20), the maximum intensity of compressive stress is

$$f_c = \frac{M_0}{Z} - p_0 \text{ or } \frac{M_1 y_1}{I + \frac{Pl^2}{10E}} - \frac{P}{A} \text{ approximately} \quad . \quad . \quad (28)$$

and corresponding to (19) or (31), the maximum intensity of tensile stress is

$$f_t = \frac{M_0}{Z} + p_0 = \frac{M_1 y_1}{I + \frac{Pl^2}{10E}} + \frac{P}{A} \text{ approximately} \quad . \quad . \quad (29)$$

EXAMPLE.—A round bar of steel 1 inch diameter and 10 feet long has axial forces applied to the centres of each end, and being freely supported in a horizontal position carries the lateral load of its own weight (0·28 lb. per cubic inch). Find the greatest tensity of compressive and tensile stress in the bar: (*a*) under an axial thrust of 500 lbs.; (*b*) under an axial pull of 500 lbs.; (*c*) with no axial force. ($E = 30 \times 10^6$ lbs. per square inch.)

$w = 0{\cdot}28 \times \frac{\pi}{4} = 0{\cdot}22$ lb. per inch length. $M_1 = \frac{wl^2}{8} = 396$ lb.-inches.

$y_1 = 0{\cdot}5$ inch. $I = \frac{\pi}{64} = 0{\cdot}04909$.

$\frac{Pl^2}{10E} = \frac{500 \times 120 \times 120}{10 \times 30 \times 10^6} = 0{\cdot}024$. $p_0 = \frac{500}{0{\cdot}7854} = 637$ .lbs per sq. inch.

(*a*) Maximum intensity of bending stress by (16) is approximately

$$p_b = \frac{M_0 y_1}{I} = \frac{396 \times \frac{1}{2}}{0{\cdot}04909 - 0{\cdot}024} = \frac{198}{0{\cdot}02509} = 7900 \text{ lbs. per square inch.}$$

Maximum compressive stress, $f_c = 7900 + 637 = 8537$ lbs. per square inch.

Maximum tensile stress, $f_t = 7900 - 637 = 7263$ lbs. per square inch.

(*b*) Maximum intensity of compressive stress by (28) is approximately

$$f_c = \frac{198}{0{\cdot}04909 + 0{\cdot}024} - 637 = 2710 - 637 = 2073 \text{ lbs. per. sq. inch.}$$

Maximum intensity of tensile stress by (29) is approximately, similarly

$$f_t = 2710 + 637 = 3347 \text{ lbs. per square inch.}$$

(*c*) $$f_t = f_c = \frac{198}{0{\cdot}04909} = 4030 \text{ lbs. per square inch.}$$

The values of the bending stress by the more exact rules (5) and (24) are for (*a*) 8100 lbs. per square inch, and for (*b*) 2666 lbs. per square inch; they are worked out in the Author's "Strength of Materials."

**120. General Case of Combined Bending and Thrust or Pull.**—An empirical approximate formula covering any case of combined bending and longitudinal load corresponding to (14), Art. 118, and to (20) and (29), Art. 119, for the greatest intensity of stress is

$$\text{maximum } f = \frac{M_1 y_1}{I \mp \frac{Pl^2}{cE}} + \frac{P}{A}$$

where $M_1$ is maximum bending moment resulting (algebraically) from the eccentricity of P (neglecting flexure) and lateral loads, and where the negative sign is to be used when P is a thrust, and the positive sign when P is a pull. The *constant* $c$ for a strut hinged at both ends is about 8 for eccentric loads, 10 for uniformly distributed loads, and rather higher values for concentrated loads. It may be taken as, say, 10 in all cases.

A more exact but unwieldy result might, of course, be obtained by combinations of such equations as (1), Art. 118, and (1), Art. 119. The value of $c$ applicable to other forms of ends depends on the type of

loading and also upon whether the maximum stress occurs at the ends or at some intermediate point.[1] When both ends of a strut are fixed, $c$ may be taken as roughly about 34 for estimating the stress about the middle of its length, and as about 57 for the ends. When one end is hinged and the other is fixed, $c$ may be taken as about 24 for maximum stress at any section. The other extreme stress at the section of maximum stress is found in any case by reversing the sign of $\frac{P}{A}$.

**121. Bending Couples acting on Stanchions.**—In steel structures stanchions are frequently subjected to bending moments greatly in excess of any arising from slight accidental eccentricity in the application of the thrust or from flexure such as was considered in Arts. 118 and 119. For example, a stanchion supporting a roof may also serve as a crane post or carry a heavy load on a bracket. In such a case the effect of flexure will be negligible, and in the present article it will not be taken into account. To estimate the maximum stress it will be necessary to find the maximum bending moment at any section of the stanchion, and then to combine the stresses due to bending and thrust by the usual method given in (1), Art. 111.

The effects produced by a given bending couple depend upon the end conditions of the stanchion, and as the investigations follow fairly closely the lines of those of beams given in Chapters VII. and VIII., they are here stated somewhat briefly.

When an eccentric load is applied by a bracket, the couple is reducible to two equal and opposite forces perpendicular to the stanchion, and an axial thrust. To simplify the calculations the depth of the bracket will be supposed small in comparison with the length of the stanchion, so that the couple is taken as acting wholly at a single section; any error involved will be on the safe side, for the bending moment will be slightly over-estimated, any increase in the depth of a bracket reducing the maximum value.

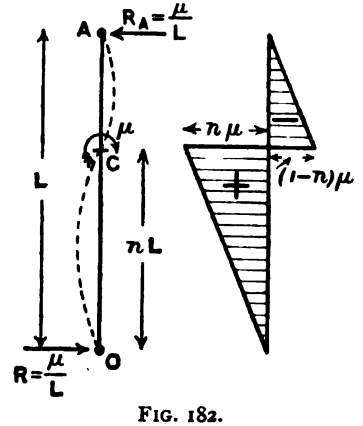



FIG. 182.

CASE 1.—*Stanchion hinged at both ends subjected to a couple* $\mu$ *at* C, *distant* $n$L *from the base* (Fig. 182).

Notation as in the figure and as used for beams in Art. 95, etc., except that length of stanchion is L. Taking the origin at O the base, and using the principles of Art. 93,

[1] These values may be obtained by expanding the transcendental functions involved in more exact values of the bending moments or by the method illustrated between equations (8) and (9) of Art. 119, but in the latter case deflections must be measured from a line joining points of contraflexure.

From O to C,

$$M = EI\frac{d^2y}{dx^2} = \frac{\mu}{L}.x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating,

$$EI\frac{dy}{dx} \text{ or } EI\,.\,i = \frac{\mu x^2}{2L} + A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$EIy = \frac{\mu x^3}{6L} + Ax + o \text{ (since } y = o, \text{ for } x = o) \; . \quad (3)$$

From C to A,

$$M = EI\frac{d^2y}{dx^2} = -\mu + \frac{\mu}{L}x \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$EI\,.\,i = \mu\left(\frac{x^2}{2L} - x\right) + B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

And for $x = nL$ equations (2) and (5) must give the same value for $i_C$, hence

$B = nL$ and (5) becomes

$$EIi = \mu\left(\frac{x^2}{2L} - x + nL\right) + A \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

$$EIy = \mu\left(\frac{x^3}{6L} - \frac{x^2}{2} + nLx\right) + Ax + C \quad . \quad . \quad . \quad (6)$$

which must agree with (3) when $x = nL$, hence $C = -\mu\frac{n^2L^2}{2}$ and

$$EI\,.\,y = \mu\left(\frac{x^3}{6L} - \frac{x^2}{2} + nLx - \frac{n^2L^2}{2}\right) + Ax \quad . \quad . \quad (6a)$$

This is zero for $x = L$, hence $A = \frac{\mu L}{6}(2 - 6n + 3n^2)$, and by substitution in the previous equations the values of $i$ and $y$ at all points in the elastic line may be found. Particular values are—

$$i_o = \frac{\mu L}{6EI}(2 - 6n + 3n^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$i_C = \frac{\mu L}{3EI}(1 - 3n + 3n^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$i_A = -\frac{\mu L}{6EI}(1 - 3n^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$y_C = \frac{\mu L^2}{3EI}.n(1 - n)(1 - 2n) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If $n = 1$ $\qquad i_C = i_A = \frac{\mu L}{3EI}$ . . . . . (11)

*Bending Moments.*—The general type of bending-moment diagram is shown, in accordance with equations (1) and (4) on the right-hand side of Fig. 182. The least possible bending moment to which the stanchion must somewhere be subjected is $\frac{1}{2}\mu$. When the maximum bending moment is as small as possible it will change from $-\frac{1}{2}\mu$ to $+\mu\frac{1}{2}$ at the application of the external couple $\mu$ hence putting $x = nL$,

equating (1) to $+\frac{1}{2}\mu$, or (4) to $-\frac{1}{2}\mu$, $n = \frac{1}{2}$ which is obvious in this case from the symmetry of the end conditions. The bending-moment diagrams for $n = \frac{1}{2}$ and for $n = 1$ are shown in Fig. 183.

If the couple $\mu$ consists of a horizontal pull $\frac{\mu}{d}$ and an equal horizontal thrust at a distance $d$ below it, the total change in bending moment will be at a uniform rate over the distance $d$, the point of inflexion occurring at some intermediate section, and a similar remark

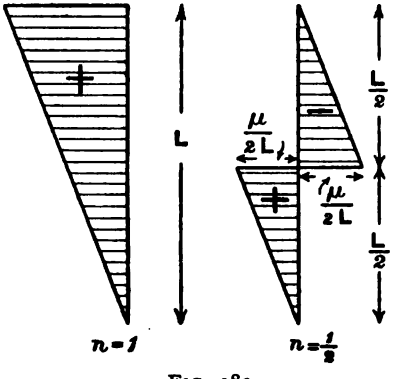



FIG. 183.

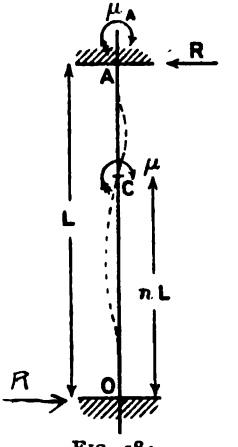



FIG. 184.

will apply to subsequent bending-moment diagrams; the conditions for, and amount of least bending moment on the stanchion may easily be found. It will occur when the length $d$ is symmetrically placed in the middle of the length L, and the amount will be $\frac{\mu}{2} \cdot \frac{L - d}{L}$.

*Case II. Stanchion fixed at both ends, acted upon by a couple $\mu$ at C distant $n$L from one end* (Fig. 184).

The conditions are $i_A = 0$, $i_0 = 0$, $y_A = 0$, $y_0 = 0$, $i_C$ and $y_C$ to be the same for the length AC as for the length OC. From O to C,

$$M = EI\frac{d^2y}{dx^2} = \mu + M_A - R(L - x) \quad . \quad . \quad . \quad . \quad (12)$$

$$EI \cdot i = \mu x + M_A \cdot x - R\left(Lx - \frac{x^2}{2}\right) + o \quad . \quad . \quad . \quad (13)$$

$$EI \cdot y = \tfrac{1}{2}\mu x^2 + \tfrac{1}{2}M_A x^2 - R\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + o \quad . \quad . \quad (14)$$

From C to A,

$$M = EI\frac{d^2y}{dx^2} = M_A - R(L - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$EI \cdot i = M_A \cdot x - R\left(Lx - \frac{x^2}{2}\right) + A$, and from (13) at $x = nL$,

$$A = \mu nL \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$EI.y = M_A.\frac{x^2}{2} - R\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + \mu nLx + B, \text{ and from (14) at } x = nL,$$

$$B = -\frac{\mu}{2}n^2L^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

$i_A = 0$, $y_A = 0$, hence—

$$RL = 6\mu n(1 - n) \quad \ldots\ldots\ldots\ldots \quad (18)$$

$$M_A = \mu n(2 - 3n) \quad \ldots\ldots\ldots\ldots \quad (19)$$

$$i_C = \frac{\mu L}{EI}n(i - n)(1 - 3n + 3n^2) \quad \ldots\ldots\ldots\ldots \quad (20)$$

Just below C,

$$M = \mu(-6n^3 + 9n^2 - 4n + 1) \quad \ldots\ldots\ldots\ldots \quad (21)$$

which must be equal to $\frac{1}{2}\mu$ for least value of M at C, hence

$$n = \tfrac{1}{2} \text{ or } \tfrac{1}{2} \pm \sqrt{\tfrac{1}{12}}, \textit{ i.e. } 0{\cdot}5,\ 0{\cdot}7887 \text{ or } 0{\cdot}2113 \quad \ldots \quad (22)$$

The complete results for these cases are shown in Fig. 185. The inflections below C are at $x = \frac{3n - 1}{6n}.L.$

If $n = 1$, this gives $x = \frac{1}{3}L$ . . . . . (23)

Also $M_0 = \mu(1 - 4n + 3n^2)$ . . . . . (24)

which varies between $+\mu$ and $-\frac{1}{2}\mu$; the least value of the bending moment at the base is zero, viz. when $n = \frac{1}{3}$ or $n = 1$. The results in

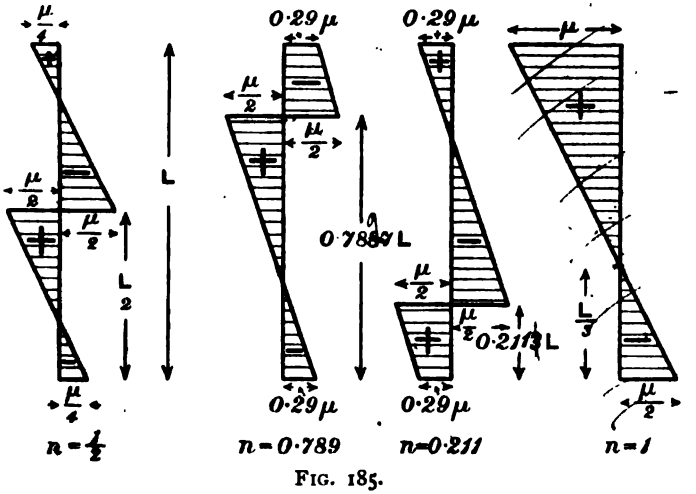



FIG. 185.

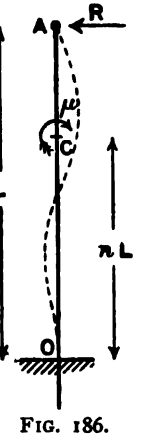



FIG. 186.

this case can also conveniently be obtained by the alternative method given in Art. 103 (14) and (15), the deflections and slopes being found from Art. 95.

*Case III. Stanchion fixed at the base and hinged at the top, acted upon by a couple* $\mu$ *at* C *distant* $nL$ *from the base* (Fig. 186).

The conditions are $i_0 = 0$, $y_0 = 0$, $y_A = 0$, $i_C$ and $y_C$ the same for length AC as for length OC. From O to C,

$$M = EI\frac{d^2y}{dx^2} = \mu - R(L - x) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$EI \cdot i = \mu x - R\left(Lx - \frac{x^2}{2}\right) + o \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$EI \cdot y = \mu\frac{x^2}{2} - R\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + o \quad . \quad . \quad . \quad . \quad . \quad (27)$$

From C to A, $M = EI\frac{d^2y}{dx^2} = -R(L - x)$

$EIi = -R\left(Lx - \frac{x^2}{2}\right) + A$, and from (26) at $x = nL$,

$$A = \mu nL \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$EIy = -R\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + \mu nLx + B$, and from (27) at $x = nL$,

$$B = -\frac{\mu n^2 L^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$y_A = o$, hence— $$R = \frac{3\mu n(2 - n)}{2L} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

If $n = 1$ $$i_A = \frac{\mu L}{4EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

For the least bending moment anywhere on the stanchion, M just below C must be $+\frac{1}{2}\mu$, and from (25) this occurs when

$$3n^3 - 9n^2 + 6n - = o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

*i.e.* when $n = 0{\cdot}605$ or $0{\cdot}258$ . . . . . (33)

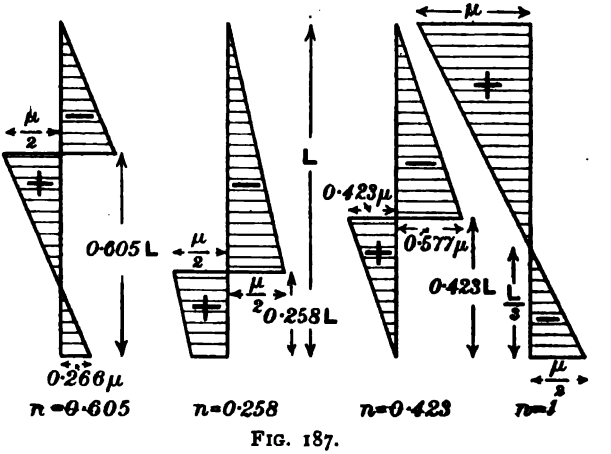



FIG. 187.

The bending-moment diagrams for these cases are shown in Fig. 187, which also shows the condition to make the bending moment at

the base zero and the particular case of $n = 1$. The point of inflection below C is at

$$x = \frac{3n^2 - 6n + 2}{3n(n-2)} \cdot L, \text{ which is } \tfrac{1}{3}L \text{ if } n = 1 \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The results in this case can be obtained as in Art. 95 (14), which agrees with (30) of the present article, slopes and deflections then being found by the method of Art. 97 or otherwise.

Ex. 3 of Art. 95 and Fig. 144 illustrate the modifications when the couple consists of two forces perpendicular to the stanchion.

*Case IV. Stanchion fixed at the base and free at the top acted upon by a couple* $\mu$.

In this case the stanchion is subject to the bending moment of the full amount $\mu$. The slopes and deflections have been given in Art. 95 (*d*), Fig. 143.

*Other Cases.* Another possible case would be that of a stanchion fixed at the base, and fixed in direction only but not in position at the top. Practical stanchions will have conditions intermediate to those given. For example, with good foundations and rigid base plates securely fastened a stanchion may be taken as "fixed" at the base. Where the top of the stanchion is only connected to other parts of a structure of appreciable flexibility the reaction R of Fig. 186 will be less than that given by (30), and the conditions will lie between Cases III. and IV.

EXAMPLE.—Solve Example 3 of Art. 95 (*a*) if both ends of the stanchion are hinged, (*b*) if both ends are fixed.

(*a*) Taking moments about either hinge

$$R_A = R_0 = \frac{\mu}{L} = \tfrac{10}{15} = \tfrac{2}{3} \text{ ton.}$$

At B, $M_B = -R \times AB = -\tfrac{2}{3} \times 5 = -3{\cdot}3$ tons-feet.
At C, $M_C = \tfrac{2}{3} \times 7{\cdot}5 = 5$ tons-feet.

(*b*) The reaction and bending moment at say, the upper end may be calculated by (14) and (15) of Art. 103, $\delta$ and $i$ being calculated by applying (5) and (3) of Art. 95 as indicated at (12) and (13) of Art. 95. Or by using the values given in Ex. 2 of Art. 103 for the thrust and pull of 4 tons each,

$$R_A = \frac{4}{15^3}\{100 \times 25 - (7{\cdot}5)^2 \times 30\} = \tfrac{26}{27} \text{ ton.}$$

$$M_A = \tfrac{4}{225}\{500 - (7{\cdot}5)^3\} = \tfrac{25}{18} = 1\tfrac{7}{18} \text{ tons-feet.}$$

At B (Fig. 144) $M_B = 1\tfrac{7}{18} - \tfrac{26}{27} \times 5 = -\tfrac{185}{54} = -3{\cdot}426$ tons-feet.
At C $M_C = \tfrac{25}{18} - \tfrac{26}{27} \times \tfrac{15}{2} + 10 = 4\tfrac{1}{6}$ tons-feet.
At O $M_0 = \tfrac{25}{18} - \tfrac{26}{27} \times 15 + 10 = -3\tfrac{1}{18}$ tons-feet.

The changes in bending moment being linear from point to point.

## EXAMPLES IX.

1. In a short cast-iron column 6 inches external and 5 inches internal diameter, the load is 12 tons, and the axis of this thrust passes $\frac{1}{8}$ inch from the centre of the section. Find the greatest and least intensities of compressive stress.

2. The axis of pull in a tie-bar 4 inches deep and $1\frac{1}{2}$ inch wide passes $\frac{1}{10}$ inch from the centre of the section and is in the centre of the depth. Find the maximum and minimum intensities of tensile stress on the bar at this section, the total pull being 24 tons.

3. The vertical pillar of a crane is of I section, the depth of section parallel to the web being 25 inches, area 24 square inches, and the moment of inertia about a central axis parallel to the flanges being 3000 (inches)$^4$. When a load of 10 tons is carried at a radius of 14 feet horizontally from the centroid of the section of the pillar, find the maximum intensities of compressive and tensile stress in the pillar which is fixed at the base and quite free at the top.

4. If a cylindrical masonry column is 3 feet diameter and the horizontal wind pressure is 50 lbs. per foot of height, assuming perfect elasticity, to what height may the column be built without causing tension at the base if the masonry weighs 140 lbs. per cubic foot?

5. A mild-steel strut 5 feet long has a T-shaped cross-section $6 \times 4 \times \frac{1}{2}$, see B.S.T., 21 in., Table VI, Appendix. Find the ultimate load for this strut, the ends of which are freely hinged, if the crushing strength is taken as 21 tons per square inch and the constant $a$ of Rankine's formula $\frac{1}{7500}$.

6. Find the greatest length for which the section in problem No. 5 may be used, with ends freely hinged, in order to carry a working load of 4 tons per square inch of section, the working load being $\frac{1}{4}$ of the crippling load and the constants as before.

7. A mild-steel stanchion, the cross-sectional area of which is 53·52 square inches, is as shown in Fig. 176, the least radius of gyration being 4·5 inches. The length being 24 feet and both ends being fixed, find the crippling load by Rankine's formula, using the constants given in Art. 116.

8. Find the ultimate load for the column in problem No. 7, if it is fixed at one end and free at the other.

9. Find the breaking load of a cast-iron column 8 inches external and 6 inches internal diameter, 20 feet long and fixed at each end. Use Rankine's constants.

10. Find the working load for a mild-steel strut 12 feet long composed of two T-sections $6'' \times 4'' \times \frac{1}{2}''$, the two 6-inch cross-pieces being placed back to back, the strut being fixed at both ends. Take the working load as $\frac{1}{4}$ the crippling load by Rankine's rule.

11. Find the ultimate load on a steel strut of the same cross-section as that in problem No. 10, if the length is 8 feet and both ends are freely hinged.

12. Find the necessary thickness of a metal in a cast-iron pillar 15 feet long and 9 inches external diameter, fixed at both ends, to carry a load of 50 tons, the ultimate load being 6 times greater.

13. Find the external diameter of a cast-iron column 20 feet long, fixed at each end, to have a crippling load of 480 tons, the thickness of metal being 1 inch.

14. A latticed stanchion is built of two standard channel sections 7 in. by 3 in. (see B.S.C. 9, Table II, Appendix) placed back to back. How far apart should they be placed in order to offer equal resistance to buckling in all directions?

15. Solve problem No. 1 if the column is 10 feet long, one end being fixed and the other having complete lateral freedom. (E = 5000 tons per square inch.)

16. With the ultimate load as found by Rankine's formula in problem No. 5, what eccentricity of load at the ends of the strut (in the direction of the least radius of gyration and towards the cross-piece of the T) will cause the straight homogeneous strut to reach a compressive stress of 21 tons per square inch, assuming perfect elasticity up to this load? The distance from the centroid of the cross-section to the compression edge is 0·968 inch. (E = 13,000 tons per square inch.)

17. With the eccentricity found in problem No. 16 and a load of 16 tons per square inch of section, of what length may the strut be made in order that the greatest intensity of compressive stress shall not exceed 21 tons per square inch? What is then the least intensity of stress, the distance from the centroid of the cross-section to the tension edge being 3·032 inches?

18. Find the load which will cause an extreme compressive stress of 21 tons per square inch in a stanchion of the section given in problem No. 7, 12 feet long and freely hinged at the ends, if the depth of section in the direction of the least radius of gyration is 16 inches, and the deviation of the load from the centre of the cross-section is 1 inch in the direction of the 16-inch depth. (E = 13,000 tons per square inch.)

19. What load will the column in Problem No. 1 carry if it is fixed at one end, and has complete lateral freedom at the other, if the column is 10 feet long, the eccentricity of loading $\frac{1}{4}$ inch, and the greatest tensile stress 1 ton per square inch. What is the greatest intensity of compressive stress? (E = 5000 tons per square inch.)

20. Find the necessary diameter of a mild-steel strut, 5 feet long, freely hinged at each end, if it has to carry a thrust of 12 tons with a possible deviation from the axis of $\frac{1}{10}$ of the diameter, the greatest compressive stress not to exceed 6 tons per square inch. (E = 13,000 tons per square inch.)

21. Solve problem No. 19 if the deviation may amount to 1 inch.

22. A round straight bar of steel 5 feet long and 1 inch diameter rests in a horizontal position, the ends being freely supported. If an axial thrust of 2000 lbs. is applied to each end, find the extreme intensities of stress in the material. Weight of steel, 0·28 lb. per cubic inch. ($E = 30 \times 10^6$ lbs. per square inch.)

23. Find what eccentricity of the 2000-lbs. thrust in the previous problem will make the greatest intensity of compressive stress in the bar the least possible, and the magnitude of the stress intensity.

# CHAPTER X

## *FRAMED STRUCTURES*

**122. Frames and Trusses.**—The name *frame* is given to a structure consisting of a number of bars fastened together by hinged joints; the separate bars are called members of the frame. Such structures are designed to carry loads mainly applied at their joints, the members being simple ties or struts although the structure as a whole may be subjected to bending.

The external forces acting on a framed structure are the loads, and the supporting forces or reactions at its points of support. In many important framed structures the centre lines of all members and of all loads and reactions lie approximately in one plane; such structures may be called plane frames. In other cases, of which we shall notice a few, the members and forces do not lie in one plane, but are more generally distributed in space; such frames may be called space frames. The most important frames are *trusses*, which act as a whole as beams; they include braced girders of bridges called bridge trusses and roof principals called *roof trusses*.

Although the name *frame* has been applied to hinge jointed structures, it is the usual British practice to make most framed structures with riveted joints. In America and elsewhere pin-jointed structures are in many cases employed, and in such cases the force or stress in members can be determined by the principles of statics with more certainty than where the more rigid riveted joints are used. It is usual, however, to estimate the stresses in structures of which the members are riveted together, or in some cases two or more members form one continuous piece, as if the bars were all freely hinged at every joint. Such a computation neglects *secondary* (*bending*) *stresses* arising from resistance to free angular movement at the joints. The secondary stresses are sometimes separately estimated (see Art. 174).

**123. Perfect and Imperfect Frames.**—A *perfect* frame is one which has just sufficient members to keep it stable in equilibrium under any system of external forces acting at its joints without change of shape. If the frame has either more or fewer than this number it is said to be *imperfect*. If it has fewer members it is said to be *deficient* or unstable. If it has more it is said to be a *redundant* or over rigid frame. Fig. 188 represents examples of perfect plane frames; they have the property that the length of any one member may be slightly altered (as by change of temperature or error of workmanship) without inducing stress in any of the other members.

Fig. 189 represents deficient frames: while they may be stable under a certain system of loads any change in direction or magnitude of the applied loads may render them unstable, and change their shape except in so far as such change is resisted by rigid joints. A member joining either AB *or* CD would make the frames perfect.

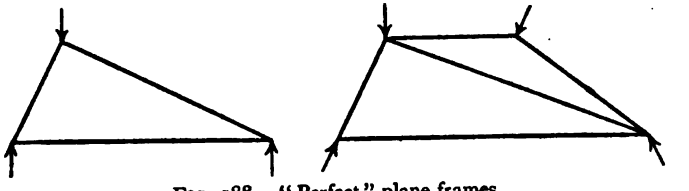


FIG. 188.—"Perfect" plane frames.

Fig. 190 represents redundant frames formed by the addition of members AB and CD to Fig. 189. Such frames are generally stressed if an alteration of length takes place in any one member due to change in its temperature or error in construction, and the frame is then said to be self-strained. The stresses in redundant frames are not calculable

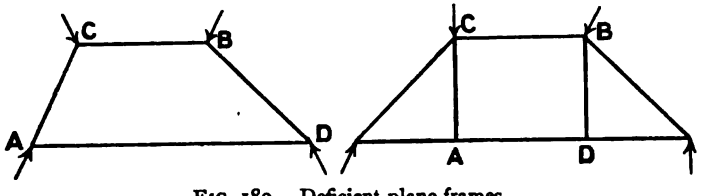



FIG. 189.—Deficient plane frames.

by the simple statical principles applicable to perfect frames; the frames are called statically indeterminate structures (see Chap. XIV.).

*Use of Counterbraces.*—Such frames as those shown in Fig. 190 are frequently used; although redundant they may serve as practically perfect frames if the ties or braces AB and CD are long, because their resistance

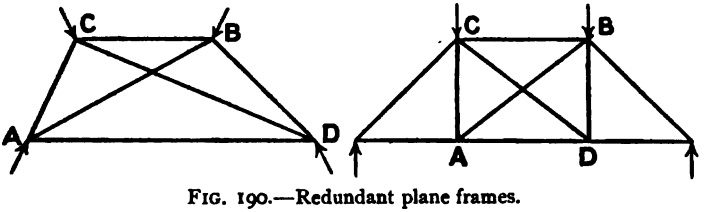



FIG. 190.—Redundant plane frames.

to compression (as struts) is then negligible. Thus excess of external thrust at B say, puts CD in tension, and AB out of use, while excess of thrust at C puts tension in AB while CD is idle. Thus a structure counterbraced with flexible ties may resist the changing action of a moving load employing the braces alternately.

**124. Number of Members in a Perfect Frame.**—The basis of the

perfect plane frame is the triangle which has three members and three joints (Fig. 188). For every additional joint two more bars will be required in building up a more complex perfect frame which is always divisible up into triangles, hence for four joints the number of members is $3 + 2$, for five joints $3 + 4$, and for $n$ joints

$$3 + 2(n - 3) = 2n - 3 \text{ members.}$$

This criterion serves to show on inspection whether a plane frame is perfect, deficient, or redundant.

Similarly the basis of the space frame is the tetrahedron, having four joints and six members; for each additional joint three additional members will be required, and for $n$ joints the number of members will be,

$$6 + 3(n - 4) = 3n - 6.$$

**125. Roofs and Roof Trusses.**—Roofs of considerable span are supported at intervals by principals or trusses, which resist the bending resulting from the loads applied to the roof. Fig. 191 shows five roof principals, the first of which is ABCDE, and the second A'B'C'. The roof covering is attached to the *purlins* which transfer the load to the joints of the principals. Fig. 192 shows a number of roof trusses suitable for various spans, and indicates to some extent the evolution of large roof trusses. The thick lines indicate struts and the thin ones ties. (*a*) represents two rafters with a single tie forming a roof principal suitable for small spans; (*b*) represents the King Post Truss which has a suspension rod from the apex to the cross tie; (*c*), (*d*), and (*e*) represent suitable types of frames for larger spans; (*d*) is sometimes a timber truss, excepting the vertical ties which are steel; (*e*) represents a very common steel truss, the struts being shorter than in (*d*). The length of main rafter between successive purlins (at joints) is usually limited to about 8 feet, which helps to determine the type of truss to be used. The total rise of a roof with straight rafters is usually $\frac{1}{4}$ of the span, and for large spans a crescent shape such as (*f*) is sometimes adopted to obviate a high roof.

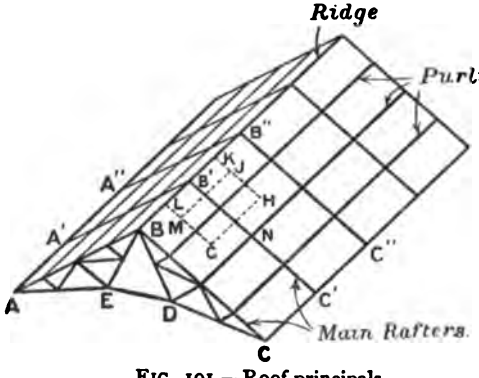


FIG. 191.—Roof principals.

Types (*g*), (*h*), (*i*) and (*j*) may be looked upon as a different line of development for steel roofs, each main rafter being supported by its own truss, and the two trusses tied together by the main horizontal tie bar. The struts are short, being in many cases perpendicular to the rafters.

All these roofs may be made with the main horizontal tie bar slightly

*cambered* (*i.e.* raised above the points of support of the roof) as shown,

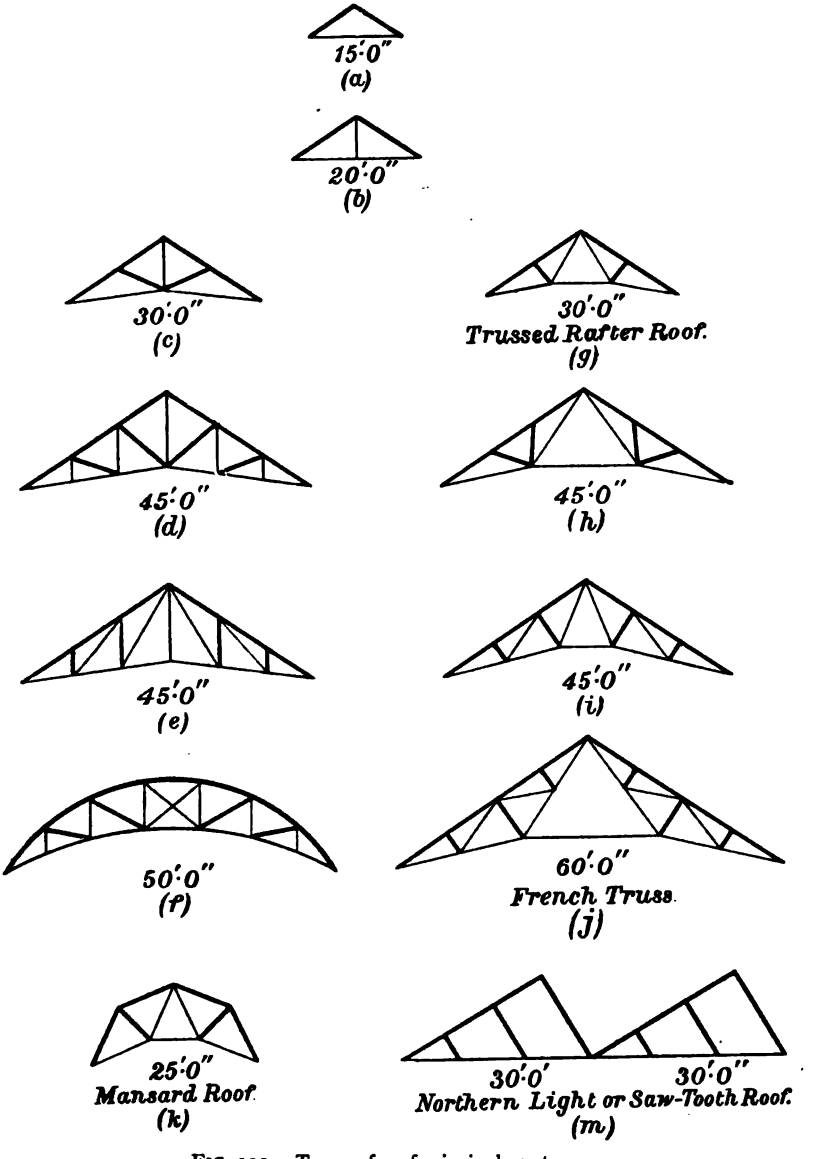



FIG. 192.—Types of roof principals or trusses.

say, $\frac{1}{40}$ of the span, or with the lower ties all in one horizontal line adjoining the two points of support. A cambered tie admits of shorter struts.

The form (*k*) represents a Mansard roof sometimes used when roof space is to be utilized for rooms. (*m*) represents a very common form of roof for workshops or sheds, the short side being glazed to admit a northern light without direct sunshine.

**126. Braced Girders.**—A braced girder or open webbed girder consists of tension and compression flanges to withstand the pull and thrust arising, as explained in resistance to bending moments (Chap. V.), connected by bracing or web members which withstand the shearing force. The flanges, called the upper and lower booms or chords, are often continuous, although neglecting secondary stresses, the stresses in the members are calculated as if the portions of the chords were discontinuous at the joists with the web members. Fig. 193 shows diagrammatically the parts of a simple braced girder single track

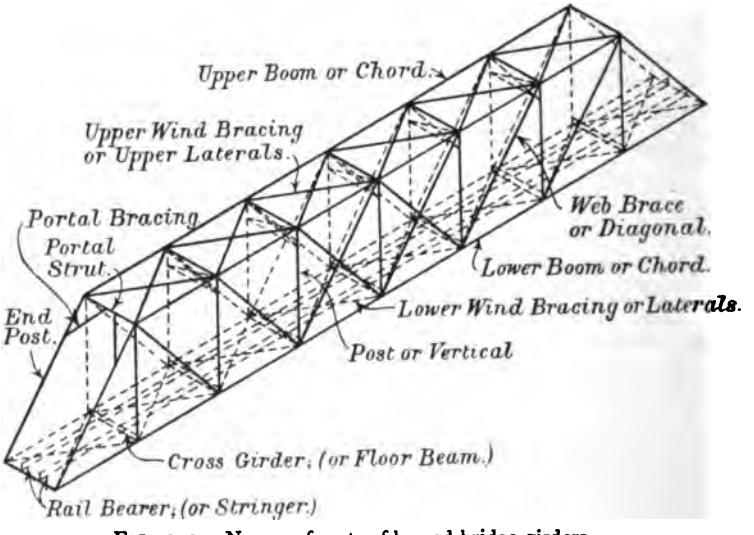



FIG. 193.—Names of parts of braced bridge girders.

railway bridge of the through type. When the load on a bridge is carried by the girders at the joints of the lower boom the bridge is called a *through* bridge; when at the joints of the upper chord a *deck* type. In the former case the load passes *through* the bridge, and in the latter case *over* it. The floor systems of bridges vary, but Fig. 193 shows a case in which the train load is carried on *rail bearers* which are supported by *cross girders* which transfer the weight to the main girders at the joints or panel points of the lower boom, which is divided into a number (in this case 8) of equal panels or bays.

To resist wind pressure on the side of the main girders, *wind bracing* (crossed) is placed below the track and if head room allows also connecting the upper booms. If the head room is insufficient curved or arched girders sometimes connect the top booms. Where head room is ample,

crossed braces in a vertical plane called sway bracing sometimes connect the vertical post and assist in resisting side pressure of the wind and centrifugal force (if any) of the moving load, and in reducing distortion of the bridge due to deflection of the cross girders or floor beams. The end posts are also usually connected by a substantial strut called the portal strut which, particularly in the absence of upper wind bracing, transfers a considerable part of the wind load from one main girder to the other. The portal formed by the end posts and the connecting strut is usually braced when possible.

*Chief types*—(a) *Parallel type.*—The commonest forms of girders with parallel chords are shown in Fig. 194; the struts are shown by thick lines and the ties by thin ones.

The **N** or Pratt type is the commonest type of braced girder for moderate spans; it is also sometimes made with end posts vertical instead of sloping as shown in the figure and in Fig. 193 (see Fig. 204). The central bay or bays being counterbraced the frame is strictly speaking redundant, but the counterbraces serving as ties only, the frame is virtually perfect. The necessity for counterbracing near the middle of the span arises from the change in sign of the shearing force (see Art. 86 and Fig. 129) which is taken by the diagonals.

The Warren girder, the diagonals of which are inclined at 45° or 60°, also represents a fairly common form and is a perfect frame. The Howe truss which is fairly common in America is used for combinations of steel and timber construction, the sloping struts being timber. In the **N** type with diagonals sloping the other direction the steel struts are vertical and as short as possible.

The shorter panels of double intersection trusses allow a shorter railbearer to be used in a large bridge with a fixed inclination of the diagonals, but require more, although slightly lighter, cross girders. The Baltimore truss is a simple modification of the **N** type with sub-divided panels and is used largely in America for long spans. The double Warren or single lattice girder has one redundant member. Double lattices are also used.

(b) *Curved type.*—For long spans (above say 180 or 200 ft.) a braced girder with a curved or broken chord becomes more economical although more expensive to construct than the parallel type. Examples of hog-back girders, *i.e.* girders with the upper chord curved convex upwards, are shown in Figs. 198, 208, 214 and 215.

**127. Dead Loads on Roofs.**—The coverings may be taken as about the following weights per square foot of horizontal ground area covered :—

| | |
|---|---|
| Tiles on boarding, with steel purlins . . . . . | 24 lbs. |
| Slates on boarding, with steel purlins . . . . . | 14 lbs. |
| Corrugated iron, and steel purlins . . . . . | 6 lbs. |
| Glazed covering and purlins . . . . . | 8 lbs. |

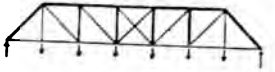
In addition to this there is the weight of the truss itself to be carried. This cannot be known accurately until it has been designed, but various formulæ have been devised from existing roofs to give a preliminary estimate which may be checked after the roof is designed and if

## BRACED BRIDGE GIRDERS—PARALLEL CHORDS.

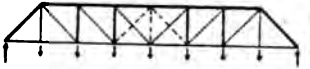
### THROUGH.

#### Single web systems.

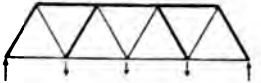
N or Pratt. (*a*) Odd number of bays.



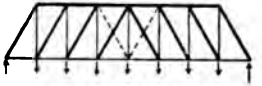
(*b*) Even number of bays.



Warren.



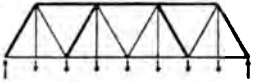
Howe.



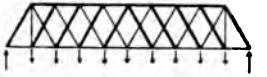
#### Divided panels or double web systems.

Whipple Murphy or Linville.

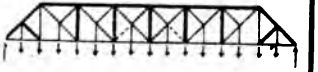
Warren (with vertical suspenders).



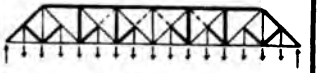
Double Warren or single lattice.



Baltimore (*a*).



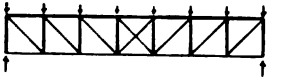
(*b*).



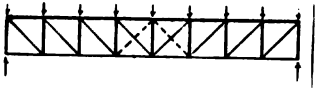
### DECK.

#### Single web systems.

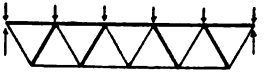
N or Pratt. (*a*) Odd number of bays.



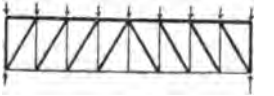
(*b*) Even number of bays.



Warren.

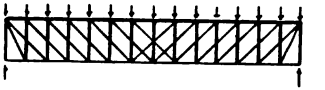


Howe.



#### Divided panels or double web systems.

Whipple Murphy or Linville.



Warren (with verticals).

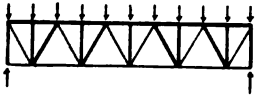


FIG. 194.

necessary the design modified accordingly. The following such formulæ are in use for pine and steel roofs: Ricker's formula[1]

$$w = \frac{s}{25} + \frac{s^2}{6000}$$

where $s$ = span in feet, $w$ = weight of truss in pounds per sq. ft. of horizontal projection of roof. This varies from about 1 to 15 lbs. per sq. ft. for spans from 20 to 200 ft. For spans under 100 ft. roofs entirely of steel are somewhat heavier.

Howe's formula,

$$w = \tfrac{3}{4}\left(1 + \frac{s}{10}\right).$$

For moderate spans inclusive dead loads some 2 to 5 pounds per sq. ft. greater than those given for coverings alone are commonly adopted.

*Special loads.*—Any load suspended from the truss must be separately allowed for in estimating the stress in the members.

*Occasional loads. Snow.*—The allowance to be made for snow on a roof depends upon the climate. In Great Britain the usual allowance is 5 pounds per sq. ft. of horizontal projection of the roof on which snow can collect, taken in addition to dead and wind loads.

**128. Wind Loads on Structures.**—The pressure of the wind is often one of the most important loads which exposed structures such as roofs have to bear.

Many experiments have been made to determine the pressure on surfaces resulting from wind pressure. Of these we notice particularly three series.

(1) Experiments made during the construction of the Forth Bridge[2] 1883–1890.

Pressures were recorded by gauges on small areas of 1–5 sq. ft. and also on a larger area of 300 sq. ft. The most notable fact recorded was that the maximum pressure per sq. ft. reached on the small area was much greater than the average reached on the whole of the large area, the highest value being 41 lbs. per sq. ft. on the small area and 27 lbs. per sq. ft. on the large one with average maximum values for 12 violent gales of 29·8 and 16·9 lbs. respectively. The maximum values on the areas were not necessarily reached simultaneously and later experiments referred to below support the explanation that the greater pressure on the smaller area results mainly from the very localised intensity of gusts.

(2) Records made on the Forth Bridge since its erection.[3]

On 1·5 sq. ft. gauges, these experiments show the great difference of pressure at different heights above ground varying from a maximum of 65 lbs. per sq. ft. at 378 ft. elevation to 20 lbs. per sq. ft. at 50 ft.

[1] "A Study of roof Trusses," Bulletin No. 16, Univ. of Illinois, Eng. Experiment Station.

[2] See *Engineering*, Feb. 28, 1890.

[3] See paper by Mr. Adam Hunter, M.Inst.C.E., in the Transactions of the Junior Inst. of Engs., 1906.

with average values during 15 storms (1890–1906) of 50 and 13 lbs. per sq. ft. respectively.

(3) Experiments made at the National Physical Laboratory.[1]

The earlier experiments indicate a normal pressure intensity P on small circular and square surfaces a few square inches in area perpendicular to the direction of an artificial air current of—

$$P = kV^2 = 0{\cdot}0027\ V^2 \text{ lbs. per sq. ft.} \quad . \quad . \quad . \quad . \quad (1)$$

where V = velocity of the wind in miles per hour; other experimenters have obtained a rather higher value of the coefficient $k$. Various interesting results were obtained relating to pressures on surfaces of different shapes, and model lattice girders on which the intensity of pressure was higher than on square plates.

It also appears that the wind pressure on flat plates consists partly of the pressure on the windward side and partly of a suction on the leeward side. On small roof models the suction on the leeward slope appeared to be of equal importance with the pressure on the windward slope.

The later experiments in the open air with wind pressure on surfaces 25 to 100 sq. ft. in area indicate a normal pressure on rectangular surfaces of about

$$P = k \cdot V^2 = 0{\cdot}0032\ V^2 \text{ pounds per sq. ft.} \quad . \quad . \quad . \quad (2)$$

with little or no difference in pressure per sq. ft. with difference in area.

Experiments on a large model lattice girder in the open air show a pressure of

$$0{\cdot}00405 \cdot V^2 \text{ pounds per sq. ft.} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or 1·26 times as great a pressure as on a rectangular board of equal area.

The later experiments on roof slopes 56 sq. ft. in area in the open air indicate important suction effects on the leeward slopes of roofs of buildings the internal pressure of which may be affected by wind, and negligible suction effects on the leeward slopes if the roof is mounted on columns through which the wind can pass freely. The normal pressure on the roof being

$$P = k \cdot V^2 \text{ pounds per sq. ft.} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the values of $k$ for three slopes are given as follows for the case in which internal pressure of a building may be affected by the wind (*e.g.* openings on windward side, and none on the leeward side).

| | Values of $k$ for slopes of | | |
|---|---|---|---|
| | 60° | 45° | 30° |
| Windward side . . . . . | +0·0034 | +0·0028 | +0·0015 |
| Leeward side . . . . . | −0·0032 | — | −0·0022 |

[1] Proc. Inst. C.E., vol. clvi, "The Resistance of Plane Surfaces in a Uniform Current of Air," by Dr. T. E. Stanton, and later, "Experiments on Wind Pressure," vol. clxxi.

The values of $k$ for the case of a building open on both sides are the same for the windward slope and zero for the leeward slope. There is considerable advantage in being able to state the intensity of pressure on a surface, which is either perpendicular to, or oblique to the direction of the wind in terms of the wind velocity, as in (2), (3) or (4), since to predetermine the probable pressure which a proposed structure will have to bear, it is only necessary to measure the maximum velocity of the wind at the site.

*Actual Wind Load Allowances.*—The usual allowance for wind pressure perpendicular to the wind (*i.e.* on a vertical surface normal to an assumed horizontal wind) is from 30 to 56 lbs. per sq. ft. according to the exposure of the situation. So low an allowance as 30 lbs. should be treated as a live load, but 56 lbs. might be taken as an equivalent dead load (see Art. 41). The value given by (3) for V = 100 miles per hour (about a maximum value for Great Britain) would be 40·5 lbs. per sq. ft. The British Board of Trade require an allowance of 56 lbs. per sq. ft. for girders in exposed situations, while the building laws of several American cities require an allowance of 30 lbs. per sq. ft. horizontal wind pressure on buildings. A common allowance for bridge designs is 30 lbs. per sq. of train (taken at 10 sq. ft. per lineal foot) for the travelling wind load. The most commonly quoted value for the pressure $P_n$ normal to a roof sloop inclined at an angle $\alpha$ to the horizontal, in terms of the horizontal wind pressure P (neglecting leeward suction) is that given by Unwin's formula based on experiments by Hutton, viz. :—

$$P_n = P \cdot \sin \alpha^{1 \cdot 84 \cos \alpha - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Another formula in common use is that of Duchemin, viz. :—

$$P_n = P \cdot \frac{2 \sin \alpha}{1 + \sin \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The relative complication of such formulæ does not appear to be justified by experimental results, and a simpler formula reasonably correct would be

$$P_n = \frac{P \cdot \alpha}{45} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for values of $\alpha$ up to 45°, and above that slope, $P_n$ may be taken as equal to P. This agrees with Unwin's formula for the almost standard rise of $\frac{1}{4}$ span for which $\alpha = 26° 34'$, and $P_n = 0 \cdot 59$ P. Comparisons of (5), (6), and (7) may easily be made by plotting $P_n$ on a base of $\alpha$ for any value of P, such as P = 1 which gives the coefficient of P in the values of $P_n$.

**129. Dead Loads on Bridges.**—These consist of the weight of the steel superstructure, roadway, ballast, permanent way, etc.

Some of these items can be fairly accurately estimated before the design is complete from the known volume and density of the materials carried. The following are usual values—

| | | |
|---|---|---|
| Ballast (normally about 1 ft. deep) | 120 | lbs. per cubic ft. |
| Concrete . . . . . | 140 | " " |
| Brickwork . . . . . | 120 | " " |
| Masonry . . . . . | 140 | " " |
| Asphalte . . . . . | 156 | " " |
| Timber . . . . . | 45 | " " |

Permanent way for single line of railway 175 lbs. per ft. run (excluding ballast).

The *actual* weight of cross girders, rail bearers, etc., should be taken into account in designing the main girders, or if a preliminary estimate is used the design should afterwards be checked by the actual values. The weight of the main girders depends upon the type of bridge, and the actual weight should be calculated after a preliminary design; before this can be made a preliminary estimate of the dead weight of the main girders is required, and is based on the known weight of bridges of similar types. This must be largely a matter of experience and available data of similar designs. Various formulæ have been devised to give for various types of bridges approximations to the dead weight of either the main girders, or of the whole of the steelwork including the floor. The following may be cited—

Unwin's Formula:

$$\text{Weight of girder in tons per foot run} = w = \frac{Wr}{cs - lr} \quad . \quad (1)$$

where W = total equivalent uniformly distributed dead load in tons.

$r$ = ratio of span to depth.

$l$ = clear span in feet.

$s$ = working stress in tons per square inch in the booms.

$c$ = a constant of about 1400 in small plate girders to about 1800 for braced girders, or may be deduced for any type of girder from examples of known size, weight, and working stress.

Anderson's Formula (for plate girders) —

$$w = \frac{W}{500} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

*American Formulæ.*—These are generally attempts to approximate to all the dead load of the structure including the floor and are of the type

$$w = al + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $a$ and $b$ are constants depending on the type of bridge, and whether for single or double track railway, on the traffic to be borne, and upon the working intensity of stress allowed. Evidently the variables $s$ and $r$ in (1) must affect the value of $w$, and a formula such as (3) can only be used under fairly restricted values of $s$ and $r$ which are established practice. Thus the values of $a$ and $b$ applicable to say an American bridge company's usual design would give a much smaller value of $w$ than would correspond to the practice of say a British railway company for a similar rolling load.

**130. Moving Loads on Bridges.**—These vary greatly according to the class of traffic to be borne, and some values have been given in Arts. 84 and 85.

The wind load on a moving train is sometimes treated separately as a moving load, or allowed for by an increase in the uniformly distributed wind load on the girders.

*Load due to Centrifugal Force.*—The lateral pressure on the rails

due to the centrifugal force exerted by any part of a train if the line of rails crossing a bridge is on a curve is calculated from the formula $\frac{W \cdot v^2}{gr}$ where W is the weight of the portion considered, $v$ its speed in feet per second, $g = 32{\cdot}2$ feet per sec. per sec., and $r$ is the radius of the curve in feet. This lateral pressure is added to the wind pressure on the loaded boom of the bridge and affects the stress in the lateral or wind bracing. The eccentricity of the centre of gravity of the train loads due to elevation of the outer rail on a curve will also cause some slight modification in the stresses produced in the structure.

*Load due to Braking Forces.*—The (forward) horizontal forces exerted by a train on the rails when brakes are applied may amount to about one fifth of the weight of the train distributed in the same way as the wheel loads. The most important effect will be to cause bending stress in the cross girders which bend in a horizontal plane.

**131. Incidence and Distribution of Loads on Framed Structures.**—A frame is designed to resist forces applied at its joints, and in framed structures means are taken to insure that the loads are applied at the joints. Thus in a roof the loads due to the covering and the wind are carried on purlins (Fig. 191) resting on the joints of the rafters and the purlins transfer the load to the joint.

The load taken at any joint, such as that between B′ and N (Fig. 191), is regarded as the load falling on the surface MGHJ extending half way to each of the neighbouring joints B′ and N on the same principal A′B′C′ and half way to the neighbouring principals ABC and A″B″C″. The load carried at B′ is that on a similar area extending on either side of the ridge, while that carried at C′ is on an area equal to that between two consecutive principals and extending from C′ half way to the nearest purlin.

Again, in a through bridge (Fig. 193) the floor load carried by a cross girder is that on the area extending half way to each of the neighbouring cross girders and is transferred by the cross girders to the joint of the loaded (lower) chord of the main girder. The rolling load is transferred from the railbearers to the cross girders, the amount borne by the latter being the reactions of the railbearers calculated by the principles of statics for a beam resting *freely* on supports at its ends (see Art. 83).

The weight of the main girders is actually a distributed load, but where there are many cross girders and therefore many panels their weight may, like the loads, be generally divided up for convenience and with sufficient accuracy into concentrated loads at the joints; the load at each joint being that on the half panel on either side of it and that at an end joint being the load on half an end panel. The dead load exclusive of the weight of the girder is carried by the same chord as the live load. Consequently it is often assumed that two-thirds of the total dead load comes on the loaded chord joints and one-third (due to part of the weight of the girder) comes on the joints of the unloaded chord. In large girders the proportion on each should be carefully estimated.

Where a load is applied other than at a joint (as where purlins are placed between joints, or in the case of the weight of the members of a frame, such load is divided between joints according to the principles of statics (Arts. 46 and 47), but in addition to the simple stresses there is bending stress in the members carrying such loads, and this, unless negligible, must be taken into account in estimating the stresses in members of the structure (see Arts. 119 and 120).

In some cases a load is shared by two or more parts of a structure in a way which cannot very simply be calculated, the proportion borne by each depending upon the relative stiffness of the parts. Examples of such distribution are given in Arts. 158 to 164, but frequently some assumption as to the distribution greatly simplifies calculation and is sufficient for a reasonably approximate estimate of stresses. For example, if a horizontal wind load is carried by one side of the girders of the through bridge in Fig. 193, the load on the upper flange is transferred to the end supports of the bridge partly by the main girder's end posts, the upper horizontal girder or wind bracing being thereby stressed in passing some of the load to the leeward main girder. But some load on the upper boom is transferred to the lower or loaded boom by the verticals at each joint (resisting bending), and consequently the lower wind bracing may be taken to carry somewhat over half the wind load. Nevertheless it would be well to allow for the full half of the wind load being transferred from the upper to the lower flange at the ends and for the full half wind load being carried by the upper wind bracing. Various assumptions are in use.

## EXAMPLES X.

1. A roof of the type shown in Fig. 195, 28 ft. span and 7 ft. rise with principals 8 ft. apart has a covering weighing 14 lbs. per sq. ft. of covered area. Find the total dead load assignable to each of the five outer joints of the principals. If in addition there is a wind exerting a pressure of 30 lbs. per sq. ft. normal to the roof, find the normal wind loads assignable to each of the three outer joints on the windward side of the roof.

2. Find the total wind load per principal on the slope of a roof of 40 ft. span, 10 ft. rise, principals 10 ft. apart when the horizontal wind pressure is 56 lbs. per sq. ft. using Unwin's formula or formula (7) of Art. 128.

3. With the same wind pressure as in Problem 2, find the wind loads on each of the five joints on the windward side of a French roof truss of 50 ft. span, 12½ ft. rise, principals 12 ft. apart.

# CHAPTER XI

## *STRESSES IN FRAMES*

**132. Methods of Determining Stresses in Members of Perfect Frames.**—The stresses in individual members of a perfect frame which are all either struts or ties are determined by application of the principles of statics stated in Chapter III. Either graphical or algebraic methods or a combination of both may be employed, but in any case the following are the guiding principles. (1) The frame as a whole is a rigid body and the external forces (load and reactions) acting upon it form by themselves a system of forces (generally non-concurrent) in equilibrium. (2) The pulls or thrusts of the several members meeting in any joint form a system of concurrent or nearly concurrent forces in equilibrium. (3) Any portion of the structure may be taken as a rigid body held in equilibrium by the external forces acting upon it together with the forces exerted upon it, through members, by the remainder of the structure.

**133. Stress Diagrams.**—If force polygons are drawn for the external forces on a plane frame and for each joint of the frame, the polygons can all be fitted together in a single vector figure called a stress diagram. In this vector diagram each line, taken in opposite directions, represents two forces, viz. a side in each of the two separate force polygons which go to make up the whole stress diagram.

*Simple Roof Truss.*—An example will make this clear. Let the simple roof truss shown in Fig. 195 be acted upon by the vertical forces AB, BC, CD, at its joints as shown. The vertical reactions DE and EA may be found by the method of Art. 48, Fig. 49, but in this case from the symmetry, DE and EA are each half of the sum of the three loads. The line *abcd* is set out to represent the loads, and its point of bisection at *e* gives the magnitude of *de* and *ea* the reactions, *abcdea* constituting the closed polygon for the external forces on the frame. The force polygon for the joint at the left-hand support may now be drawn, since only the two sides *af* and *fe* are unknown. Indicating joints by the space letters for the members or force lines radiating from it, the polygon for the joint ABGF may now be drawn, for the thrust of the member AF is equal and opposite at its two ends. The sides *fa*, *ab* are already drawn, and the polygon *fabgf* is completed by drawing through *b* a line parallel to BG, and then through *f* a line parallel to FG to meet in *g*. Proceeding in this way the whole stress diagram *abcdefghkl* may be

drawn in, and includes force polygons for each joint. When the polygon for either the joint LEHK or LKCD has been drawn there remains only one side to complete the stress diagram: if the former joint is solved first the remaining side is *ld*; this may be drawn parallel to LD from say *l*, and if it passes through *d* this fact checks the accuracy of the previous drawing. The polygon for the joint DEL will have been drawn (unconsciously) in drawing the polygons for the external forces and the two neighbouring joints. In the completed figure each line represents as previously stated two forces; thus the vector *bg*

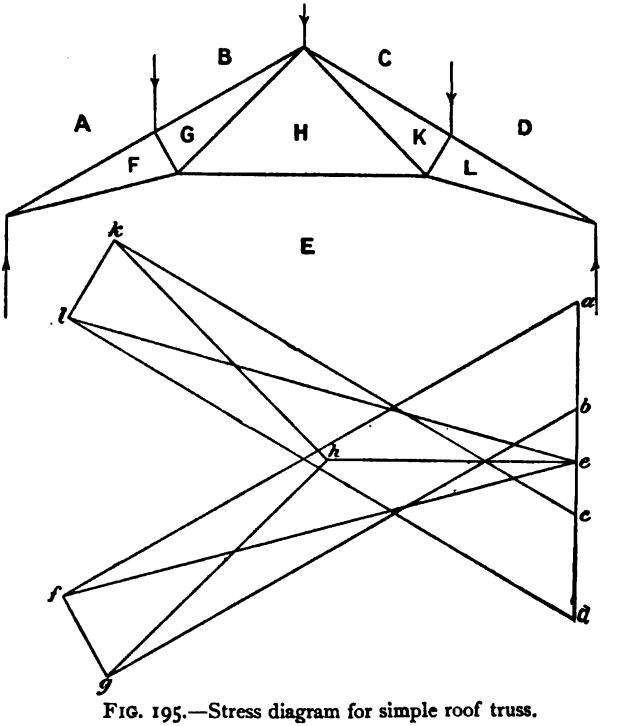



FIG. 195.—Stress diagram for simple roof truss.

represents the thrust of the rafter BG on the joint ABGF, while the vector *gb* represents the thrust of the same rafter BG on the joint BCKHG. Or again, *he* represents the pull in the rod HE at the joint HEFG, while *eh* represents the pull of the tie rod HE at the joint HKLE.

*Reciprocal Figures.*—The frame or space diagram of, say, Fig. 195, and the stress or vector diagram, form reciprocal figures which have certain reciprocal properties; to each node or vertex from which lines radiate in one figure there is a corresponding closed polygon in the other bounded by sides corresponding to the radiating lines and respectively parallel to them. To each line joining two nodes in either

figure there is a corresponding line in the other forming a common side to the polygons corresponding to the two nodes.

*To distinguish between Ties and Struts from the Stress Diagram.*—Knowing the direction of, say, the force EA (upward) at the joint EAF, it is evident from Fig. 195, that the correct order of letters in the vector polygon for this joint is *eaf* (not *efa*), hence the force at this joint exerted by the rafter AF is represented by *af* (not *fa*), and is a thrust, *i.e.* the member is a strut. The correct order of sides *eaf* being *ea*, *af*, *fe* the corresponding order of the lines EA, AF, FE radiating from this joint is a *clockwise* order. When this order is clockwise for one joint it immediately follows that it must be the same for the neighbouring joints, for a thrust, *af*, must be associated with a balancing thrust, *fa*, at the next joint of the rafter. Similarly, it follows that the correct order is clockwise for *all* the joints. Hence if we wish to know whether the member HK say is a strut or a tie, we know that for the joint HKLE the force in HK is in the direction *hk* (not *kh*), and reference to the vector diagram shows that the direction *hk* is a *pull* at the joint HKLE, *i.e.* HK is a *tie*.

This characteristic order of space letters round the joints is a very convenient method of picking out the kind of stress in one member of a complicated frame. Note that it is the characteristic order of space letters round a joint that is constant in a given diagram—*not* the direction of vectors round the various polygons constituting the stress diagram.

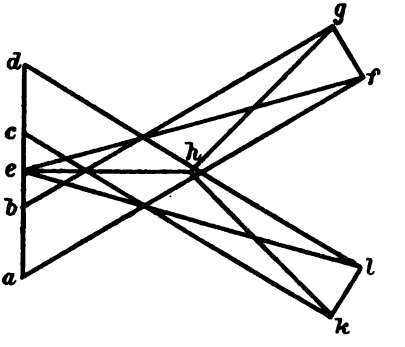


FIG. 196.

Fig. 196 represents the stress diagram for exactly the same frame diagram and lettering as Fig. 195, but is the contra-clockwise vector diagram, *e.g.* the left-hand reaction AE is now represented by *ae* (instead of *ea*), and the force of KH at the joint HELK is represented by *kh* (instead of *hk*), which still indicates the member to be a tie.

*Warren Girder.*—A second example of a simple stress diagram is shown in Fig. 197, viz. that of a Warren girder, all members generally being of the same lengths, the diagonals inclined 60° to the horizontal.

Two equal loads, AB and BC, have been supposed to act at the joints 1 and 2, and the frame is supported by vertical reactions at 3 and 4, which are found by a funicular polygon or may be very easily calculated by taking moments about the points of support. The remaining forces in the bars are found by completing the stress diagram *abc* . . . *klm*.

Note that the force AB at joint 1 is downward, *i.e.* in the direction *ab* in the vector diagram corresponding to a contra-clockwise order, A to B, round joint 1. This is, then, the characteristic order (contra-clockwise) for all the joints, *e.g.* to find the nature of the stress in KL,

the order of letters for joint 5 is K to L (contra-clockwise), and referring to the vector diagram, the direction *k* to *l* represents a thrust of the bar KL on joint 5; the bar KL is, therefore, in compression.

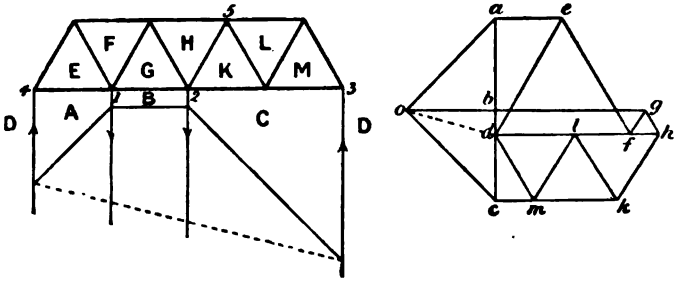



FIG. 197.—Stress diagram for simple Warren girder.

*Curved or Hog-back* N *Girder.*—Fig. 198 shows the stress diagram for a girder of the Hog-back or Curved Top Chord type, the span being divided into eight equal panels each carrying a load W. Half the load on the end panels is carried directly at the supports, and may be ignored in the reactions used for calculating the stress in the members. The following are the stresses scaled from Fig. 198 in terms of the panel loads W:—

| Members | AL, AY | AN, AW | AP, AU | AR, AS | AK, AZ | LM, YX | NO, WV |
|---|---|---|---|---|---|---|---|
| Compression | 4·07W | 5·6W | 6·25W | 6·375W | 3·5W | 1·375W | 0·417W |

| Members | HM, CX | GO, DV | FQ, ET | KL, ZY | MN, XW | OP, VU | PQ, UT | QR, TS | RS |
|---|---|---|---|---|---|---|---|---|---|
| Tension | 3·88W | 5·5W | 6·2W | 4·57W | 2·13W | 1·00W | 0·25W | 0·267W | 0·583W |

The members JK and BZ are not stressed.

*French Roof Truss.*—The stress diagram of this roof truss (known in America as the Fink roof truss) involves an interesting special point such as may be met with in other structures. The vertical loads are shown in Fig. 199 as symmetrical, but the methods are the same in any case. When the reactions HJ and JA have been determined, the polygons for the joints JAK, KABL, KLMJ, may successively be drawn. On attempting to draw the vector polygon for either of the joints BCPNML or MNRJ, it will be noticed that more than two sides are unknown, and the plane polygon is therefore not determinate. If a start be made to draw the stress diagram from the other end the same difficulty is apparent. To overcome it, the stress in one or more members must be determined by some other method, and several are available, such as the method of sections (see Art. 138). The method adopted in Fig. 199 is known as the method of substitution. By it the

stress in QD is determined from the fact that the thrust in QD is not affected by the form of the internal bracing consisting of the members QP and PN. Hence to find the stress in QD, replace (temporarily or in imagination) the bars QP and PN by a single bar QY, connecting the joints marked 1 and 2, thus reducing the number of bars radiating from the joint BCNML by one, the polygon *bcyml* may now be completed by drawing *my* and *cy* parallel to MY and CY respectively to intersect in *y*, and the polygon *cdqy* may then be completed by drawing *dq* and *yq* parallel to DQ and YQ respectively to intersect in *q*. The stress *dq* in DQ is now known, and the previous bracing may be replaced and the polygon *cdqb* drawn, and the whole stress diagram completed. The point *y* is not a vertex or node in the completed diagram.

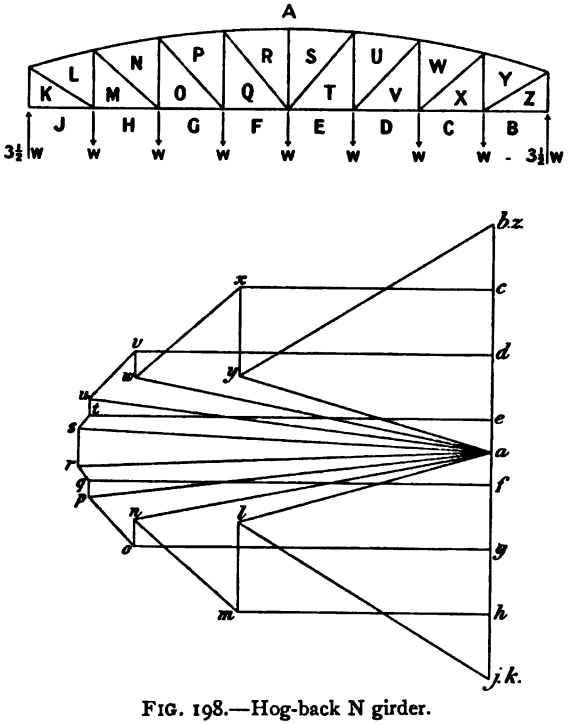



FIG. 198.—Hog-back N girder.

*Island Station Roof.*—This is shown in Fig. 200, the loads being given in tons, and is an example of a structure which is not wholly a perfect frame. The post is continuous from the base Y to the apex V. *abcdeml* is the stress diagram for the left side and *ghkon* that for the right side. The post is subjected to bending moments proportional to the ordinates shown to the right of the space diagram. The principal magnitudes of these bending moments, found by taking component forces perpendicular to the post (neglecting the effect of flexure), are

$$M_X = VX \times \text{horz. component of } mn = \frac{15}{\sqrt{3}} \times 1{\cdot}60 = 13{\cdot}9 \text{ ton-feet}$$

$$M_Y = VY \times \text{horz. comp. of } mn - XY \times \text{horz. comps. of } am \text{ and } nh$$
$$= (VX + XY) \times \text{horz. comp. of } mn - XY \times \text{horz. comps. of } am \text{ and } nk$$
$$= M_X - XY \times \text{horz. comp. of } ak = 13{\cdot}9 - 1{\cdot}61 \times 15$$
$$= -10{\cdot}2 \text{ ton-ft.}$$

where the positive sign corresponds to contra-clockwise bending moments above the section considered.

The point of contraflexure being distant $z$ from X

$$13{\cdot}9 - 1{\cdot}61z = 0, \qquad z = 8{\cdot}66 \text{ feet}$$

which may also be found from the bending moment diagram.

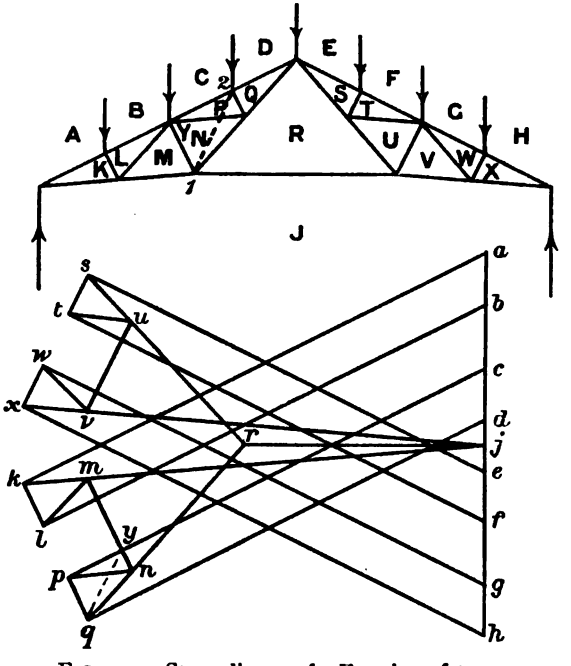



FIG. 199.—Stress diagram for French roof truss.

The bending moments between X and Y might be found by considering the roof as a rigid body, the oblique forces only producing bending moment on XY. The resultant oblique force is evidently $0{\cdot}8 + 1{\cdot}6 + 0{\cdot}8 = 3{\cdot}2$ in the line CD. Then

$$M_X = 3{\cdot}2 \times x = 13{\cdot}9 \text{ ton-feet}$$
$$M_Y = -\,3{\cdot}2 \times y = -\,10{\cdot}2 \text{ ton-feet}$$

and the line CD cuts XY in Z, which measures 8·67 feet from X and gives the point of inflexion.

**134. Stress Diagrams for Roofs with Wind Loads.**—When in addition to vertical loads a roof is subjected to oblique forces such as wind loads distributed as explained in Art. 131, the reactions at the supports of a principal will not be vertical. The magnitude and direction of the reactions will depend partly upon the way in which the roof principal is supported. In roofs of considerable span it is not uncommon to support

one end on horizontal rollers, while the other end is horizontally hinged; this admits of expansion of the principal, and also makes the supporting forces determinate, for, neglecting friction, the supporting force at the end resting on rollers, or the "free" end, must be vertical. The other

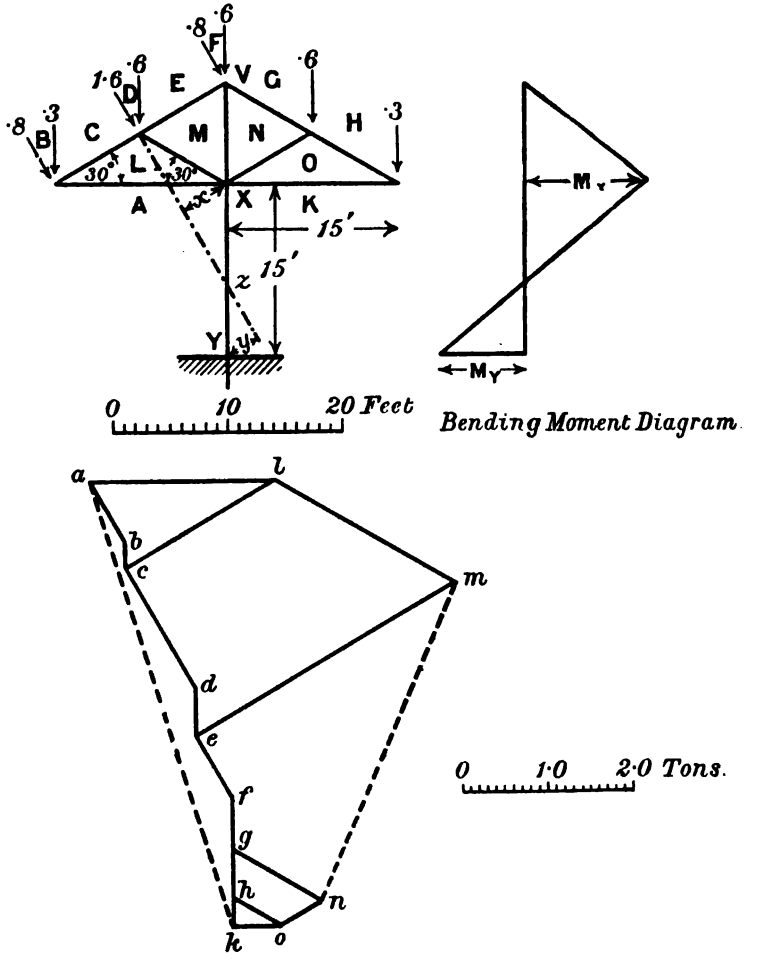



FIG. 200.—Open stress diagram for roof on column.

reaction is known by a point in its line of action (the hinge), and therefore both reactions may be determined as explained in Art. 47. Fig. 201 shows an example of a roof hinged at the left side, and "free" or freely supported on rollers at the right-hand side. The lower stress diagram is the vector diagram for the roof with the wind blowing on the right

slope or "free" side of the roof. The oblique and the vertical loads at the joints may either be combined by a triangle of forces on the space or frame diagram as in Fig. 201, and their vector sum used in the stress

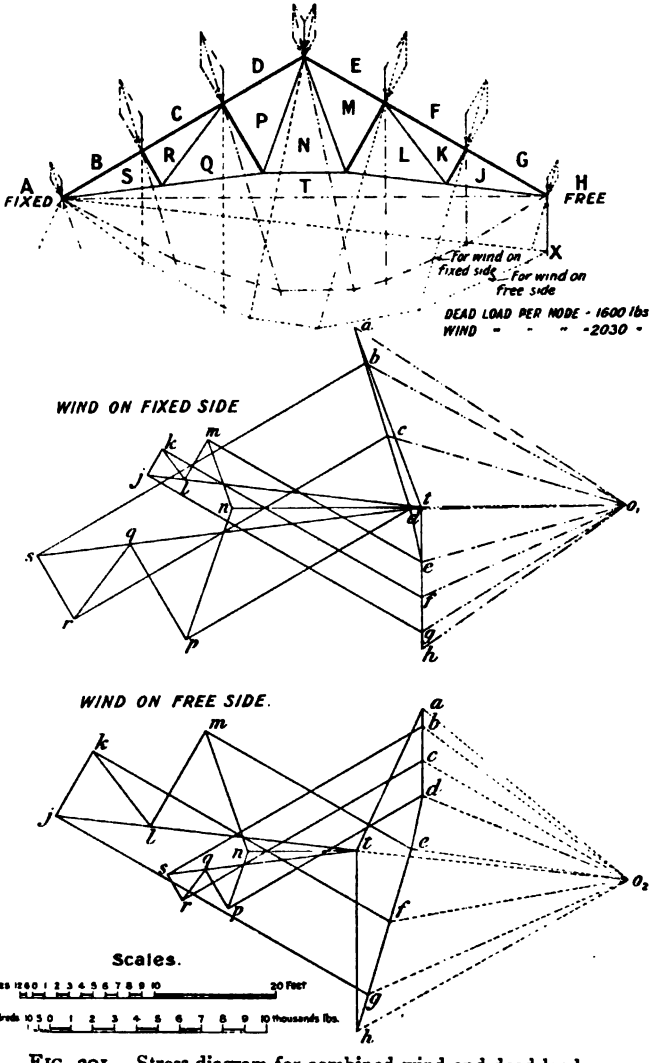
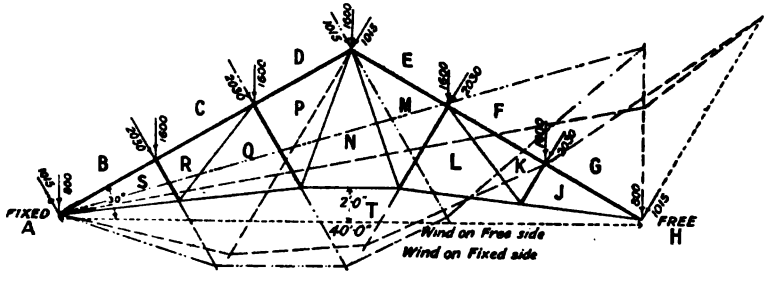



FIG. 201.—Stress diagram for combined wind and dead loads.

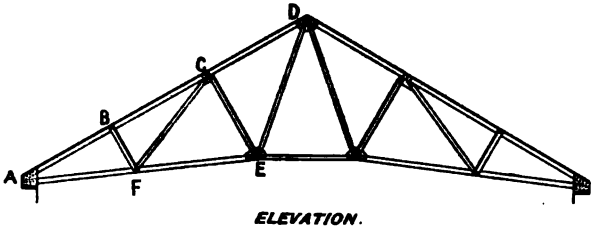
diagram, or the two loads may be set down separately in the vector diagram as in the example shown in Fig. 200.

To find the reactions in Fig. 201 for the case of wind from the

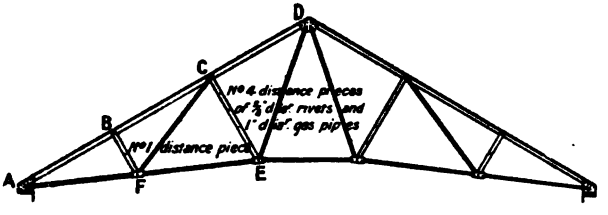
right the vertical forces AB, BC, and CD are set down at *abcd* in the lower stress diagram and the remaining oblique forces on the right are set off at *defgh*. Then an indefinite line *ht* is set up to represent the direction of the (vertical) reaction at the right-hand side, and then from any pole $o_2$ radiating lines are drawn, and the corresponding funicular polygon drawn *starting from the fixed end*, and finally meeting the




FRAME DIAGRAM.




ELEVATION.




ALTERNATIVE ELEVATION

FIG. 201A.—Relating to Fig. 201B and Plate I.

vertical through the free end in X. This point X is joined to the fixed end, thus closing the funicular polygon, and through $o_2$ a line $o_2t$ parallel to this closing side is drawn which determines *t*, then *ta* is the reaction at the fixed end and *ht* that at the free end. The whole stress diagram is then easily completed. The diagram for the wind on the other side is similarly drawn. If the maximum and minimum stresses are required

it will be necessary to draw a stress diagram for the vertical loads acting alone. Sometimes the stress diagrams for the wind loads alone for either side are drawn without inclusion of the dead vertical loads and a separate diagram for the vertical loads. This plan is shown in Fig. 201B (reaction from top of Fig. 201A), which relates to the same roof and loads as shown in Fig. 201. The stresses as read off from the stress diagrams are shown in tabular form as determined by the two methods, which of course agree.

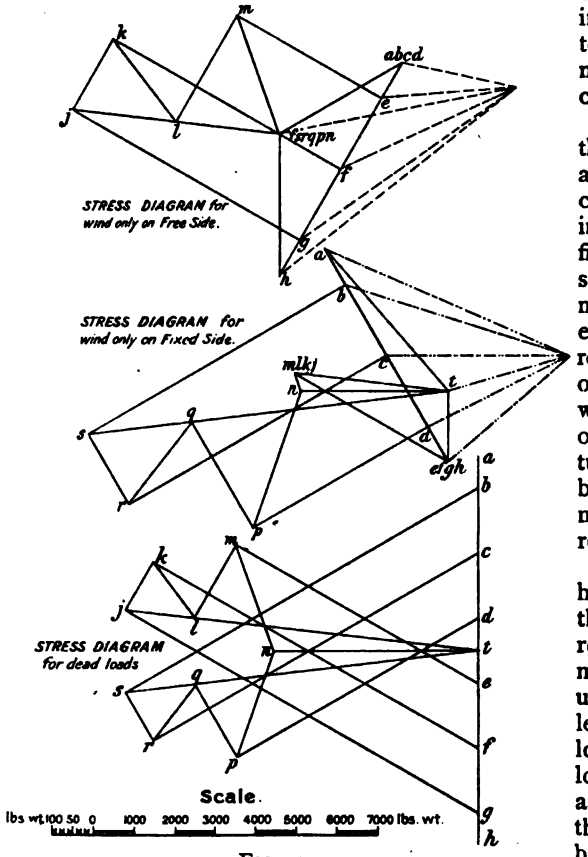

STRESS DIAGRAM for wind only on Free Side.
STRESS DIAGRAM for wind only on Fixed Side.
STRESS DIAGRAM for dead loads
Scale.
lbs wt. 100 50 0 1000 2000 3000 4000 5000 6000 7000 lbs. wt.

FIG. 201B.

It will be noticed that three diagrams are required in either case for the complete information. If a fixed working unit stress is used only the maximum stress in each member will be required for purposes of design, but if the working stress is based on the degree of fluctuation (see Chap. II.) both maximum and minimum will be required.

When a roof is hinged at both sides the reactions are not really statically determinate. They are usually taken as parallel; but if the wind loads and vertical loads are combined, as in Fig. 201, and the reactions taken both parallel to the resultant load, the result is not the same as if wind loads and vertical loads are drawn on separate diagrams and the reactions taken parallel to the resultant in each case. The vertical components of the reactions are the same in either case, but the horizontal components arising from the oblique wind pressures differ; it may be shown that the most probable distribution of horizontal pressure on the hinges is half the horizontal wind pressure on each hinge. However, if either of the other two methods are used the resulting stress determinations for the members is not in practical cases greatly

TABLE OF STRESSES FROM SEPARATE DIAGRAMS (FIG. 201B).

| Member. | Wind on left (fixed side). | Wind on right. | Dead load. | Maximum. | Minimum. |
|---|---|---|---|---|---|
| BS | + 7400 | + 3525 | + 10100 | + 17500 | + 10100 |
| CR | + 7400 | + 3525 | + 9300 | + 16700 | + 9300 |
| DP | + 5050 | + 3525 | + 6850 | + 11900 | + 6850 |
| EM | + 4400 | + 4110 | + 6850 | + 11250 | + 6850 |
| FK | + 4400 | + 6460 | + 9300 | + 15760 | + 9300 |
| GJ | + 4400 | + 6460 | + 10100 | + 16560 | + 10100 |
| SR | + 2030 | — | + 1390 | + 3420 | + 1390 |
| QP | + 3040 | — | + 2060 | + 5100 | + 2060 |
| ML | — | + 3040 | + 2060 | + 5100 | + 2060 |
| KJ | — | + 2030 | + 1390 | + 3420 | + 1390 |
| ST | − 9000 | — | − 8800 | − 17800 | − 8800 |
| QT | − 6430 | — | − 7040 | − 13470 | − 7046 |
| NT | − 3670 | — | − 5000 | − 8670 | − 5000 |
| LT | − 3820 | − 2550 | − 7040 | − 10860 | − 7040 |
| JT | − 3820 | − 5100 | − 8800 | − 13900 | − 8800 |
| RQ | − 2580 | — | − 1770 | − 4350 | − 1770 |
| PN | − 3600 | — | − 2780 | − 6380 | − 2780 |
| NM | − 470 | − 3090 | − 2780 | − 5870 | − 2780 |
| LK | — | − 2560 | − 1770 | − 4330 | − 1770 |

TABLE OF STRESSES FROM THE TWO COMBINED DIAGRAMS (FIG. 201) AND THE DEAD LOAD DIAGRAM (FIG. 201B).

| Member. | Maximum stress. | Condition for maximum. | Minimum stress. | Condition for minimum. |
|---|---|---|---|---|
| BS | + 17500 | Wind on left | + 10100 | Dead load only. |
| CR | + 16700 | ,, ,, | + 9300 | ,, ,, |
| DP | + 11900 | ,, ,, | + 6850 | ,, ,, |
| EM | + 11250 | ,, ,, | + 6850 | ,, ,, |
| FK | + 15760 | ,, on right | + 9300 | ,, ,, |
| GJ | + 16560 | ,, ,, | + 10100 | ,, ,, |
| SR | + 3420 | ,, on left | + 1390 | ,, ,, |
| QP | + 5100 | ,, ,, | + 2060 | ,, ,, |
| ML | + 5100 | ,, on right | + 2060 | ,, ,, |
| KJ | + 3420 | ,, ,, | + 1390 | ,, ,, |
| ST | − 17800 | ,, on left | − 8800 | ,, ,, |
| QT | − 13470 | ,, ,, | − 7040 | ,, ,, |
| NT | − 8670 | ,, ,, | − 5000 | ,, ,, |
| LT | − 10860 | ,, ,, | − 7040 | ,, ,, |
| JT | − 13900 | ,, on right | − 8800 | ,, ,, |
| RQ | − 4350 | ,, on left | − 1770 | ,, ,, |
| PN | − 6380 | ,, ,, | − 2780 | ,, ,, |
| NM | − 5870 | ,, on right | − 2780 | ,, ,, |
| LK | − 4330 | ,, ,, | − 1770 | ,, ,, |

different, and both separate and combined diagrams are in frequent use. Fig. 202 shows the stress diagram for a curved roof hinged at one side and freely supported on rollers at the other. The roof principals are spaced 12 ft. 6 ins. apart, and a wind load of 40 lbs. horizontally and a vertical load of 25 lbs. per square foot has been assumed.

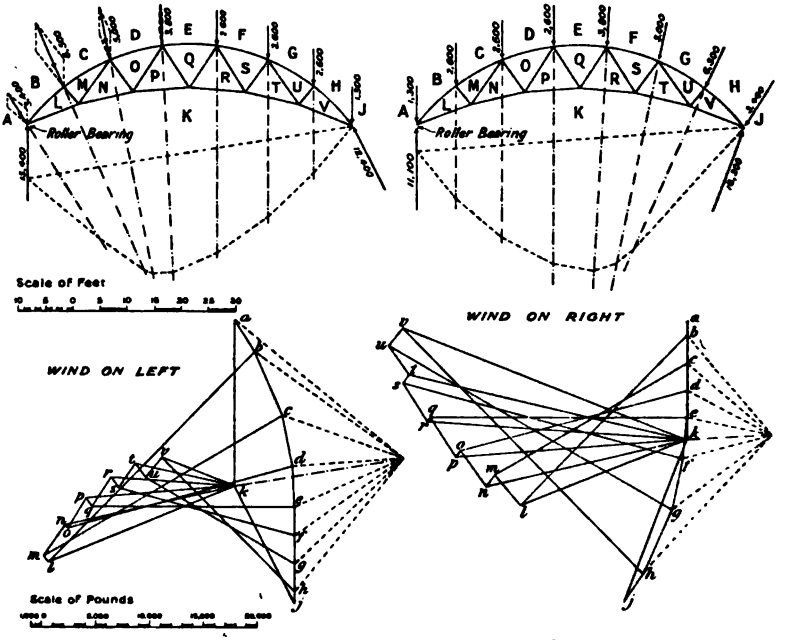



FIG. 202.—Stress diagrams for curved roof truss.

**135. Simple Roof Design.**—A simple example of roof design is shown in Fig. 201A and Plate I. The calculated stress in the members are those given in the previous article, and the particulars of loads and working stresses are as follow:—

Effective span, 40 feet. Principals, 8 feet apart. Rise = $\frac{1}{4}$ span = 10 feet.

*Dead loads.*—

| | | | |
|---|---|---|---|
| Rafters and purlins | 6 | lbs. per | super. foot. |
| Slate battens | 2 | ,, | ,, |
| Slates | 9 | ,, | ,, |
| Truss (Howe's formula, Art. 127) | 4 | ,, | ,, |
| Snow | 5 | ,, | ,, |
| Total | 26 | ,, | ,, |

*Wind load.*—56 lbs. horizontal = 33 lbs. normally, (7) Art. 128.

*Working stresses.*—6½ tons per square inch in tension, 5 tons per

square inch in shear, 10 tons per square inch for bearing. Principal rafters and angle struts taken as having one end fixed and one end hinged by Rankine's formula (Art. 116), and the constants there given for mild steel with factors of safety between 3 and 4. Bowed struts taken as hinged both ends. Two forms of truss are shown in Fig. 201A, and details for both in Plate I. The first is now the more common form, the struts being made of angle sections instead of flats with distance pieces, and the tie rods are flats, instead of rounds with forged ends, and the joints are formed by gusset plates. Two examples will suffice to illustrate the design of the members.

*Member* ST (joining points A and F).

Maximum tension 17,800 lbs. requires $\frac{17800}{6{\cdot}5 \times 2240} = 1{\cdot}22$ sq. in. section, which is provided by 4 ins. by $\frac{3}{8}$ in. flat (less rivet hole) or by $1\frac{1}{4}$ in. round tie rod.

*Member* BS (connecting points A and B).

Maximum thrust in the rafter 17,500 lbs. = 7·82 tons. Length = 7·7 ft. = 92·5 ins., and the equivalent length of a strut hinged both ends is $92{\cdot}5 \div 1{\cdot}4 = 66$ ins. Struts being limited to a ratio $\frac{l}{k} = 100$, the minimum radius of gyration $k$ must be at least 0·66 in. Referring to the Table IV. Appendix, the next Tee section above this is $3\frac{1}{2} \times 3\frac{1}{2} \times \frac{3}{8}$ ins., for which the least $k = 0{\cdot}717$ in. and area = 2·496 sq. ins. The crippling load for this is by Art. 116,

$$\frac{21 \times 2{\cdot}496}{1 + \frac{66 \times 66}{7500 \times (0{\cdot}717)^2}} = 46{\cdot}4 \text{ tons}$$

The factor of safety for this is $46{\cdot}4 \div 7{\cdot}82 = 6$, which is satisfactory.

The struts PQ and RS are made $2\frac{1}{4} \times 2\frac{1}{4} \times \frac{5}{16}$ ins. angles; $2\frac{1}{4} \times 2\frac{1}{4} \times \frac{1}{4}$ ins. angle would fulfil the requirements of Rankine's rule, but $\frac{5}{16}$ in. is a minimum size for use with $\frac{5}{8}$ in. rivets, the smallest diameter used in such work. Often $\frac{3}{8}$ in. thickness would be preferred on the ground of durability. The ratio of $\frac{l}{k}$ in any case exceeds the equivalent of 100 for hinged ends, but this may be permitted when the sectional area for some practical reason greatly exceeds the requirements of Rankine's or other strut formula used.

**136. Statically Indeterminate Frames — Principle of Superposition.**—The stresses in the members of a frame containing redundant members (see Arts. 123 and 124) are frequently difficult to determine and depend upon the relative stiffness of the various parts. But simple approximate methods may frequently be applied; for example, a structure and its loads may be subdivided into two or more perfect frames containing some members in common, so that when the perfect frames are superposed they form the actual structure. The stresses in these perfect frames having been determined, the stresses in the actual imperfect frame may be found by adding algebraically the

stresses in the component frames. This method, which is an approximation, is called the method of superposition, and its accuracy is tested by an example and commented upon in Art. 159. The method of

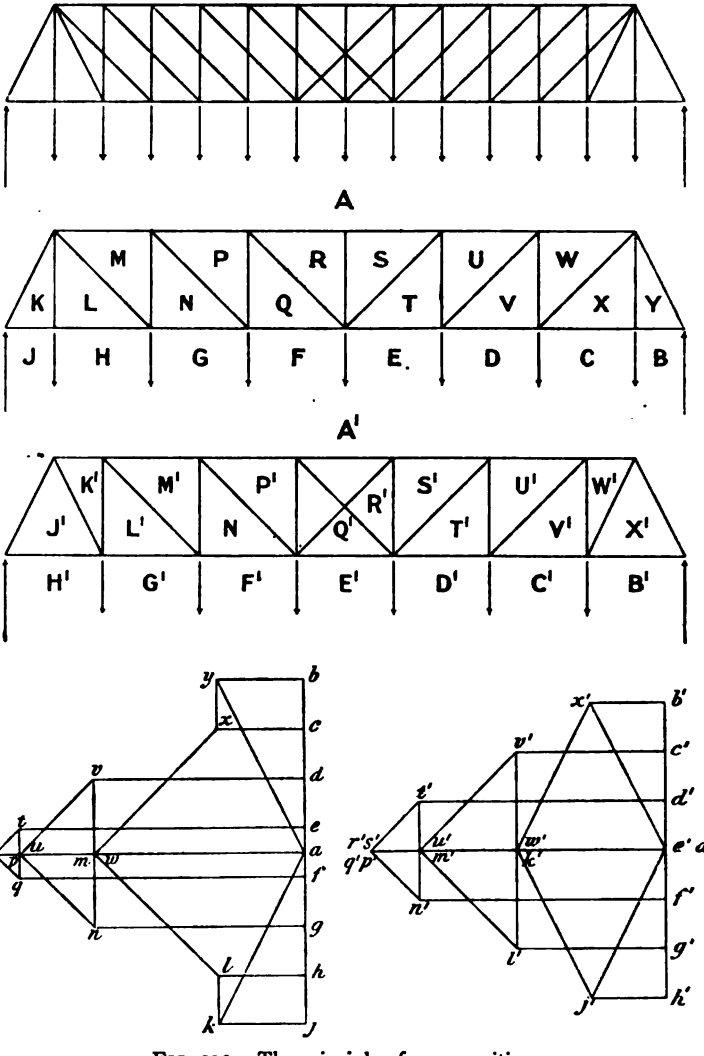



FIG. 203.—The principle of superposition.

superposition is illustrated in Fig. 203, which shows the stress diagrams for a double intersection N or Whipple Murphy girder equally loaded at each panel point; the frame and forces are resolved into the parts

shown by unaccented and accented letters in the frame and vector diagrams. The members AK and A'J' are identical, and the thrust in this member is found by adding *ak* and *a'j'*. Again, the pull in the bottom boom for the two central panels is found by adding the tensions *qf* (or *te*) and *q'e'*. The member J'K', on the other hand, appears in the second only, and the pull in it is *j'k'*. A second illustration is given in Fig. 237, which represents the method applied to the girder in (*a*) Fig. 236. Table B in the example at the end of Art. 150 shows how nearly correct the stresses conventionally calculated by the method of superposition are in this case; also that for symmetrical loading the results are exact.

**137. Method of Resolution.**—When a small number (say three) of members of a frame meet at a common joint, all but two of the forces being known, the others may often be found easily by simple resolution of these concurrent forces into components and application of equations (1) and (2) of Art. 44. Taking the simple N girder in Fig. 204 as an example, the reactions $R_1$ and $R_2$ at the supports being calculated by moments, the vertical thrust of AB at A must just balance $R_1$, and the stress in AS must be zero, since there is no other horizontal force at A. Then proceeding to joint B, the vertical component of the force BS say in BS must balance the upward thrust $R_1$ in AB, since these are the only vertical forces at B or BS sin $\theta = R_1$, and BS $= R_1$ cosec $\theta$ (a pull at B). And the wholly horizontal force (BC) in the member BC must balance the horizontal component of the pull BS at B, or

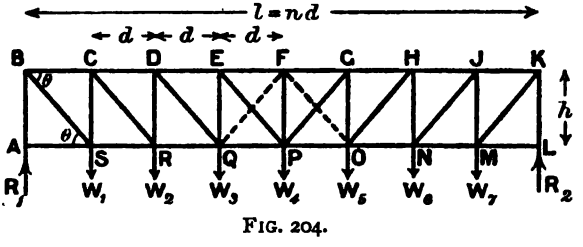


FIG. 204.

$$BC = BS \cdot \cos\theta = R_1 \cdot \operatorname{cosec}\theta \cdot \cos\theta = R_1 \cdot \cot\theta$$

Proceeding to joint S, resolving vertically, if SC = tension in SC

$$BS \sin\theta + SC = W_1,$$

or $$SC = W_1 - BS \sin\theta = W_1 - R_1 \text{ (or thrust } R_1 - W_1)$$

And resolving horizontally, BS cos $\theta$ = tension in SR, or SR $= R_1 \cot\theta$.

Similarly proceeding from joint to joint, the stresses in all the members of the girder may be found. A simpler method for such a frame is given in the next article.

**138. Method of Sections.**—This method, due to Rankine, enables the stress in any member of a simple frame to be calculated without first calculating the stresses in a great number of other members. The principle of the method is that if a structure be divided by an ideal surface into two parts, the forces in the bars cut by the ideal surface, together with the external forces on either part of the divided structure, form a system of forces in equilibrium. If the external forces on either

part of the structure are known, the forces in the members cut may be determined by applying to either portion of the structure the principles of Art. 45; and frequently a single equation will suffice to determine the stress in any one member; the determination may, of course, be made graphically or algebraically, according to convenience in a particular case.

EXAMPLES.—(1) *French Roof Truss.*—The difficulty mentioned in Art. 133 in connection with drawing the stress diagram for the French roof truss (Fig. 199) may conveniently be overcome by finding the stress in a single member by the method of sections. For example, to find the stress in one or more of the members DQ, QR, RJ in Fig. 205, take an imaginary plane of section XX. Then the structure to the left of XX is in equilibrium under the external forces AB, BC, CD, and JA, together with the three forces exerted by DQ, QR, and RJ, which may therefore be determined by any of the principles of Arts. 45, 46, and 47. The most convenient method of finding the stress in one of these three members (avoiding simultaneous equations) will be to apply equation (1), Art. 46, taking clockwise moments about the intersection of the other two, *e.g.* to find the pull of RJ on joint (1), taking moments about point (2)

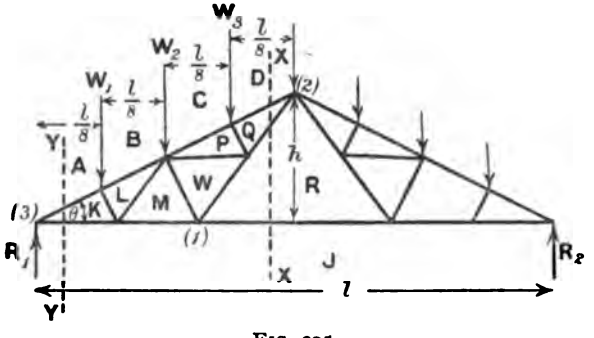



FIG. 205.

$$R_1 \times \tfrac{1}{2}l - W_1 \times \tfrac{3}{8}l - W_2 \times \tfrac{1}{4}l - W_3 \times \tfrac{1}{8}l - RJ \times h = 0$$

or,
$$RJ = +\frac{1}{h}(\tfrac{1}{2}R_1 l - \tfrac{3}{8}W_1 l - \tfrac{1}{4}W_2 l - \tfrac{1}{8}W_3 l)$$

Similarly, the force in QR might be found by a single equation of moments about point (3), the intersection of DQ and RJ (produced). Again, if the tie KJ is horizontal, the method of sections might be very simply applied to find the stress in AK by assuming a section surface YY; for resolving vertically upwards the forces on the portion of the structure to the left of YY by (1) or (2), Art. 44

$$R_1 + AK \sin\theta = 0, \quad \text{or } AK = -R_1 \operatorname{cosec}\theta$$

*i.e.* the force in AK thrusts downwards at point (3) with a force $R_1$ cosec $\theta$. If the tie KJ were not horizontal two simultaneous equations corresponding to (1) and (2), Art. 44, with horizontal and vertical components respectively, might be employed.

(2) *N Girder.*—The method of sections is particularly simple in the case of girders with parallel flanges or booms. For a diagonal member

such as *ab* (Fig. 206) assume a section XX cutting the three members *ab*, *ac*, and *be*, then taking the vertical forces on the left of the section upwards, say, *ab* being the stress in *ab*

$$R + ab \sin\theta - W_1 - W_2 - W_3 = 0$$
$$ab = (W_1 + W_2 + W_3 - R) \operatorname{cosec}\theta, \text{ or } F_3 \operatorname{cosec}\theta$$

thrust toward *a* where $F_3$ is the shearing force in the panel *eb* according to the sign given in Art. 59. The tension in *ab* is, of course, $-F_3 \cdot \operatorname{cosec}\theta$, and if R is greater than $W_1 + W_2 + W_3$, *ab* is then in tension. For a vertical member such as *cb*, take a section such as YY, then resolving vertically upwards to the left of YY, if *bc* = thrust of *bc* on *b*

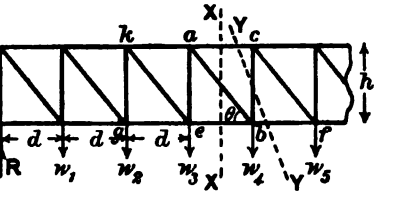



FIG. 206.—Method of sections.

$$R - W_1 - W_2 - W_3 - W_4 - bc = 0$$
$$bc = R - (W_1 + W_2 + W_3 + W_4) = -F_4$$

where $F_4$ is the shearing force on the panel *bf*.

For a horizontal member *ka* of the top chord take a plane section through the bottom joint *e* passing just to the left of the joint *a*; then considering clockwise moments about *e* of forces on the part of the structure to the left of the section

$$R \times 3d - W_1 \times 2d - W_2 \times d + ka \times h = 0$$
$$ka = \frac{1}{h}(-3Rd + 2W_1 d + W_2 d), \text{ or } \frac{1}{h} \cdot M_e$$

where $M_e$ is the bending moment on the girder at *e* with sign according to Art. 59, and *ka* is the pull of the member *ka* on the joint *k*. In this case $M_e$ is negative, and the tension in *ka* is negative, *i.e.* it is a thrust $\frac{1}{h}(3Rd - 2W_1 d - W_2 d)$.

The force in the lower chord is similarly found by taking a nearly vertical section through a joint of the top chord; thus by moments about *k*

$$R \times 2d - W_1 \cdot d - h \times ge = 0$$

$$ge = \text{pull in member } ge = \frac{1}{h}(2Rd - W_1 d), \text{ or } -\frac{1}{h}M_k$$

where $M_k$ = the bending moment at *k* with sign according to Art. 59.

The stresses in the web members are shown in Fig. 207 by drawing vertical and oblique lines across the shearing force diagram parallel to the members. The stresses in the vertical members are given by the lines vertically below the members, and those in oblique members by oblique lines crossing the space vertically below the corresponding bay. Similarly, the ordinates of the bending-moment diagram give the stresses in the upper and lower chords to a scale dependent upon the depth of

the girder. Stresses in four members, A, B, C, D, are shown by the lines *a*, *b*, *c*, *d* respectively.

(3) *Warren Girder.*—This may be similarly dealt with, by vertical sections clear of joints for all web members, and vertical sections through opposite joints for all chord members. The web members resist shearing force, and the chord members resist bending moments.

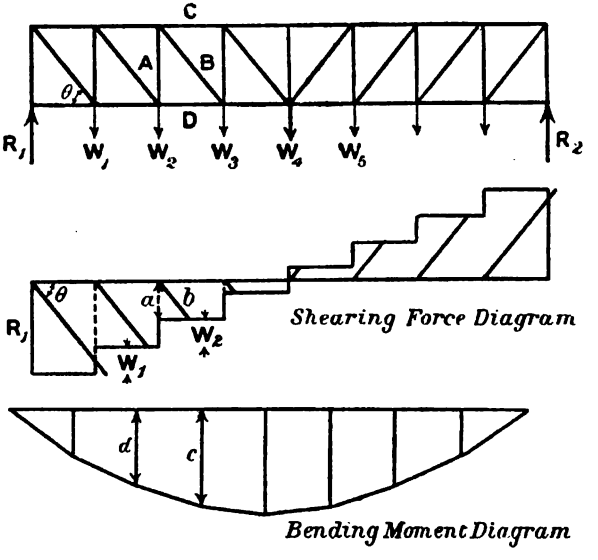



FIG. 207.

(4) *N Girder with Inclined Chords.—Diagonals.*—For a diagonal member such as BD (Fig. 208) take a section XX, and take moments of the forces on the portion to the right of XX about point Z, the intersection of two of the three members cut by XX. Let $r$ = perpendicular distance of BD produced from Z.

$$(\text{Pull of BD at B} \times r) + W_1(d + z) - R_1 \cdot z = 0$$

$$\text{Pull in BD} = \frac{1}{r}\{R \cdot z - W_1(d + z)\}$$

*Chord Members.*—For the thrust in AB use the same section, and take moments about D. Let $s$ = perpendicular distance of AB from D.

$$(\text{Thrust of AB at B} \times s) + W_1 \cdot d - R_1 \times 2d = 0$$

$$\text{Thrust in AB} = \frac{1}{s}(2R_1 \cdot d - W_1 \cdot d) = \frac{-M_D}{s}$$

Or, again, the horizontal component of the thrust in AB $= -\frac{M_D}{AD}$ from which the thrust in AB is obtained by multiplying by the secant of the inclination of AB.

The tension in DC may be found by using the section XX and considering moments about B as in the case of parallel chords, viz.

Tension in DC × BC = $R_1 . d$; hence pull in DC = $\frac{1}{BC} \times R_1 . d$, or more generally $-\frac{M_B}{CB}$.

*Verticals.*—For the thrust in AD the section YY may be used, taking moments about Z

$$\{\text{Thrust of AD at D} \times (z + 2d)\} + W_2(2d + z) + W_1 . (d + z) - R_1 z = 0$$

$$\text{Thrust in AD} = \frac{1}{z + 2d}\{R_1 . z - W_2(2d + z) - W_1 . (d + z)\},$$ which may be negative.

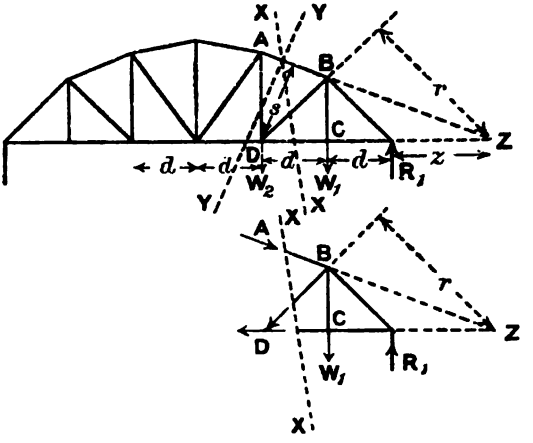



FIG. 208.

*Alternatives.*—As an alternative, equations of forces may be used. The vertical components of AB and BD jointly balance the shearing force in the bay ABCD; hence when the pull in BD has been determined the vertical component, and hence (multiplying by the cosecant of the inclination) the actual stress, in AB may be found.

Again, if the chord stresses in AB and DC have been determined (say, by moments about D and B), the horizontal component of the stress in BD must equal the difference of the horizontal components of the chord stresses, and the stress in BD is found from its horizontal component by multiplying by the secant of its inclination.

(5) *Parabolic Girder.*—This is a particular case of the previous one, in which the vertical heights of the top chord from the lower chord are proportional to the ordinates of a symmetrical parabola, and therefore also (Art. 57, Fig. 81) to the bending moments for a uniformly distributed load on all the spans. Hence, from the previous case, the

tension in the lower chord $-\left(\frac{M_B}{BC}\right)$ is constant throughout, and equal to the horizontal component of the thrust in the top chord for a uniform dead load of $w$ per foot, viz. $\frac{1}{8}wl^2 \div$ central depth. For this load the stress in the diagonals will be zero, for considering such a joint as A or D (Fig. 208) the horizontal component of the diagonal stress is equal to the *difference* of the horizontal chord tensions on either side of it, which is zero. Further, considering any lower chord joint under these conditions of load, it immediately follows that the tension in the vertical member is equal to the panel load, the sole function of such members being to transfer the load to the top chord. The vertical component of the thrust of the top chord at any section then balances the shearing force.

More generally for any type of dead load similar conditions would hold if the height of the girder at every cross-section is proportional to the bending moment at that section.

(6) The Baltimore truss, Fig. 194, is a modification of the N frame suitable for long spans, and can conveniently be solved for given positions of the load by the method of sections, the treatment being almost exactly as for the N girder.

**139. Stresses from Coefficients.**—In simple types of girders carrying uniform loads the stresses may be tabulated from general expressions for the members of any panel. The stresses in two similar members of a truss may be resolved into different coefficients (dependent only on the number of panels and position in the girder), multiplied by the same constant, and for the same number of panels, but different proportions and loadings, other constants with the same coefficients will be applicable. Taking Fig. 204 as an example, consider any panel such as DEQR. Let $m$ be the number of panels between it and the left support, and $n$ be the total number of panels (say, even). Let W be the load per panel, $\frac{W}{2}$ from the end panels being carried directly at each end support and W at each panel point. Then the effective reaction $R_1 = R_2 = \frac{n-1}{2}.W$. The (negative) shear in panel DEQR is $\left(\frac{n-1}{2} - m\right)W$, and by the method of sections the tensile stress in DQ is

$$\left(\frac{n-1}{2} - m\right)W \operatorname{cosec}\theta, \text{ or } W\frac{1}{h}\sqrt{h^2 + d^2}\left(\frac{n-1}{2} - m\right)$$

or W cosec $\theta$ multiplied by the *coefficient* $\frac{n-1}{2} - m$, the coefficients for diagonals from the supports to the centre forming an arithmetical progression.

The thrust in the vertical DR to the left of the panel DEQR is equal to $\left(\frac{n-1}{2} - m\right)W$ or W multiplied by the coefficient $\frac{n-1}{2} - m$. The (negative) bending moment at D is

$$\frac{n-1}{2}\mathrm{W}\times md-\mathrm{W}d\left(m\frac{m-1}{2}\right)=\frac{\mathrm{W}dm}{2}(n-m)$$

hence the stress in the bottom chord RQ is

$\frac{\mathrm{W}dm}{2h}(n-m)$, or W cot $\theta$ multiplied by the coefficient $\frac{m}{2}(n-m)$.

The (negative) bending moment at Q is

$$\frac{n-1}{2}\mathrm{W}(m+1)d-\mathrm{W}d\frac{m}{2}(m+1)=\frac{\mathrm{W}d}{2}(m+1)(n-m-1)$$

hence the stress in the top chord DE is

$\frac{\mathrm{W}d(m+1)}{2h}(n-m-1)$, or W cot $\theta$ multiplied by the coefficient $\frac{m+1}{2}(n-m-1)$

the coefficients for the left-hand half of Fig. 204 in which $n=8$, for example, are

| Member of panel. | Constant. | Panel. BC ($m=0$) | CD $m=1$ | DE $m=2$ | EF $m=3=\left(\frac{n}{2}-1\right)$ |
|---|---|---|---|---|---|
| Left vertical | W | $\frac{7}{2}$ | $\frac{5}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ |
| Diagonal | W cosec $\theta$ | $\frac{7}{2}$ | $\frac{5}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ |
| Top chord | W cot $\theta$ | $\frac{7}{2}$ | 6 | $1\frac{5}{2}$ | $\frac{5}{2}$ |
| Bottom chord | W cot $\theta$ | 0 | $\frac{7}{2}$ | 6 | $1\frac{5}{2}$ |

When $n$ is odd, and for other simple types of girder, the coefficients may be similarly tabulated.

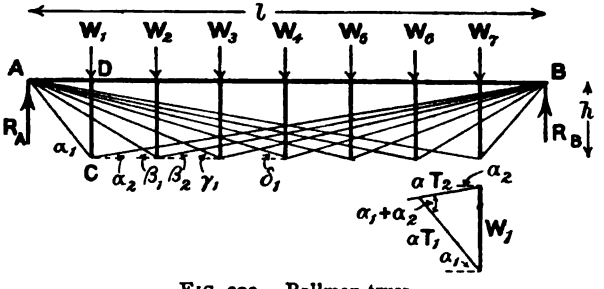



FIG. 209.—Bollman truss.

## 140.—Some Special Framed Girders.

*Bollman Truss* (Fig. 209).—This type of girder, which is really a trussed beam (see Art. 164), carries its load at the top chord; it was

most frequently used for deck bridges. The stresses, neglecting any flexual rigidity of the top chord, are easily found as follows. Thrust in DC = $W_1$. Let ${}_aT_1$ and ${}_aT_2$ be the tensions in AC and BC respectively. Then from the triangle of forces shown,

$${}_aT_1 = W_1 \frac{\cos \alpha_2}{\sin (\alpha_1 + \alpha_2)} \qquad {}_aT_2 = W_2 \frac{\cos \alpha_1}{\sin (\alpha_1 + \alpha_2)}$$

and so on for all the oblique ties. Or for eight panels as in Fig. 209, reaction at A due to $W_1 = \frac{7}{8}W_1$ = vertical component of ${}_aT_1$, or

$${}_aT_1 \sin \alpha_1 = \tfrac{7}{8}W_1 \text{ and } {}_aT_1 = \tfrac{7}{8}W_1 \operatorname{cosec} \alpha_1, \qquad \text{where } \cot \alpha_1 = \frac{l}{8h}$$

$${}_aT_2 = \tfrac{1}{8}W_1 \operatorname{cosec} \alpha_2 \qquad \text{where } \cot \alpha_2 = \frac{7l}{8h}$$

And similarly

$${}_\beta T_1 = \tfrac{6}{8}W_2 \operatorname{cosec} \beta_1 \qquad \text{where } \cot \beta_1 = \frac{l}{4h}$$

$${}_\beta T_2 = \tfrac{2}{8}W_2 \operatorname{cosec} \beta_2 \qquad \text{where } \cot \beta_2 = \frac{3l}{4h}$$

and so on.

Thrust in AB due to $W_1$ is ${}_aT_1 \cos \alpha_1$ or ${}_aT_2' \cos \alpha_2$; hence the total thrust in AB is

$$\tfrac{7}{8}W_1 \cot \alpha_1 + \tfrac{6}{8}W_2 \cot \beta_1 + \tfrac{5}{8}W_3 \cot \gamma_1 + \tfrac{4}{8}W_4 \cot \delta_1 + \text{etc.}$$

$$= \tfrac{7}{64}W_1 \frac{l}{h} + \tfrac{12}{64}W_2 \frac{l}{h} + \tfrac{15}{64}W_3 \frac{l}{h} + \tfrac{16}{64}W_4 \frac{l}{h} + \text{etc.}$$

$$= \frac{l}{64h}(7W_1 + 12W_2 + 15W_3 + 16W_4 + 15W_5 + 12W_6 + 7W_7)$$

*Fink Truss* (Fig. 210).—This is a later form of the Bollman Truss and its solution is similar. The solution is shown in Fig. 210, in which

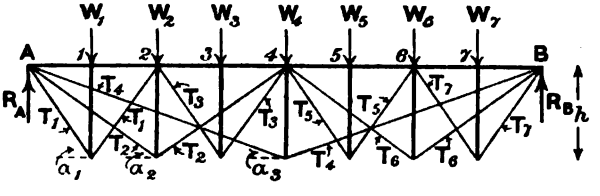



FIG. 210.—Fink truss.

$T_1$ is the tension in each tie to the foot of the post under panel point number 1, and so on. Resolving at the foot of posts 1, 3, 5, and 7, $2T_1 \sin \alpha_1 = W_1$ etc., and $T_1 = \frac{W_1}{2} \operatorname{cosec} \alpha_1$, $T_3 = \frac{W_3}{2} \operatorname{cosec} \alpha_1$, etc. Evidently when there are no oblique forces at the top of the posts, the thrusts in the first, third, fifth, and seventh verticals respectively are $W_1$, $W_3$, $W_5$, and $W_7$. The second post carries half the vertical component

of $T_1$ and $T_3$, viz. $\frac{W_1}{2}$ and $\frac{W_3}{2}$ and

Thrust in second post $= W_2 + \frac{W_1 + W_3}{2}$

„ fourth post $= W_4 + \frac{W_3 + W_5}{2} + \frac{W_2 + W_6}{2} + \frac{W_1 + W_3 + W_5 + W_7}{4}$

$= W_4 + \frac{1}{2}(W_2 + W_6) + \frac{3}{4}(W_3 + W_5) + \frac{1}{4}(W_1 + W_7)$

„ sixth „ $= W_6 + \frac{W_5 + W_7}{2}$

$T_2 = \frac{1}{2}\left(W_2 + \frac{W_1 + W_3}{2}\right)$ cosec $\alpha_2$

$T_4 = \frac{1}{2}\{W_4 + \frac{1}{2}(W_2 + W_6) + \frac{3}{4}(W_3 + W_5) + \frac{1}{4}(W_1 + W_7)\}$ cosec $\alpha_3$

Thrust in top chord.

First two panels, $T_1 \cos \alpha_1 + T_2 \cos \alpha_2 + T_4 \cos \alpha_3$

Third and fourth panels, $T_2 \cos \alpha_2 + T_4 \cos \alpha_3 + T_3 \cos \alpha_1$

which may be reduced by writing cot $\alpha_1 = \frac{1}{8}\frac{l}{h}$, cot $\alpha_2 = \frac{1}{4}\frac{l}{h}$, cot $\alpha_3 = \frac{1}{2}\frac{l}{h}$.

*Very long Span Trusses.*—For very long spans the Baltimore trusses, Fig. 194 (in which the panel of the N girder is subdivided), are modified by having the top chord curved. Fig. 211 shows such a truss as is

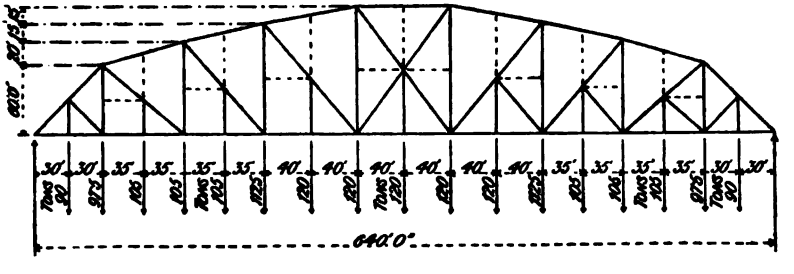


FIG. 211.—Modified Baltimore truss for long spans.

used for the centre span in the design for the Quebec Bridge (1911) and of approximately the same dimensions. The stress diagram presents no special difficulty. The members shown dotted support other members, and are not to be considered as members of the truss.

## EXAMPLES XI.

1. A roof truss of the type shown in Fig. 195 has a span of 28 feet, rise 7 feet, and no camber of the tie rods, the joints in the main rafter bisecting its length. For the loads given in Problem No. 1, Examples X., find the maximum stresses in all members due to dead loads, and to wind loads separately, assuming fixed hinges at both supports.

2. If the roof principals in Problem No. 2, Examples X., are of the types shown in (*h*), Fig. 192, but the ties have no camber, find the maximum stresses due to wind pressure when both sides have fixed hinges.

3. A French roof truss (see Fig. 199) has a span of 50 feet and a rise of 12 feet 6 inches, and the lower ties have no camber. Find the stresses in all members due to a dead load of 25 lbs. per square foot of covered area, the principles being 12 feet apart.

4. Find the maximum stresses in the roof of Problem No. 3 when the horizontal wind pressure is 56 lbs., adopting the formula (7) of Art. 128, and taking one side "free" and the other "fixed." Assume that the wind may be from either side.

5. A Warren girder having web members all inclined 60 degrees has eight panels, the first five from the left end being loaded with 5 tons per panel uniformly distributed. Find the stress in all members.

6. An N girder having seven panels each 6 feet long and 8 feet high has the first five from the left-hand end loaded with a uniformly distributed load of W per panel. Find the stress in each member if the central bay is counter-braced by members capable of bearing tension only.

7. Find an expression for the compression in the top chord of a Bollman truss fully loaded with a load W at each panel point if there are $n$ panels of height $h$, the total span being $l$.

8. Find the maximum compression in the top chord of a Fink truss having a load W at each panel point, there being eight panels and the span being $l$ and the height $h$. Find also the thrust in the central vertical post.

9. Find the chord stresses for the truss and loading shown in Fig. 211.

# CHAPTER XII

## *MOVING LOAD STRESSES IN FRAMES*

**141. Stresses due to Rolling Loads.**—The methods of finding stresses given in the preceding chapter are applicable to known loads on the various parts of the truss. But in order to compute the maximum stresses to which a member of a bridge will be subjected by a travelling load crossing the span, the position of the load to produce this maximum effect has to be considered. This matter has been partially dealt with in Chapter VI., but the application to framed girders will require further notice and illustration.

**142. Chord Stresses.**—The chord stresses may be found (Art. 138) by taking moments about joints in the opposite chord, to which the stress is proportional, and the stress in the chord is a maximum when the moment about the opposite joint is a maximum. For a uniformly distributed load this occurs (see Art. 76 and end of Art. 90) when the whole span is loaded. Hence, if an equivalent uniformly distributed load is adopted the determination of chord stresses due to rolling loads is precisely similar to that for uniformly distributed dead loads.

In the case of concentrated loads arising from axle loads or from conventional train loads (Art. 85) the maxima occur when the bending moments at opposite joints reach maximum values and the corresponding positions of the load (for joints of both loaded and unloaded chords) are given in Art. 81. The calculation of maximum bending moments in such a case may be accomplished for determined positions of the load by moving the span length under the loads as in Art. 80 or by algebraic calculation. In either case the calculation is much more tedious than when an equivalent uniformly distributed load is employed.

**143. Conventional Calculation of Web Stresses for Uniform Rolling Loads.**—We have seen that for girders with horizontal chords (Art. 138, section 2) the web member stresses are proportional to the shearing force at the member, and hence the maximum (positive or negative) stresses due to rolling loads will occur in such members when the maximum positive or negative shearing forces occur. A simple conventional method (Fig. 212) is to assume that the maximum positive shearing force at any section occurs when all panel points to the left of that section are fully loaded and those to the right are unloaded: and that maximum negative shearing force occurs when all panel points to the right are fully loaded and those to the left unloaded. This is an approximation which will be shown to be on the safe side,

but is clearly an impossible condition, for no panel point can be entirely unloaded if the adjacent panel point carries the *full* panel load ($w \times d$) due to a uniformly distributed load.

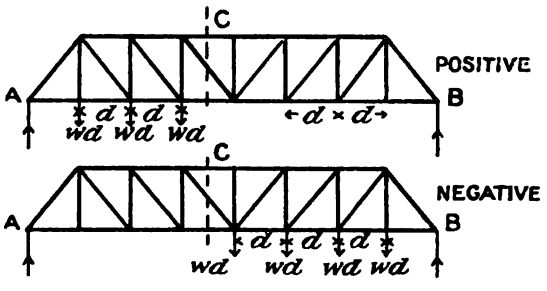



FIG. 212.—Conventional loading for extreme shearing forces and stresses in web members.

Adopting such conventional loading for both horizontal and curved chord girders, maximum positive and negative stresses in web members due to a uniformly distributed rolling load can be calculated as in Art. 138, sections (2), (3) and (4). If these be added algebraically to the stress due to dead load the extreme maximum and minimum stresses are obtained.

EXAMPLE 1.—A through **N** girder of 80-ft. span has eight bays of 10 ft. each, the height throughout being 12 ft. The uniformly distributed dead load is 0·6 ton per foot run, and the rolling load is equivalent to 2 tons per foot. Find the maximum and minimum stresses in each diagonal and vertical member. All dead loads as well as live loads to be taken as at the bottom chord joints.

The panel loads are, for dead loads 6 tons, and for rolling loads 20 tons. Using the reference letters of Fig. 204 or 213, the dead load reactions (effective) are $R_1 = R_2 = \frac{42}{2} = 21$ tons. The dead load shears in the main panels, altering by the panel load of 6 tons at each lower chord joint, are—

| Panel . . . . . | AS | SR | RQ | QP | PO | ON | NM | ML |
|---|---|---|---|---|---|---|---|---|
| Dead load shear in tons . | −21 | −15 | −9 | −3 | +3 | +9 | +15 | +21 |

The extreme shears due to rolling load on, say, the panel RQ are for maximum positive shear, loads of 20 tons at S and R only, then $R_2 = \frac{40 \times 1{\cdot}5}{8} = 7{\cdot}5$ tons = maximum positive shear. For panel QP the value is $R_2 = \frac{60 \times 2}{8} = 15$ tons. The full values are

| Panel . . . . . | AS | SR | RQ | QP | PO | ON | NM | ML |
|---|---|---|---|---|---|---|---|---|
| Maximum positive shear | 0 | +2·5 | +7·5 | +15 | +25 | +37·5 | 52·5 | 70 |
| Maximum negative shear | −70 | −52·5 | −37·5 | −25 | −15 | −7·5 | −2·5 | 0 |

combining these with the dead loads shears, the extreme shears in each panel are—

| Panel . . . . . | AS | SR | RQ | QP | PO | ON | NM | ML |
|---|---|---|---|---|---|---|---|---|
| Maximum positive or negative shear . . . | −91 | −67·5 | −46·5 | −28 / +12 | +28 / −12 | +46·5 | +67·5 | +91 |
| Minimum positive or negative shear . . . | −21 | −12·5 | −1·5 | — | — | +1·5 | +12·5 | +21 |

The maximum positive shear due to the rolling load and the positive shear due to dead load are shown on Fig. 213, together with the resultant for half the span. The negative quantities for the other half are symmetrical.

The maximum thrusts in AB, CS, DR are 91, 67·5 and 46·5 tons respectively, that is, the shearing forces in the panels AS, SR, and RQ, while the minimum thrusts are 21, 12·5 and 1·5 tons respectively. The

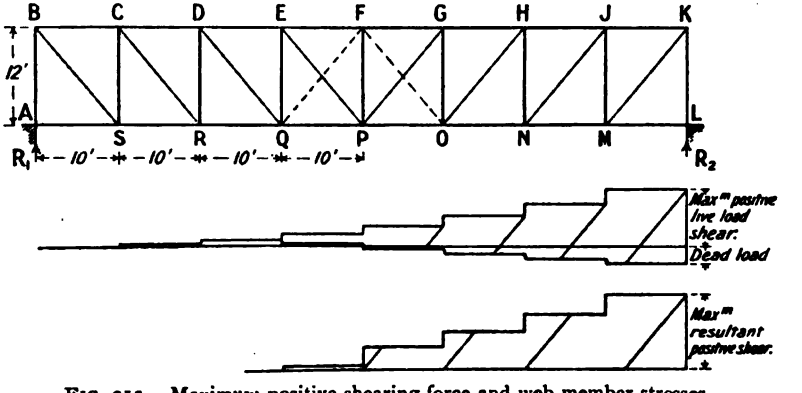



FIG. 213.—Maximum positive shearing force and web member stresses.

stress borne by EQ varies from a thrust of 28 tons to a tension of 12 tons. The stress in FP is always zero. If the dead load is so distributed that $\frac{1}{3}$ of the total comes on the joints of the top chord, 2 tons thrust will have to be added to the results for CS, DR, EQ, and FP, and 1 ton to that for AB.

The stresses in the diagonals are found by multiplying the extreme shears by cosec $\theta$, *i.e.* by $\frac{BS}{AB} = \frac{1}{12}\sqrt{144 + 100} = 1{\cdot}30$, giving

| Member . . . . . | BS | CR | DQ | EP | GP | HO | JN | KM |
|---|---|---|---|---|---|---|---|---|
| Maximum tension . . . | 118·3 | 88·0 | 60·5 | 36·4 | 36·4 | 60·5 | 88·0 | 118·3 |
| Minimum tension . . . | 27·3 | 16·3 | 1·95 | — | — | 1·95 | 16·3 | 27·3 |
| Maximum thrust . . . | — | — | — | 15·6 | 15·6 | — | — | — |

These are also shown by the diagonal lines parallel to the diagonal members, across the shear diagram in Fig. 213. The change from tension to thrust in EP and GP may of course be prevented by counter-bracing the bays QP and PO as shown by the dotted lines (see Art. 148).

**144. Exact Method for Girders with Horizontal Chords and Single Web Systems.**—The exact load position for and amount of the maximum shear has been dealt with fully in Art. 90, section (1). It is interesting to compare the results from the conventional loading above with the true value. Using Fig. 134, the maximum positive shearing force in any panel DC having $m$ panels to the left of it (out of $n$ panels total), with a rolling load $w$ per foot and span $l$ feet, is by (4) Art. 90

$$\frac{w}{2} \cdot \frac{m^2}{n(n-1)} \cdot l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

According to the conventional loading above, the right-hand side would be (taking moments about A, Fig. 134)

$$m \cdot \frac{l}{n} w \left( \frac{m+1}{2} \cdot \frac{l}{n} \right) \div l = \frac{w}{2} \cdot \frac{m(m+1)}{n^2} \cdot l \quad . \quad . \quad . \quad (2)$$

For the extreme right-hand side panel at B, Fig. 134, $m = n - 1$, and both (1) and (2) give

$$\frac{w}{2} \cdot \frac{n-1}{n} \cdot l$$

But for smaller values of $m$ (nearer the middle of the span) (2) gives a slightly higher value than (1), *i.e.* it is a trifle on the safe side Applying both methods to such a truss as Fig. 204, where $n = 8$, we get the following maximum positive shears (with corresponding negative values), taking $\frac{wl}{2} = 100$:—

| Panel . . . . . | AS | SR | RQ | QP | PO | ON | NM | ML |
|---|---|---|---|---|---|---|---|---|
| $m =$ : . . . . . | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| Conventional maximum + shear . . . . . | 0 | 3·12 | 9·4 | 18·8 | 31·2 | 46·8 | 65·5 | 87·5 |
| Actual maximum + shear | 0 | 1·78 | 7·13 | 16·1 | 28·5 | 44·5 | 64·1 | 87·5 |

The difference in the results from the two methods is very small, as will be realized if the results are plotted on such shear diagrams as Fig. 213 or Fig. 134, which are too small to show the difference.

EXAMPLE.—Find the exact maximum stress in the member HO (Fig. 204 or 213), with the loading given in the example at the end of Art. 143.

The actual maximum positive shearing force in the panel ON, putting $m = 5$, and $n = 8$, is from (1),

$$\frac{2}{2} \cdot \frac{25 \times 80}{8 \times 7} = 35{\cdot}71 \text{ tons}$$

and adding the dead-load shear of + 9 tons gives 44·71 tons (instead of 46·5 tons), and multiplying by sec. $\theta$ or 1·30 gives the maximum stress in HO,

$$1{\cdot}3 \times 44{\cdot}71 = 58{\cdot}1 \text{ tons instead of } 60{\cdot}5 \text{ tons.}$$

When the shearing force has been determined, the stresses in the web members follow, as for the conventional method, viz. from Art. 138, section (2).

**145. Exact Method with Curved Top Chord and Single Web Systems.**—For a girder with curved top chord the stress in a web member such as BD (Fig. 214) is found as shown in Art. 138, section (4),

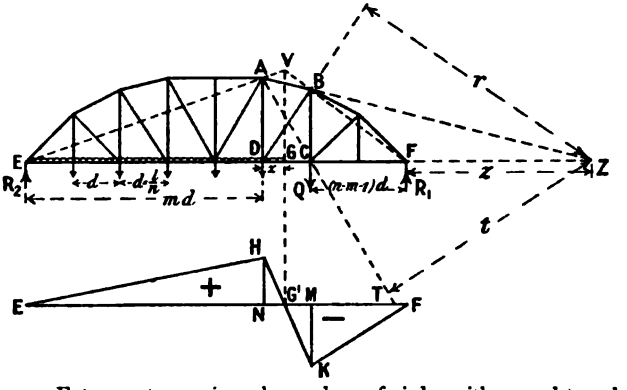



FIG. 214.—Extreme stresses in web members of girder with curved top chord.

and Fig. 208.[1] To find the position of the uniform load $w$ per foot to give maximum pull in BD, suppose it covers entirely all the $m$ panels to the left of DC (Fig. 214) and extends a distance DG or $x$ beyond D towards C, and the pull in BD is P. By moments about E

$$R_1 = \frac{w}{2l}(md + x)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And the joint load Q at C, from moments about D, is

$$Q = \frac{wx^2n}{2l} = \frac{wx^2}{2d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then by moments about the centre Z

$$P = \frac{I}{r}[R_1 \cdot z - Q\{z + (n - m - 1)d\}] \quad . \quad . \quad . \quad . \quad (3)$$

$$= \frac{w}{2rl}[z(md + x)^2 - nx^2\{z + (n - m - 1)d\}] \quad . \quad (4)$$

[1] It is assumed that the two chord members cut by the section taken for finding the stress in BD will, if produced, intersect outside the span of the girder. If they intersect inside the span, as often occurs in the braced arch (see Fig. 292, but omit central hinge), the span must be fully loaded for maximum stress in the member BD.

And for a maximum value of P, $\frac{dP}{dx} = 0$, hence

$$x = \frac{m}{n - 1 + n(n - m - 1)\frac{d}{z}} . d \quad . \quad . \quad . \quad . \quad (5)$$

which reduces to $\frac{m}{n-1} \cdot \frac{l}{n}$ when $z$ is infinite, *i.e.* for a horizontal top chord as in (2), Art. 90. Substituting this in (1) and (2), equation (3) gives P.

The distance CG from C for maximum thrust when the load extends from F to G is—

$$\frac{n - m - 1}{n - 1 - nm\frac{d}{l + z}} . d$$

The position of the load for maximum stress in any vertical may similarly be found.

*Alternative Method.*—The influence method may also be used. Considering unit load rolling over the span EF, when it has moved a distance $y$ from E (short of D), $R_1 = \frac{y}{l}$, and as in (3), $P = \frac{y}{l} \cdot \frac{z}{r}$, which is proportional to $y$, *i.e.* the influence line EH is a straight line, such that $HN = \frac{md}{l} \cdot \frac{z}{r} = \frac{m}{n} \cdot \frac{z}{r}$, similarly for the load between C and F the influence line is the line KF, such that $KM = \frac{n - m - 1}{n} \cdot \frac{z + nd}{r}$. Also it is easy to show that the rate of change as the unit load moves from D to C is proportional to $x$, *i.e.* the influence line is a straight line through H and K, which is the line HG'K. As in Art. 90, the stress due to a uniform load $w$ per foot extending from E to G' is found from the area EHG' under the influence line, viz.—

$$\tfrac{1}{2}w(HN \times EG') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$EG' = md + NG'$, and NG' may be found (confirming (5)), by dividing DC in the known ratio $\frac{HN}{KM} = \frac{mz}{(n - m - 1)(z + nd)}$, and substituting in (6), this gives

$$\text{max. pull in DB} = \frac{w}{2} \cdot \frac{m^2}{n} \cdot \frac{z}{r} \cdot d \left\{ 1 + \frac{1}{n - 1 + n(n - m - 1)\frac{d}{z}} \right\} \quad (7)$$

Similarly, the maximum thrust in DB due to live load is

$$\frac{w}{2} \cdot \frac{(n - m - 1)^2}{n} \cdot \frac{(z + nd)}{r} d \left\{ 1 + \frac{1}{n - 1 - n \cdot m \cdot \frac{d}{z + l}} \right\} \quad (8)$$

Similar methods may be used to draw the influence lines for the stress in the verticals.

An alternative simple method of finding the point G′ is to join AE and FB. These lines produced meet at V, and G′ is the projection of V. For G′ is the point at which a load would produce no stress in DB; to the left of G′ a load would produce tension in DB, and to the right it would produce thrust. That no stress is produced in DB by a load under V is proved by the fact that EVB would be a bending moment diagram for such a load, and hence $M_D \div AD = M_B \div BC$, therefore the tension in DC equals the horizontal component of the thrust in AB. Hence by the method of sections the horizontal component of the stress in DB equals zero. When the point G is determined, the stress in DB may be found by various means, such, for instance, as a stress diagram.

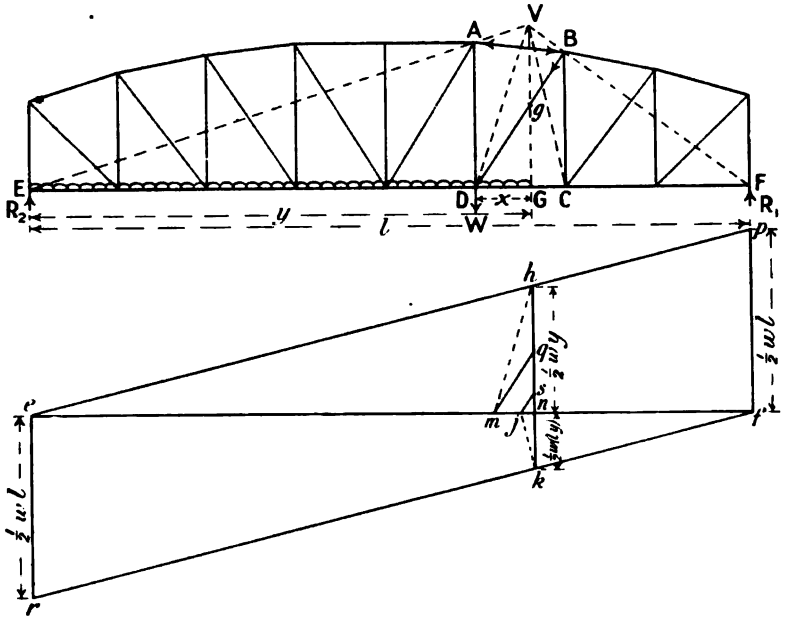



FIG. 215.

*Graphical Construction.*—The maximum stress in a diagonal DB due to a rolling load may be determined graphically by dropping a perpendicular VG (Fig. 215) from V. Then D*g* represents the pull in DB on the same scale that VG represents half the load on EG. Similarly, *g*B represents the minimum stress or maximum thrust in DB on the same scale that VG represents half the load on GF. When these stresses are required for each diagonal it is convenient to set off the diagram *pfer* with the dimensions shown for a load *w* per foot. Then *nh* represents half the load *w*.EG or $\frac{1}{2}w.y$, hence by drawing *hm* parallel to VD, and then *mq* parallel to DB, *mq* gives the maximum stress to a known scale, viz. that on which *pf* has been made equal to

$\frac{1}{2}wl$. The correctness of the construction may be proved as follows. The load $w$. ED on ED may be replaced by $\frac{1}{2}w$. ED at E, and $\frac{1}{2}w$. ED at D; similarly, the part $w$. DG may be replaced by $\frac{1}{2}$. $w$. DG at D, and $\frac{1}{2}$. $w$. DG at G. The load at E may be ignored, and the load at G, we have just seen, causes no stress in DB. Hence the effect in DB is that of a load $\frac{1}{2}(ED + DG)w = \frac{1}{2}EG . w = W$, say, at D. Let T' be the thrust in AB, T the tension in DC, and S the tension in DB, of which $h$ and $v$ are horizontal and vertical components respectively. Taking, say, a vertical section through the panel DC, the forces on the right-hand part of the structure are, firstly, T' and S at B, and, secondly, T and $R_1$ meeting at F; these pairs must balance, and therefore act through B and F, hence the resultant of T' and S is in the line VF. Again, considering forces on the left-hand portion, there are, firstly, T' and (downwards) $W - v$ through A; secondly, $T + h$ and $R_2$ through E. Since these balance, their resultants act through E and A, hence the resultant of T' and $W - v$ are in the line VA. Finally, taking, say, clockwise moments about V of forces on the structure to the left of the section, since the resultant of T' and S is through V, the sum of the moments of T' and S about V is zero, and resolving S at D

If $a$ is the perpendicular distance of V from AB,

$$-T' . a - v . x + h . VG = 0 \quad . \quad . \quad . \quad . \quad (1)$$

and since the resultant of T' and $W - v$ is through V,

$$+T' . a - (W - v)x = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

hence adding (1) and (2).

$W . x = h . VG$ or $\frac{x}{VG} = \frac{h}{W}$, or if VG represents W, $x$ or DG represents $h$ to the same scale, and consequently D$g$ represents S, of which $h$ is the horizontal component to the same scale. Similarly $g$B represents the thrust in BD for a load $w$. GF on GF to the same scale that VG represents a load $\frac{1}{2}w$. GF.

EXAMPLE 1.—Taking Fig. 214 to represent a girder 80-feet span subjected to a uniform rolling load of 1 ton per foot, and the heights of successive verticals from either end to the centre being 0, 10, 15, 17·5, and 17·5 feet, find the maximum tension and the maximum thrust due to rolling load in the member BD by the conventional, and the exact methods when the panel DC is not counterbraced

The lengths of $s$ and $r$ may conveniently be measured from a drawing to scale and used in, say, inches as measured; they may also be calculated in feet as follows. Fall in AB = 2·5 feet in 10 feet horizontally; hence the length CZ for a fall of 15 feet $= 15 \times \frac{10}{2 \cdot 5} = 60$ feet, and $s = 40$ feet, DB $= \sqrt{15^2 + 10^2} = 18 \cdot 03$ feet, $r = 70 \sin \hat{BDC}$ $= 70 \times \frac{15}{18 \cdot 03} = 58 \cdot 2$ feet.

Using the conventional method, all joints E to D having full panel loads of 10 tons, by moments about E—

$$R_1 = \tfrac{10}{80}(10 + 20 + 30 + 40 + 50) = \tfrac{10}{8} \times 15 = 18 \cdot 75 \text{ tons}$$

Hence by taking a vertical section in the panel DC, and considering the structure to the right of it, and taking moments about Z of the external forces—

$$\text{Pull in BD} = \frac{1}{58{\cdot}2}(18{\cdot}75 \times 40) = 12{\cdot}87 \text{ tons}$$

Using the exact method, $m = 5$, $n = 8$, then from (5),

$$x = \frac{5 \times 10}{7 + (8 \times 2 \times \frac{10}{40})} = \frac{50}{11} = 4{\cdot}55 \text{ feet}$$

$$R_1 = \frac{1}{2 \times 80}\left(\frac{54{\cdot}55^2}{2}\right) = 18{\cdot}6 \text{ tons}$$

$$Q = \frac{1}{2 \times 10} \times 4{\cdot}55^2 = 1{\cdot}04 \text{ tons.}$$

Hence by moments about Z

$$\text{Pull in BD} = \frac{1}{58{\cdot}2}(18{\cdot}6 \times 40 - 1{\cdot}04 \times 60) = 11{\cdot}71 \text{ tons,}$$

which may also be checked by (7).

It may be noted that the conventional method is more in error (although on the safe side) for curved chord girders than for the horizontal chord type (see Art. 144). The errors are greater the more inclined the chords are to the horizontal.

For maximum thrust in BD by the conventional method, joints C to F are loaded, giving $R_2 = \frac{10}{80}(10 + 20) = 3{\cdot}75$ tons. Hence by moments about Z for the structure to the left of the previous section

$$\text{Thrust in BD} = \frac{1}{58{\cdot}2} \times 3{\cdot}75 \times 120 = 7{\cdot}74 \text{ tons.}$$

Using the exact method, $R_2 = \frac{1}{2 \times 80}(25{\cdot}45)^2 = 4{\cdot}05$ tons.

$$\text{Load at D} = \frac{1}{2 \times 10}(5{\cdot}45)^2 = 1{\cdot}49 \text{ tons.}$$

Hence by moments about Z, thrust in BD $= \frac{1}{58{\cdot}2}(4{\cdot}05 \times 120 - 1{\cdot}49 \times 70) = 6{\cdot}55$ tons, which may be checked by equation (8).

EXAMPLE 2.—What uniformly distributed load in the above example would be sufficient to prevent a reversal of stress in the member BD?

Let $w$ be the uniform load per foot. Then the dead load must be just sufficient to cause a tension 7·74 tons in BD. Right-hand reaction $= \frac{70w}{2} = 35w$. Taking moments of the dead load about Z

$$35w \times 40 - 10w \times 50 - 10w \times 60 = 7{\cdot}74 \times 58{\cdot}2 = 450$$
$$w = \tfrac{450}{300} = 1{\cdot}5 \text{ tons per foot}$$

or more exactly, taking 6·55 tons thrust to be neutralized

$$35w \times 40 - 10w \times 50 - 10w \times 60 = 6{\cdot}55 \times 58{\cdot}2 = 382$$
$$w = \tfrac{382}{300} = 1{\cdot}273 \text{ tons per foot}$$

**146. Trusses with Multiple Web Systems.**—In finding the maximum stresses in multiple web trusses due to rolling loads the same principles as have been used for single web systems are applicable. Usually the conventional loading will be sufficiently near for finding the maximum stresses in web members in accordance with the methods given in Art. 138, section (5), for fixed loads. Thus in the Whipple-Murphy truss the method of superposition (Art. 136) may be employed to find, as in the present article, the maximum web stresses in each of two girder systems into which the Whipple-Murphy girder may be split (Fig. 203).

Again, in the Baltimore truss, Fig. 194, and Art. 138, section (5), the methods are those which have been dealt with already. An exact solution by means of the influence line is also possible, but the conventional system of complete panel loads is sufficiently near for most purposes, and much less complicated.

**147. Stress Calculations for Concentrated Loads.**—The maximum stress in a web member of a truss with horizontal chords occurs when the shearing force in its panel is a maximum, and the position of the loads to give the maximum shearing force has been demonstrated in (4), Art. 82, a condition which is fulfilled when a particular wheel load passes into the panel concerned. When the position is determined the maximum shear is easily calculated. When the top chord is curved, as in Fig. 214, the position of the load for maximum stress in BD, say, instead of being given by (3), Art. 82, viz. $\frac{W}{l} = \frac{W_2}{d}$, writing $d$ instead of $k$, is given by the modified equation

$$\frac{W}{l} = \frac{W_2}{d}\left(1 + \frac{md}{z}\right) \text{ or } W_2 = \frac{\frac{W}{n}}{1 + \frac{md}{z}}$$

the term $\frac{md}{z}$ arising from the slope of the top chord, and being zero when $z$ is infinite. For examples of stress calculations in web members of curved chord girders having multiple web systems subjected to concentrated travelling loads, see "Modern Framed Structures," by Johnson, Bryan, and Turneaure.

**148. Stresses in Counterbraces.**—The reversal of live-load shear in a particular panel of a braced girder may be taken up by making the diagonal of such a section as will enable it to resist the necessary tension and thrust. But instead of this very frequently a second diagonal capable of resisting tension only is introduced to take up the shear which would otherwise put the main diagonal in compression. Such diagonals, shown dotted in Figs. 194, 204, 213 and 216, are called *counter-braces*.[1] The main diagonal is that which takes tension when the girder is fully loaded, and also the dead-load tension.

[1] The American practice is to call this a *counter* or counter-tie, and to call a diagonal capable of taking a reversal of stress a counterbrace.

The counterbracing of a panel really makes the structure statically indeterminate (see Chap. XIV.), and the stresses in the main diagonal and counterbrace depend upon their sections, and particularly on the initial conditions of attachment which determine the initial stresses which may exist due to dead load. If the main diagonal carries completely its dead-load stress when the attachment (or final adjustment) is made, any live load tending to reduce this tension will put the counterbrace in tension without first reducing the main diagonal to a state of ease or zero stress. And these conditions will be safer to assume in calculating the possible tension in a counterbrace. In some constructions the counterbrace is actually left slack during erection and then given a slight tension; in such a case it would be required to resist live-load stress plus the initial adjusting tension. Actually the stress in both counterbrace and main brace will be affected by the fact that the other one is capable of resisting considerable thrust, although nominally unable to do so. For several reasons, then, the stress calculations for counterbraced panels must be regarded as conventional in a greater degree than is true for other simpler types of girders.

The calculations of tension in the counterbraces due to the live load are made in the same way as for live-load stresses in the main braces, the other braces being ignored, and will be best illustrated by examples.

EXAMPLE 1.—Find the stress in the counterbrace FO, Fig. 204 or 213, with the data given in Ex. 1, Art. 143.

Referring to the results of the example quoted, the maximum negative live-load shear in the panel PO is 15 tons, which is assumed to be wholly borne by the counterbrace. The inclination is equal to that of the main brace, viz. $\theta$ where cosec $\theta = 1{\cdot}3$, hence the maximum tension in FO is $15 \times 1{\cdot}3 = 19{\cdot}5$ tons.

If we assume that the two diagonals jointly carry the dead-load shear (FO being slightly either shortened or bowed), but that the main brace cannot be relied upon to take thrust, the stress to be allowed for in FO calculated for the maximum negative shear of 12 tons in panel PO, given in Ex. 1, Art. 143, would be $12 \times 1{\cdot}3 = 15{\cdot}6$ tons, the same as the thrust in PO when the panel is not counterbraced. The actual stress would probably be between 15·6 and 19·5 tons, but depends upon initial conditions.

EXAMPLE 2.—Find the maximum tension in a counterbrace AC, Fig. 214, under the conditions of Ex. 1, Art. 145.

The distance $t$ of AC from Z is CZ·sin $\hat{TCZ}$.

$$AC = \sqrt{17{\cdot}5^2 + 10^2} = 20{\cdot}15$$

$$\sin \hat{ACD} = \frac{17{\cdot}5}{20{\cdot}15} = 0{\cdot}868,$$

hence $$t = 60 \times 0{\cdot}868 = 52{\cdot}1 \text{ feet.}$$

When joints C to F are fully loaded, $R_2 = 3{\cdot}75$ as before. Omitting member BD, by moments about Z

$$\text{Pull in AC} = \frac{1}{52{\cdot}1}(3{\cdot}75 \times 120) = 8{\cdot}64 \text{ tons.}$$

For a more exact value the conditions are as before, but in a counterbraced panel such are finement is unnecessary; the result would be

$$\frac{1}{52{\cdot}1}(4{\cdot}05 \times 120 - 1{\cdot}49 \times 70) = 7{\cdot}33 \text{ tons.}$$

The stress in the vertical of a counterbraced panel is a minimum thrust (or maximum tension) when the counterbrace meeting its foot is in action, *e.g.* if AC, Fig. 214, is in tension and BD out of use, consider the joint B. The resultant of the two chord stresses at B will cause a tension in the vertical BC unless a load at B is sufficient to cause thrust. The load position to give maximum tension in BC has to be found by trial; it occurs when the chord stresses at B are as great as possible, consistent with BD remaining out and AC in action, *i.e.* when the load extends from F as far beyond C as to just cause BD to have zero stress in it.

**149. Stress in Wind Bracing.**—The wind bracing or laterals of a framed girder bridge are very generally (unless head room requires arched girders) of the form shown in Figs. 193 and 216, *i.e.* N girders counterbraced in every panel (as the wind may blow from either side), and having as chords the chords of the main girders. In the case of open-floor bridges the cross girders will form the struts of the wind-bracing system.

Such a girder of course forms a statically indeterminate system, and, as in the preceding article, the wind stress can only be determined in a conventional manner by arbitrary assumptions. Again, the distribution of wind load on the girders and on moving vehicles between the upper and lower wind bracing (if both are used) is arbitrary. It is usual to assume that all the moving wind load is taken on the wind bracing of the loaded chord, and this bracing (which may consist of a plated floor) is also often taken to withstand more than half the wind load on the girders. Some stress is also caused by the overturning effect of the wind increasing the downward pressure on the supports on the leeward side of a bridge and decreasing it on the windward side. Altogether the determination of wind stresses in the lateral bracing is somewhat empirical. It may also be pointed out that quite apart from wind pressure the lateral bracing will be stressed by the strains of the main girder chords to which it is attached. An estimate of such stresses may be made from the stresses in the main chords and the geometry of the consequent deformations.[1]

The conventional method of estimating the most important wind stresses is to ignore one system of triangulation, as in counterbraced panels dealt with in Art. 148, and to find the stress due to dead and travelling wind loads by the method of Arts. 142 and 143. As the diagonals are for greater stiffness made capable of resisting thrust this cannot give accurate results, but any error involved is on the side of safety. The calculation may be shown by an example.

[1] See an article on "The Design of Wind Bracing" in *Engineering*, June 9, 1911.

EXAMPLE.—The N or Pratt girder shown in Fig. 216 has wind bracing shown at (*b*) and (*c*). Estimate the maximum stresses in the lower lateral bracing and in the lower chords due to a stationary wind load of 2000 pounds per panel and a travelling wind load of 30 pounds

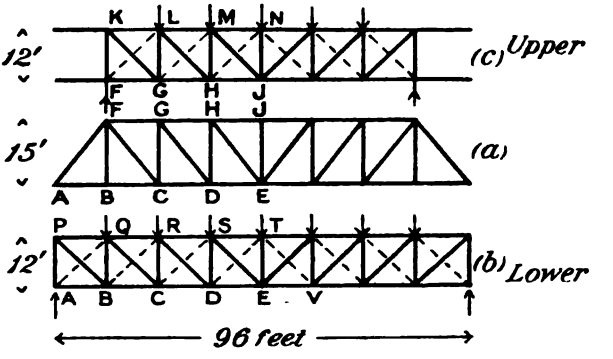



FIG. 216.—Stresses in lateral or wind bracing.

per square foot on 10 square feet per lineal foot, assuming that 60 per cent. of the stationary wind load and the whole of the travelling load is taken by the lower system.

Dead load per panel = 0·6 × 2000 = 1200 lbs.
Moving load per panel = 30 × 10 × 12 = 3600 lbs.

As the two halves are symmetrical it is only necessary to work out the stress for half the span. The wind may blow on either side, and with the wind load on the side selected in Fig. 216 the dotted braces are assumed to go out of action.

*Chord Stresses.*—For stationary load the reactions at the ends are $\frac{7 \times 1200}{2} = 4200$ lbs.

Denoting bending moments by M and a suffix and stresses by the letters at the ends of the members

$-M_B = 4200 \times 12 = 50400$ lb.-ft., hence $PQ = \frac{50400}{12} = 4200$ lbs.

Similarly, $QR = \frac{-M_C}{12} = \frac{86400}{12} = 7200$ lbs., $RS = \frac{-M_D}{12} = \frac{108000}{12} = 9000$ lbs., $ST = \frac{-M_E}{12} = \frac{115200}{12} = 9600$ lbs., $AB = 0$, $BC = \frac{-M_Q}{12} = 4200$ lbs., $CD = \frac{-M_R}{12} = 7200$ lbs., $DE = \frac{-M_S}{12} = 9000$ lbs.

For the total stress due to live and dead load it is only necessary to multiply the above by $\frac{3600 + 1200}{1200} = 4$, since the maximum chord stresses occur with full load.

*Web Stresses.*—The inclination of the braces to the chords is 45°, so from the shearing forces, falling by 1200 lbs. per panel, the dead load stresses are, *Tensile*, PB = $4200\sqrt{2} = 5940$ lbs., QC = $(4200-1200)\sqrt{2} = 4240$ lbs., RD = $1800\sqrt{2} = 2540$ lbs., SE = $600\sqrt{2} = 850$ lbs.

*Compressive.*—AP = 4200, QB = 3000, RC = 1800, SD = 600, TE = 0. The maximum live-load web stresses may similarly be written from the maximum live-load shears, as in Art. 143, and are as follows :—

*Live-load Web Stresses.*—

| Panel . . . . . | AB | BC | CD | DE | EV |
|---|---|---|---|---|---|
| Maximum negative shear . . . | 12,600 | 9450 | 6750 | 4500 | 2700 |
| Diagonal, and tension in lbs. . | PB 17,800 | QC 13,360 | RD 9540 | SE 6360 | — — |
| Cross strut, and thrust in lbs. . | AP 12,600 | BQ 9450 | CR 6750 | SD 4500 | TE 2700 |

The dead-load stresses may now be added to these to give the total stresses ; the additions are all of quantities of like sign, as change of sign in the diagonals of the middle panels is prevented by the counterbracing.

## EXAMPLES XII.

1. The girder shown in Fig. 212 is 16 ft. high and has a panel length of 12 ft. Find the maximum and minimum stresses in the upper and lower chords due to a dead load of 0·4 ton per foot run and a travelling load of 1 ton per ft. run.

2. Find the maximum and minimum stresses in the diagonals of the girder in problem No. 1. (*a*) By the conventional loadings. (*b*) By the more exact method.

3. A through Warren girder with web members inclined at 60° has 6 bays in the lower (loaded) boom and 5 in the upper. The loads are—(1) dead loads of 3 tons at each joint of the lower boom and 1 ton at each top joint ; (2) a travelling load of 1·2 tons per foot run, the bays being each 10 ft. long. Find the maximum and minimum stresses in all members.

4. A through Warren girder is the same as in problem No. 3, but has 5 bays in the lower boom and 4 in the upper one, and is subjected to the same dead load per joint. What is the maximum travelling load per foot which will only cause a reversal of stress in two web members ?

5. In a hog-back or curved top chord N girder of 6 bays, the heights in successive verticals, including the end posts, are 10, 12·5, 14, 14, 14, 12·5, and 10 ft., and the bays are each 10 ft. long. Find approximately the maximum stresses in the members of half the girder under a dead load of 0·3 ton per foot, and a live load of 1·2 ton per foot run

# CHAPTER XIII

## *SELECTED TYPICAL FRAMED STRUCTURES*

**150. Cantilever Bridges.**—The disadvantage of continuous girders have been referred to in Art. 107. In a cantilever bridge, although there may be several supports, the girders are separable into parts, each of which is statically determinate.

Two types of support for a bridge of three spans (and four supports) are shown in Figs. 217 and 218, which also show the bending-moment and shearing-force diagrams for uniformly distributed dead loads. The con-

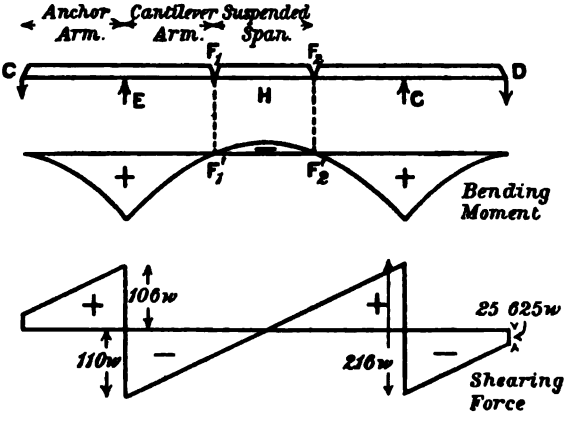



FIG. 217.

struction of the trusses is such as to form virtual hinges at $F_1$ and $F_2$, thus fixing the points of inflexion whatever the load. Then for different loadings, although the shape of the bending-moment diagram will alter, the points of inflexion will remain at $F_1'$ and $F_2'$. Either type may be looked upon as a continuous beam with hinges inserted or may be regarded as an arrangement of overhung beams simply supported at two points and carrying other simple beams suspended from their ends. Thus that in Fig. 217 consists of two beams $CF_1$ and $DF_2$, carrying the simply suspended span $F_1F_2$ from their overhung ends or *cantilever arms* at $F_1$ and $F_2$. Whether the support required at C and D is upward or downward depends upon the load between E and G compared to that

between C and E and between G and D; generally a downward anchorage will be required. The type shown in Fig. 218 consists of a central beam with overhung "cantilever" ends, which carry one end of each side span, the other ends resting on the end supports or abutments.

A great advantage of the cantilever type of bridge in many cases arises from the fact that the side spans being erected in the ordinary manner the cantilever arms can be built outwards from the piers and the central span completed from them without the use of falsework, *i.e.* support from below; erection stresses must be estimated and allowed for. For long spans where the dead load of the structure becomes very large the cantilever is advantageous because the dead weight is somewhat concentrated near the supports, and also because the average bending moments are smaller than for a simply supported span.

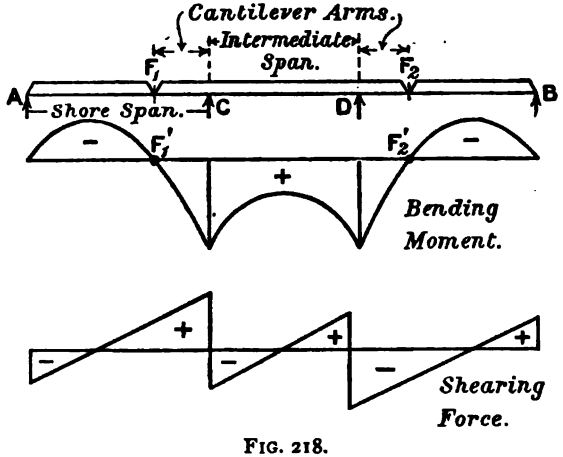



FIG. 218.

Figs. 219 and 220 show typical forms of cantilever bridges, the former being of the through and the latter of the deck type. The dotted (redundant) members which would render the structure statically indeterminate are used in erection, during which the suspended span is built out as an extension of the cantilever arms. In Fig. 220 there are two supports at the piers, but the panel between them is so lightly braced as not to transmit any shearing force, and consequently there is no change of bending moment between the two. This panel may in fact be ignored in calculating stresses and the truss may be treated as if there were a single support between the two adjoining panels. The load positions for maximum effects will have to be investigated for each type of bridge in order to find the maximum and minimum stresses due to a rolling load. It is usual to determine some form of equivalent uniformly distributed load for a cantilever bridge, but this will not be necessarily the same for the cantilever portions as for a simple span.

The conventional method of taking full panel loads may then be adopted, and the determination of live load stresses is illustrated in the following examples. The more exact methods of Art. 145 for web members are applicable, but the conventional method is simpler and sufficiently accurate, as exemplified in Example 4.

*Influence Lines for Cantilever Bridges.*—The structure being statically determinate influence lines which are straight lines are easily drawn for any section and may be used to determine maximum and minimum stresses in the members. A numerical example (No. 4) illustrates this application.

EXAMPLE 1.—The dimensions of Fig. 217, which is symmetrical, being CE = 80 ft., $EF_1$ = 70 ft., $F_1F_2$ = 80 ft., determine the dimensions of the bending-moment and shearing-force diagrams for a uniform dead load of $w$ per foot.

Since the bending-moment at $F_1$ is zero, and the shearing force from the span $F_1F_2$ at $F_1$ is $\frac{1}{2} \times 80w = 40w$, taking moments about C of the forces on $CF_1$,

$$40w \times 150 + 150w \times \tfrac{150}{2} = 80 \times R_E$$

therefore

$$R_E = 215{\cdot}6w = R_G$$

$$R_C \text{ (downwards)} = \tfrac{1}{2}(2 \times 215{\cdot}6 - 380)w = 25{\cdot}6w = R_D.$$

The shearing-force diagram can now be set out as shown in Fig. 217. $F_C = +25{\cdot}6w$, $F_E = (25{\cdot}6 + 80)w = 105{\cdot}6w$, and $105{\cdot}6w - 215{\cdot}6w = -110w$

The shearing force at mid span is zero and the other half of the diagram is symmetrical.

$$M_E = 25{\cdot}6w \times 80 + 80w \times 40 = 5248w$$

$$M_F = 0,\ M_H = -\tfrac{1}{8}w \times 80^2 = -800w$$

or checking from the span EC,

$$M_H = 5248w - \tfrac{1}{8}w \times 220^2 = -802w$$

the maximum negative bending moment or the height of the vertex of the parabola at section H. The complete bending-moment diagram is shown in Fig. 217, the signs being according to the conventions of Art. 59.

EXAMPLE 2.—The bridge girders in Fig. 219, the dimensions of which are given in terms of the equal panel lengths $d$, are subjected to a dead panel load of 5 tons and a travelling load of 15 tons per panel. Determine the maximum and minimum stresses in the members HG, EF, EG, and HE. Assume the dead as well as the live load to be carried on the lower chord.

*For dead load only.*—Taking moments about A,

$$R_B = \frac{1}{6d}(3 \times 5 \times 10d + 10 \times 5 \times 5d) = 66{\cdot}\dot{6} \text{ tons}$$

Half the downward load = 13 × 5 = 65 tons

hence $R_A = -1{\cdot}\dot{6}$ tons, *i.e.* $1{\cdot}\dot{6}$ tons downwards

which may be found directly by moments about B.

*Member HG.*—By simple geometry, length EH = $1{\cdot}3d$ and the perpendicular distance of the top chord from E = $1{\cdot}245d$.

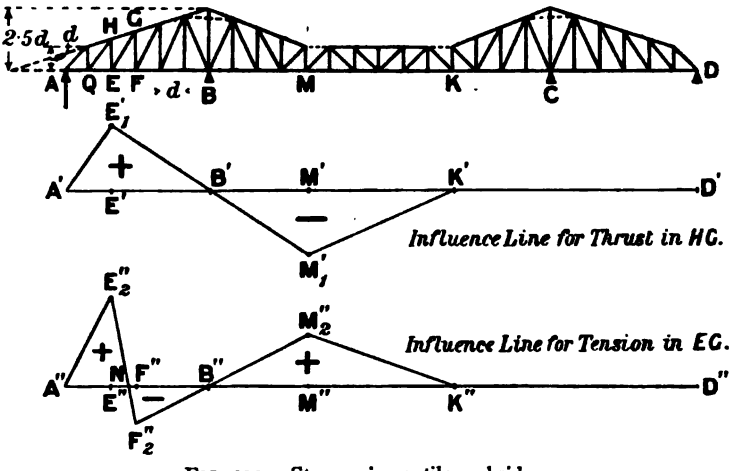



FIG. 219.—Stresses in cantilever bridge.

*For live load on AB only* (*i.e.* for maximum negative bending moment at E),

$$R_A = \tfrac{1}{2} \times 6 \times 15 - 1{\cdot}\dot{6} = 43{\cdot}\dot{3} \text{ tons}$$

maximum negative $M_E = 43{\cdot}3 \times 2d - 2(5 + 15)d = 46{\cdot}\dot{6}d$

and using E as a moment centre for a section,

$$\text{maximum thrust in HG} = \frac{46{\cdot}\dot{6}d}{1{\cdot}245d} = 37{\cdot}5 \text{ tons}$$

*For live load on BK only* (*i.e.* for maximum positive bending moment at E),

$$R_A = -\frac{1}{6d}(45 \times 4d + 60 \times 2d) - 1{\cdot}\dot{6} = 51{\cdot}\dot{6} \text{ tons (downwards)}$$

$$\text{maximum positive } M_E = 51{\cdot}\dot{6} \times 2d + 5 \times 2d = 113{\cdot}\dot{3}d$$

$$\text{maximum pull in HG} = \frac{113{\cdot}\dot{3}d}{1{\cdot}245d} = 91 \text{ tons}$$

*Member EF.*—Length GF = $1{\cdot}6d$

$$\text{maximum negative } M_G = 43{\cdot}3 \times 3d - 3(5 + 15)\tfrac{3}{2}d = 40d$$

$$\text{maximum tension in EF} = \frac{-M_G}{GF} = \frac{40d}{1{\cdot}6d} = 25 \text{ tons}$$

$$\text{maximum positive } M_G = 51{\cdot}\dot{6} \times 3d + 3 \times 5 \times 1{\cdot}5d = 177{\cdot}5d$$

$$\text{maximum thrust in EF} = \frac{177{\cdot}5d}{1{\cdot}6d} = 111 \text{ tons}$$

*Member EG.*—The stresses are found by the method of sections, taking moments about the intersection of HG and EF, the position of

which by calculation or measurement to scale is $2\frac{1}{3}d$ to the left of A and $3{\cdot}68d$ from EG produced; the section taken cuts HG, EG, and EF. For minimum tension (or maximum thrust if any) in EG, using the conventional approximation of full panel loads, see Art. 143, the panel points from F to B will be loaded, and for this live load, taking moments about B,

$$R_A = \frac{1}{6d}(d + 2d + 3d)15 - 1{\cdot}6 = 13{\cdot}\dot{3} \text{ tons}$$

Hence minimum tension in EG

$$= \frac{1}{3{\cdot}68d}(\tfrac{5}{2} \times 2\tfrac{1}{3}d + 5 \times 3\tfrac{1}{3}d + 5 \times 4\tfrac{1}{3}d - 13{\cdot}3 \times 2\tfrac{1}{3}d) = 3{\cdot}54 \text{ tons}$$

For maximum tension in EG the panel points Q and E and B to K will be loaded, and for this live load, taking moments about B,

$$R_A = \frac{1}{6d}\{(4d + 5d)15 - (4 \times 2d + 3 \times 4d)15\} - 1{\cdot}67 \text{ tons}$$
$$= 29{\cdot}17 \text{ tons (downwards).}$$

Hence maximum tension in EG

$$= \frac{1}{3{\cdot}68d}\{(29{\cdot}17 + 2{\cdot}5) \times 2\tfrac{1}{3}d + 20 \times 3\tfrac{1}{3}d + 20 \times 4\tfrac{1}{3}d\} = 61{\cdot}6 \text{ tons}$$

Similar calculations for the diagonal to the left of E would show a thrust as well as a tension, and the members would have to be made accordingly or the bay counterbraced.

*Member HE.*—The section taken cuts HG, HE, and the lower chord to the left of E and moments about the same point, $2\frac{1}{3}d$ beyond A, as for EG are taken. The minimum thrust (or maximum tension) in HE will occur when panel points from E to B are loaded. The live load reaction from moments about B is

$$R_A = \frac{1}{6d}(d + 2d + 3d + 4d)15 - 1{\cdot}67 = 23{\cdot}3 \text{ tons (upwards)}$$

Hence

$$\text{maximum tension in HE} = \frac{1}{4{\cdot}\dot{3}3d}(23{\cdot}\dot{3} \times 2{\cdot}\dot{3}d - 5 \times 3{\cdot}\dot{3}d) = 8{\cdot}74 \text{ tons}$$

The maximum thrust in HE will occur when Q and all panel points from B to K are loaded. For live load,

$$R_A = \frac{1}{6d}\{5d \times 15 - (4 \times 2d + 3 \times 4d)15\} - 1{\cdot}67 = 39{\cdot}17 \text{ tons (downwards).}$$

Hence

$$\text{maximum thrust in HE} = \frac{1}{4{\cdot}\dot{3}d}(39{\cdot}17 \times 2{\cdot}\dot{3}d + 20 \times 3{\cdot}\dot{3}d) = 36{\cdot}5 \text{ tons}$$

The maximum tension will be slightly reduced and the maximum thrust increased if part of the dead loads are taken as applied at the top chord panel points.

EXAMPLE 3.—The deck cantilever bridge girders in Fig. 220 are subject to a uniform dead load of $\frac{1}{2}$ ton per foot and a rolling load

equivalent to 2 tons per foot. Find the extreme stress in FG, HK, FK, and KG.

The main difference between this problem and the last lies in the fact that the open or lightly braced bays over the supports BC and DE

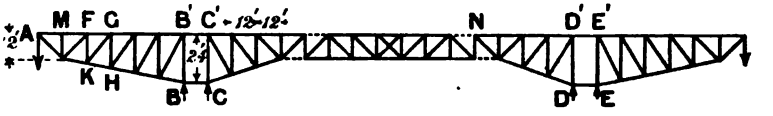



FIG. 220.

can transmit no shearing force, so that $M_B = M_C$. The panel length of 12 feet is taken as the unit of length, and the live panel load as 24 tons, and the dead panel load as 6 tons.

For dead load only, taking moments about B and ignoring the panel BC (*i.e.* for forces on the right hand using the point C instead of B),

$$R_A = \tfrac{1}{6}(6 \times 3 - 4 \times 2 - 3\tfrac{1}{2} \times 4)6 = -4 \text{ tons}$$

*i.e.* 4 tons downward. Effective reaction at C and D balance the load between them, hence

$$R_C = 7\tfrac{1}{2} \times 6 = 45 \text{ tons (upward)}$$
$$R_B = 13\tfrac{1}{2} \times 6 + 4 - 45 = 40 \text{ tons}$$

Also FK = 1·2 units, KG = 1·562 units. KH meets the top chord produced, 4 units to the left of A and 5·37 units from KG produced. Perpendicular distance of KH from G = 1·372 units.

*Member KH.*—The maximum tension will occur when A to B only is covered by the live load for which

$$R_A = \tfrac{1}{2} \times 6 \times 24 - 4 = 68 \text{ tons (upward)}$$
$$\text{maximum negative } M_G = 68 \times 3 - 3 \times 30 \times 1{\cdot}5 = 69$$
$$\text{maximum tension in KH} = \frac{69}{1{\cdot}372} = 50{\cdot}3 \text{ tons}$$

For live load on C' to N only

$$R_A = -\tfrac{1}{6}(4 \times 24 \times 2 + 3\tfrac{1}{2} \times 24 \times 4) - 4 = 92 \text{ tons downward}$$
$$\text{maximum positive } M_G = 92 \times 3 + 3 \times 6 \times 1{\cdot}5 = 303$$
$$\text{maximum thrust in KH} = \frac{303}{1{\cdot}372} = 221 \text{ tons}$$

*Member FG.*—Maximum thrust will occur when A to B' only is covered by the live load,

$$\text{maximum negative } M_F = 68 \times 2 - 2 \times 30 = 76$$
$$\text{maximum thrust in FG} = \frac{-M_F}{FK} = \frac{76}{1{\cdot}2} = 63{\cdot}3 \text{ tons}$$

Maximum tension will occur when live load extends from C' to N only,

$$\text{maximum positive } M_F = 92 \times 2 + 2 \times 6 = 196$$
$$\text{maximum tension in FG} = \frac{M_F}{FK} = \frac{196}{1{\cdot}2} = 163 \text{ tons}$$

*Member GK.*—For maximum thrust in KG all panel points from G to B′ must be loaded,

$$R_A = \tfrac{1}{6}(1 + 2 + 3)24 - 4 = 20 \text{ tons upward}$$

$$\text{maximum thrust} = \frac{1}{5{\cdot}37}\{(20 - 3) \times 4 - 6 \times 5 - 6 \times 6\} = 0{\cdot}38 \text{ ton}$$

For maximum tension in KG all panel points from C′ to N in addition to from F to A must carry live load; then,

$$R_A = \tfrac{1}{6}(4 + 5 - 4 \times 2 - 4 \times 3\tfrac{1}{2})24 - 4$$
$$= 56 \text{ tons downward}$$

$$\text{max. tension in KG} = \frac{1}{5{\cdot}37}\{(56+3)\times 4+30\times 5+30\times 6\} = 105\ 5 \text{ tons}$$

*Member KF.*—For minimum thrust or maximum tension, if any, the panel points G to B′ will carry live load,

$$R_A = \tfrac{1}{6}(1 + 2 + 3)24 - 4 = 20 \text{ tons}$$
$$\text{maximum tension in FK} = \tfrac{1}{6}\{(20-3)\times 4-6\times 5-6\times 6\} = 0{\cdot}33 \text{ ton}$$

For maximum thrust the panel points M and F and all from C′ to N will carry live load. Then

$$R_A = \tfrac{1}{6}\{5 + 4 - (4 \times 2 + 3\tfrac{1}{2} \times 4)\}24 - 4 = 56 \text{ tons downward.}$$
$$\text{maximum thrust in FK} = \tfrac{1}{6}\{(56+3)\times 4+30\times 5+30\times 6\} = 94{\cdot}3 \text{ tons}$$

EXAMPLE 4.—Determine the influence lines for stresses in HG and GE of Fig. 219, and hence with the travelling load given in Example 2, check the stresses in these members; use the dimensions in Example 2.

With unit load (1 ton) at E, $R_A = \frac{2}{3}$, $-M_E = \frac{2}{3} \times 2d = \frac{4}{3}d$, hence, thrust in HG $= -M_E \div 1{\cdot}245d = 1{\cdot}07$ ton. $E'E_1'$ is set off in Fig. 219 to represent 1·07 ton and (proportional) ordinates to the line $A'E_1'$ represent the stress in HG for corresponding positions of the unit load along AE. For positions beyond E there is a decrease at a uniform rate to zero at B, and at the same rate to M, hence the straight line $E'B'M_1'$. Beyond M there is again uniform increase of thrust (*i.e.* decrease of tension) in HG to zero at K, hence $M_1'$ is joined to K′ giving the complete influence line $A'E_1'B'M_1'K'D'$, which also represents the negative moment at E if $E'E_1'$ represents $\frac{4}{3}d$.

If the uniform live load $w = \frac{15}{d}$ tons per foot, the maximum live load thrust in HG $= w \times$ area $AE_1'B' = \frac{1}{2} \times \frac{15}{d} \times 1{\cdot}07 \times 6d = 48{\cdot}15$ tons. The dead load thrust is

$$(-1{\cdot}6 \times 2d - 10 \times d)\frac{1}{1{\cdot}245d} = -10{\cdot}6 \text{ tons}$$

Net maximum thrust in HG is therefore $48{\cdot}15 - 10{\cdot}6 = 37{\cdot}5$ tons, as in Example 2.

Similarly,

$$\text{live load tension} = 48{\cdot}15 \times \frac{\text{area } B'K'M_1}{\text{area } A'E_1'B'} = 48{\cdot}15 \times \frac{10}{6} = 80{\cdot}25 \text{ tons}$$
$$\text{net tension} = 80{\cdot}25 + 10{\cdot}6 = 91 \text{ tons nearly, as before}$$

With unit load at E,

$$\text{tension in EG} = (1 \times 4\tfrac{1}{3}d - \tfrac{2}{3} \times 2\tfrac{1}{3}d)\frac{1}{3{\cdot}68d}$$
$$= 0{\cdot}756 \text{ ton shown by } E''E_2'' \text{ (Fig. 219)}$$

With unit load at F, thrust in EG is

$$\tfrac{1}{2} \times 2\tfrac{1}{3}d \times \frac{1}{3{\cdot}68d} = 0{\cdot}317 \text{ ton represented by } F''F_2'' \text{ in Fig. 219}$$

Uniform rate of change from E to F gives $E_2''NF_2''$. Uniform rate of change from F to M gives $F_2''B''M_2''$ through zero at B'' to $M_2''M'' = \frac{4}{3} \times 0{\cdot}317 = 0{\cdot}423$ ton tension.

Uniform rate of decrease to zero at K gives $M_2''K''$.

For maximum tension in EG the load must be on A''N and B''K''.

$$E''N = \frac{0{\cdot}756}{0{\cdot}756 + 0{\cdot}317}d = 0{\cdot}705d, \quad A''N = 2{\cdot}705d$$

$$\text{maximum live load tension} = \frac{15}{d}(\text{area } A''E_2''N + \text{area } B''M_2''K'')$$

$$= \frac{15}{2d}(0{\cdot}756 \times 2{\cdot}705 + 0{\cdot}423 \times 10)d = 47{\cdot}0 \text{ tons}$$

$$\text{Dead load tension} = (1{\cdot}6 \times 2\tfrac{1}{2}d + 10 \times 3\tfrac{1}{2}d)\frac{1}{3{\cdot}68d} = 10{\cdot}0 \text{ tons}$$

Net tension = 47 + 10 = 57 tons as against 61·6 tons by the approximate method of Example 2, which gives an error on the safe side.

Similarly, since $NF'' = d - 0{\cdot}705d = 0{\cdot}295d$, maximum live load thrust $= \frac{15}{2} \times 3{\cdot}295 \times 0{\cdot}317 = 7{\cdot}83$ tons. Hence the minimum tension $= 10 - 7{\cdot}83 = 2{\cdot}17$ tons against 3·54 tons in Example 2, the latter being for a minimum stress in error on the safe side.

151. **Two-span or Centre-bearing Swingbridge.**—Swingbridges which turn on a central pivot or its equivalent, form when closed continuous girders of two spans such as illustrated in Fig. 221 The

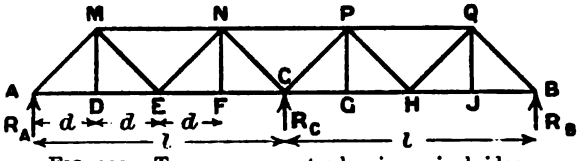



FIG. 221.—Two-span or centre-bearing swingbridge.

stresses in the members under given loads may easily be calculated by the methods already given for simple spans when the reactions have been calculated. Sometimes the ends A and B are lifted (or C depressed) when the bridge is closed, by such an estimated amount as will prevent any uplift under the live load. It is well to estimate for a lift varying from zero to one which will make the dead load upward reactions at A and B equal to those corresponding to a continuous girder. In the case of the latter full lift the reactions for both live and dead load are to be reckoned as for a continuous beam.

In other cases the ends A and B are secured by latches or pins capable of exerting upward or downward reactions due to various positions of the moving load, and the end reactions due to the dead load are zero. All stresses due to dead load are the same when the bridge is closed as when it is open and may conveniently be computed separately. Each span for the dead loads may be treated as a cantilever fixed at the pivot and free at the ends. In the case of unequal spans, in order to balance the long arm about the pivot, dead load will have to be added to the short arm.

The reactions, and consequently the stresses arising from the live load, will be those for a continuous girder of two spans. The continuous girder is a particular case of a statically intermediate structure, and the reactions, etc., for continuous beams of solid section have been dealt with in Chapter VIII. The reactions for a framed girder can only be reliably computed when the sections (or the relative sections) of the various members are known, *i.e.* when the girder has been designed. The principle of finding a reaction by equating the upward deflection due to the reaction to the downward deflection at the same point due to the load is valid, but the deflections are to be found (as in Art. 162) by the methods given in Arts. 155 to 157, which differ from the methods applicable to the deflections of a solid beam of uniform section in two important respects by taking account of (1) the variable cross-section of the girder, and (2) the shearing deflection or distortion arising from the strain of the web members, which is much greater than in a solid beam. The methods applicable to a solid beam may, however, be employed as a first approximation to design the members and then checked by the methods of Art. 155. In the case of two-span trusses of usual proportions this approximate method is found to be sufficiently accurate, and checking by the more exact method does not usually involve any important redesign. A simple example will illustrate the methods. The modifications for unequal arms do not involve any difference in principle.

EXAMPLE.—Find the stress in DE, EM, and MN (Fig. 221) if the web members are inclined at 45°, the dead load being $w_1$ per foot and the equivalent uniform live load being $w_2$ per foot, the dimensions being in feet.

*Dead Load Reactions.*—$R_A = 0$, $R_C = 2w_1 . l$, $R_B = 0$

*Live Load Reactions*—Panel load $= w_2 . d = \frac{w_2 l}{4}$

To find a general formula for the reactions, approximating by assuming the condition of a solid continuous beam, let a load W be at a distance $kl$ from A in the span AC. Following the method used in the example at the end of Art. 96, imagine the support C removed and equate the deflection at C due to W to $\frac{R_C(2l)^3}{48EI}$ (see (4) Art. 94). To find the deflection at C due to W write in (7) Art. 96, $kl$ for $b$, $(2 - k)l$ for $a$, and $l$ for $x$; this gives a deflection

$$\frac{Wl^3k(3 - k^2)}{12EI} = \frac{R_C l^3}{6EI}$$

hence $$R_C = \frac{W}{2} \cdot k(3 - k^2) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And by taking moments about B

$$R_A = \frac{W}{4}(4 - 5k + k^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and $$R_B = -\frac{W}{4} k(1 - k^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Writing in succession values of $k$ of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ for unit load at D, E, and F, and by symmetry for J, H, and G, we get then for W = 1 the following values:—

| Unit load at | D | E | F | G | H | J | D, E, and F | G, H, and J |
|---|---|---|---|---|---|---|---|---|
| $R_A$ . | 0·6914 | 0·4063 | 0·1680 | −0·0820 | −0·0938 | −0·0586 | 1·2657 | −0·2344 |
| $R_B$ . | −0·0586 | −0·0938 | −0·0820 | 0·1680 | 0·4063 | 0·6914 | −0·2344 | 1·2657 |

*Dead Load Stresses* (by method of sections)

$$M_E = 2w_1 d \times d \quad \text{tension in MN} = \frac{M_E}{d} = 2w_1 d^2 \div d = 2w_1 d$$

$$M_M = \tfrac{1}{2} w_1 d^2 \quad \text{thrust in DE} = \frac{M_M}{d} = 0{\cdot}5 w_1 d$$

shear in bay DE $= \frac{3}{2} w_1 d$ thrust in ME $= \sqrt{2} \times \frac{3}{2} w_1 d = 2{\cdot}121 w_1 d$

*Live Load Stresses.—Member MN.*—Tension $= M_E \div d$. Extreme live load stresses occur for maximum positive and negative values of $M_E$. Maximum tension occurs for CB fully loaded; from the table,

$$R_A = -0{\cdot}2344 w_2 d \quad M_E = +0{\cdot}2344 w_2 d \times 2d$$
$$\text{maximum tension in MN} = M_E \div d = 0{\cdot}4688 w_2 d$$

Maximum thrust occurs for AC fully loaded;[1] from the table

$$\text{maximum thrust} = -M_E \div d = 1{\cdot}2657 w_2 d \times 2d \div d - w_2 d = 1{\cdot}5314 w_2 d$$

*Member DE.*

$$\text{Thrust} = M_M \div d = M_D \div d$$
$$\text{maximum thrust (for CB fully loaded)} = +0{\cdot}2344 w_2 d \times d \div d = 0{\cdot}2344 w_2 . d$$
$$\text{maximum tension (for AC loaded)} = 1{\cdot}2657 w_2 d \times d \div d = 1{\cdot}2657 w_2 d$$

*Member ME.*

Maximum tension = $\sqrt{2}$ × maximum negative shear in DE

maximum thrust = $\sqrt{2}$ × maximum positive shearing force in DE

[1] Note that for more than five panels per span for the negative M at points near the central support (C) one or more panel points near the end support (A) may be unloaded, and for maximum positive M these points will be loaded (if the live load may be broken up).

For maximum thrust G, H, J, and D must be loaded, and employing conventional full panel loads

$$R_A = -0{\cdot}2344w_2d + 0{\cdot}6914w_2d = +0{\cdot}4570w_2d$$

$$\text{maximum thrust} = (w_2d - 0{\cdot}4570w_2d)\sqrt{2} = 0{\cdot}768w_2d$$

$$\text{maximum tension (E and F loaded)} = \sqrt{2}(0{\cdot}4063 + 0{\cdot}1680)w_2d$$
$$= 0{\cdot}813w_2 . d.$$

*Total Stresses.*—Adding the dead load stresses algebraically to the extreme live load stresses we get—

| Member. | Maximum tension. | Maximum thrust. |
|---|---|---|
| MN | $d(0{\cdot}4688w_2 + 2w_1)$ | $d(1{\cdot}5314w_2 - 2w_1)$ |
| DE | $d(1{\cdot}2657w_2 - 0{\cdot}5w_1)$ | $d(0{\cdot}2344w_2 + 0{\cdot}5w_1)$ |
| ME | $d(0{\cdot}813w_2 - 2{\cdot}121w_1)$ | $d(0{\cdot}768w_2 + 2{\cdot}121w_1)$ |

*Influence Lines for these Cases.*—From the fact that (1), (2), and (3) are higher than the first degree in $k$, it is evident that the influence lines for reaction, shear, and bending moment will be curved; consequently the influence line method is much less simple than for statically determinate girders, and is not here given.

**152. Rim-bearing Swingbridge.**—Swingbridges which turn on a ring of rollers form a more or less "continuous" girder over three spans, the central span being approximately the diameter of the roller

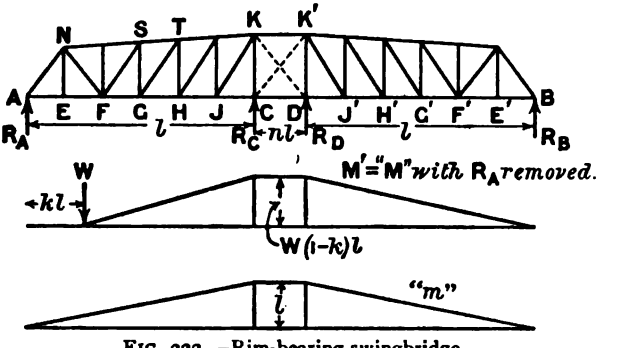



FIG. 222.—Rim-bearing swingbridge.

track. A simplified type is shown in Fig. 222. The determination of stresses in such a structure does not involve any fresh point of importance after the reactions have been found.

*Continuous Truss.*—If the girder in the central span CD (Fig. 222) is rigidly braced by very substantial web members the truss may be regarded as continuous. The dead loads and stresses may be treated as in the previous article if the free ends are simply latched when the bridge is closed. The live-load reactions may be found by assuming the girder to act as a solid beam, as in Chapter VIII., but the limita-

tions to the accuracy in such a case are considerable; the neglected effects of the distortion of the web members of the relatively short central span are really considerable, and the live load reactions found on such assumptions involve negative values at the central supports greater than the positive dead load values. It has been shown[1] by a numerical example that for a very short central span with simple bracing the more exact methods of Art. 162 give very different values. The method of finding the live-load reactions for any panel load, and hence for any combination of panel loads, is to take a single load distant, say, $kl$ from A, and (by Wilson's method, Art. 105) find $R_C$ and $R_D$ in terms of $k$ by equating to zero the deflections at C and D produced jointly by the load, and by $R_C$ and $R_D$ using the formulæ (7) and (10) of Art. 96. The reactions $R_B$ and $R_A$ are then found by simple statics. Tabulating reaction coefficients, as in the previous article for values of $k$ corresponding to each panel point, will give the reactions for any position of the moving load. Numerical examples of such bridges may be found in "Roofs and Bridges," by Merriman and Jacoby, and also, with corrections, by the more exact method in "Modern Framed Structures," where it is pointed out that the end reactions for a two-span bridge, see (2) and (3), Art. 151, may be applied to the three-span type, neglecting the short central span with fairly satisfactory results.

*Partially Continuous Truss.*—It is a common practice to make the central diagonal bracing very light, and quite inadequate to carry shear stresses which would arise from partial live loading. Such a construction may be regarded as only partially continuous. The shear stress in the span CD (Fig. 222) is nearly zero, and consequently the statical relations of the load and reactions are simplified. On account of the only partial continuity at C and D, the ordinary relations of bending moment slope and deflection are not applicable throughout the length of the beam. Applying the theory of solid beams under the assumed conditions to find the reaction $R_A$, say, due to a load W distant $kl$ from A, the upward deflection at A due to $R_A$ may be equated to the downward deflection at A, due to W when $R_A$ is removed. These deflections may most conveniently be calculated by the resilience method of Art. 108. The separate bending moments over the three ranges of length are shown on the simple bending-moment diagrams in Fig. 222, from which the ordinates M' due to W alone, or $m$ due to unit load at A, may be written for any section of the beam. Then from (10) and (11), Art. 108, assuming a constant section throughout

$$R_A \times \int m^2 dx = \int M' m dx \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Splitting these integrals into the three ranges over which they are continuous, and using convenient origins,

$$\int M' m dx = W \int_{kl}^{l} (x - kl) x dx + W(l - kl) l \times nl + W \int_{0}^{l} (1 - k) x^2 dx$$

$$= \frac{Wl^3}{6} \{4 - 5k + k^3 + 6n(1 - k)\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

[1] "Modern Framed Structures," by Johnson, Bryan, and Turneaure, Part II.

And $\int m^2 dx = \int_0^l x^2 dx + l^2 \times nl + \int_0^l x^2 dx = \frac{1}{3}l^3(2 + 3n)$ . . (3)

Hence $R_A = \int M' m dx \div \int m^2 dx = W(1 - k)\left\{1 - \frac{k(1+k)}{2(2+3n)}\right\}$ (4)

And since the shearing force in CD is zero

$$R_C = W - R_A = W\left\{k + \frac{k(1 - k^2)}{2(2 + 3n)}\right\} \quad . \quad . \quad . \quad (5)$$

And from moments about A and since $R_B = -R_D$

$$R_B = -R_D = Wk - R_C = -W\frac{k(1 - k^2)}{2(2 + 3n)} \quad . \quad . \quad (6)$$

It may be noted that as $n$ approaches zero the end reactions approach the values given in (2) and (3), Art. 151, for a two-span truss.

EXAMPLE.—Take the panel length in Fig. 222 to be 15 feet. NE = CD = 20 feet. KC = 25 feet. Live load 3000 lbs. per foot. Dead load 1000 lbs. per foot. Find the extreme stresses in ST, TG, and GH

$$n = \frac{20}{6 \times 15} = \frac{2}{9}$$

hence for unit load on various panel points, $kl$ from A (4) gives

$$R_A = (1 - k)\{1 - \tfrac{3}{16}k(1 + k)\}$$

and (6) gives

$$R_B = -\tfrac{3}{16}k(1 - k^2)$$

which gives $R_A$ for loads on the right-hand span; and taking successive values of $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, and $\frac{5}{6}$ for $k$, we get the following coefficients for the reaction at A.

| Unit load at . . | E | F | G | H | J | J' | H' | G' | F' | E' | EFGH and J | E'F'G'H' and J' |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| End reaction $R_A$ | 0·804 | 0·612 | 0·430 | 0·264 | 0·119 | −0·0477 | −0·0695 | −0·0703 | −0·0556 | −0·0304 | 2·229 | −0·2735 |

Calculated dimensions SG = 22 feet, perpendicular distance of ST from G = 21·7 feet, TH = 23 feet, ST meets HG 20 panel lengths to the left of E, *i.e.* 19 × 15 = 285 feet to the left of A; distance of this moment centre from GT produced = 276·8 feet.

*Live Load Stresses.*—Taking unit panel loads and then multiplying by 45,000 lbs., and assuming full panel loads for maximum stresses.

*Member GT.*—Maximum thrust for H and J loaded,

$$R_A = 0{\cdot}264 + 0{\cdot}119 = 0{\cdot}383$$

By moments about the intersection of ST and GH

$$\text{thrust} = \frac{1}{276{\cdot}8}(0{\cdot}383 \times 285)45{,}000 = 17{,}700 \text{ lbs.}$$

Maximum tension for E, F, G, and D to B loaded,

$$R_A = 0{\cdot}804 + 0{\cdot}612 + 0{\cdot}430 - 0{\cdot}2735 = 1{\cdot}5725$$

$$\text{tension} = \frac{1}{276{\cdot}8}(300+315+330-285\times 1{\cdot}5725)45{,}000 = 80{,}900 \text{ lbs.}$$

*Member ST.*—Maximum thrust, for A to C loaded,

$$R_A = 2{\cdot}229, \qquad -M_G = (2{\cdot}229\times 3 - 2 - 1)15\times 45{,}000$$

$$\text{thrust} = \frac{-M_G}{21{\cdot}7} = \frac{3{\cdot}687\times 15\times 45{,}000}{21{\cdot}7} = 114{,}500 \text{ lbs.}$$

Maximum tension, for D to B loaded,

$$R_A = -0{\cdot}2735$$

$$M_G = 0{\cdot}2735\times 3\times 15\times 45{,}000$$

$$\text{tension} = \frac{0{\cdot}2735\times 45\times 45{,}000}{21{\cdot}7} = 25{,}500 \text{ lbs.}$$

*Member GH.*—Maximum tension, for A to C loaded,

$$-M_T = (2{\cdot}229\times 4 - 3 - 2 - 1)15\times 45{,}000$$

$$\text{tension} = \frac{-M_T}{23} = \frac{2{\cdot}916\times 15\times 45{,}000}{23} = 85{,}600 \text{ lbs.}$$

Maximum thrust, for D to B loaded,

$$M_T = 0{\cdot}2735\times 4\times 15\times 45{,}000$$

$$\text{thrust} = \frac{1{\cdot}094\times 15\times 45{,}000}{23} = 32{,}100 \text{ lbs.}$$

*Dead Load Stresses.*—Take the loads as being all on the lower chords and the ends as not lifted when closed, then the dead load stresses are as for a cantilever. Half a panel load is taken at A. Full panel load = 15,000 lbs.

*Member GT.*—Tension $= \frac{1}{276{\cdot}8}(\frac{1}{2}\times 285 + 300 + 315 + 330)15{,}000$
$= 59{,}000$ lbs.

*Member ST.*—Tension $= \frac{M_G}{21{\cdot}7} = \frac{(45\times\frac{1}{2} + 30 + 15)15{,}000}{21{\cdot}7}$
$= 46{,}600$ lbs.

*Member GH.*—Thrust $= \frac{M_F}{23} = \frac{(60\times\frac{1}{2} + 45 + 30 + 15)15{,}000}{23}$
$= 78{,}300$ lbs.

*Total Extreme Stresses.*

Member GT—

Maximum tension = 80,900 + 59,000 = 139,900 lbs.
Minimum tension = 59,000 − 17,700 = 41,300 lbs.

Member ST—

Maximum thrust = 114,500 − 46,600 = 67,900 lbs.
Maximum tension = 25,500 + 46,600 = 72,100 lbs.

Member GH—

Maximum tension = 85,600 − 78,300 = 7,300 lbs.
Maximum thrust = 78,300 + 32,100 = 110,400 lbs.

**153. Stresses in Braced Piers.**—Fig. 223 represents a type of braced pier often used to support railway viaducts; a height of only two panels is shown, but more may be used. The panels are counter-braced by long diagonals which may be regarded as offering a negligible resistance to thrust, and the tensile stresses in any diagonals due to any cause may be safely computed as if the other diagonal intersecting it were absent, thus making the structure statically determinate. The stresses arise from the vertical loads carried by the pier, from the weight of the pier itself, and from horizontal wind pressure on the viaduct, the train, and the pier (the latter being taken as if applied at panel points), and occasionally from the centrifugal force of a train in the case of bridges built on curves. In Fig. 223 the horizontal loads are represented by $P_1$, $P_2$, and $P_3$. The stresses resulting from vertical and horizontal loads may conveniently be found separately.

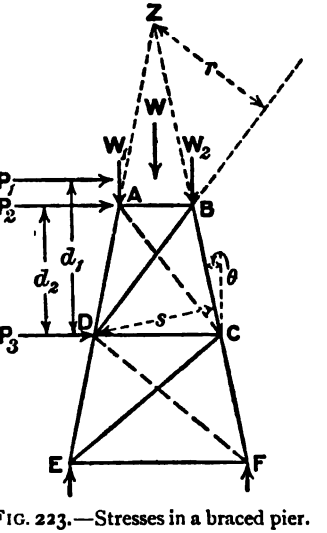


FIG. 223.—Stresses in a braced pier.

*Vertical Loads.*—The total top load W may be divided by the simple principles of statics into parts $W_1$ and $W_2$ at A and B. If W is symmetrically placed, $W_1$ and $W_2$ are equal, and the stresses in the braces AC and BD are both zero, while those in AD and BC are each $\frac{W}{2}$ secant $\theta$ and that in AB is $\frac{W}{2}\tan\theta$. If W is eccentrically placed towards A so that $W_1$ is greater than $W_2$, the additional stress may be found y taking a downward force $W_1 - \frac{1}{2}W$ at A and an upward force $\frac{W}{2} - W_2$ at B, and drawing a stress diagram after removing the member AC.

In any case the stresses may conveniently be determined by the method of sections, *e.g.* if W is eccentric by an amount $e$ towards A, taking a horizontal section through the top panel for member BD and using the moment centre Z

$$\text{tension BD} = \frac{1}{r}.W.e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And using the moment centre D

$$\text{thrust BC} = \frac{1}{s}(\text{moment of W about D}) \quad . \quad . \quad . \quad (2)$$

*Horizontal Loads.*—The stresses for horizontal loads are simply those for a braced cantilever; using the same methods

$$\text{thrust BC} = \frac{1}{s}(P_1 . d_1 + P_2 . d_2)$$

$$\text{tension BD} = \frac{1}{s}(\text{moments of } P_1 \text{ and } P_2 \text{ about Z})$$

**154. Space Frames.**—The polygon of forces as stated in Art. 44 is not limited to the case of coplanar forces but is applicable to general cases of concurrent forces in space by means of solid geometry. The stress diagram for structures not in one plane can be drawn and used by means of a plan and elevation, but for some simple structures a simple resolution of forces often reduces the problem to that of a plane frame.

*Shear Legs.*—For example, in the shear legs BD and BC (Fig. 224) stayed by the guy rope AB and carrying the load W the stresses are

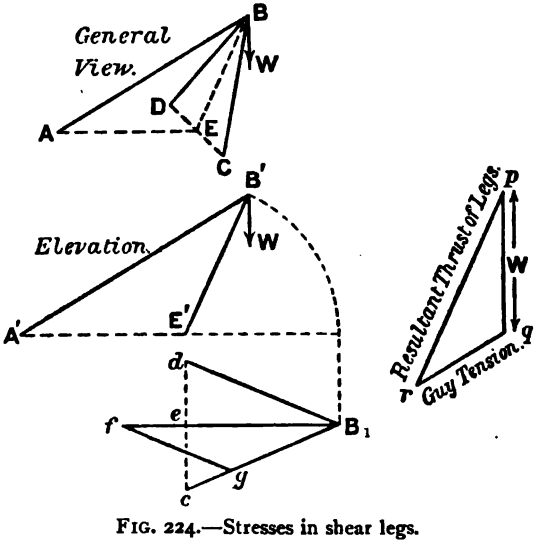



FIG. 224.—Stresses in shear legs.

readily found by replacing the two legs by an imaginary single or resultant leg BE in the plane of AB and W, which carries the resultant thrust of the two legs. The length of BE is found by setting off $ed$ = ED, $ec$ = EC, and striking arcs from $d$ and $c$ with radii BD and CB respectively to intersect in $B_1$, then $eB_1$ = EB = E'B'. A plane triangle of forces $pqr$ for the plane AEB from the elevation gives this resultant thrust of the two legs and the tension in the guy rope AB. It only remains to resolve the resultant thrust $pr$ (which is in the plane BDC of the legs) along BD and BC. The length $B_1f$ is set off equal to $pr$ and then $fg$ is drawn parallel to $dB_1$, then $gf$ represents the thrust in DB and $B_1g$ represents the thrust in BC.

*Derrick Crane.*—Fig. 225 shows a common derrick crane in which the vertical post QR is braced by two ties TQ and SQ, the pull in which is resisted at their feet by the thrust of RS and RT together with the dead weights placed at S and T to balance the load W. The stress in the tie rod QP and jib PR are found by the triangle of forces or by moments, but for the remainder of the structure the simplest plan is to replace QT and SQ by a single intermediate tie in the planes of QPR and TQS, the pull in which gives the resultant of the tensions in TQ and SQ. This resultant in the plane TQS may then be resolved into its components along QT and QS. The jib and tie may turn horizonally about RQ; so long as their common

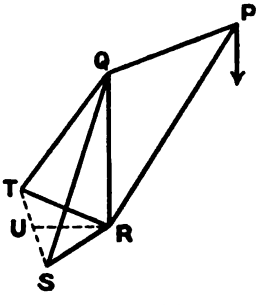



FIG. 225.—Derrick crane.

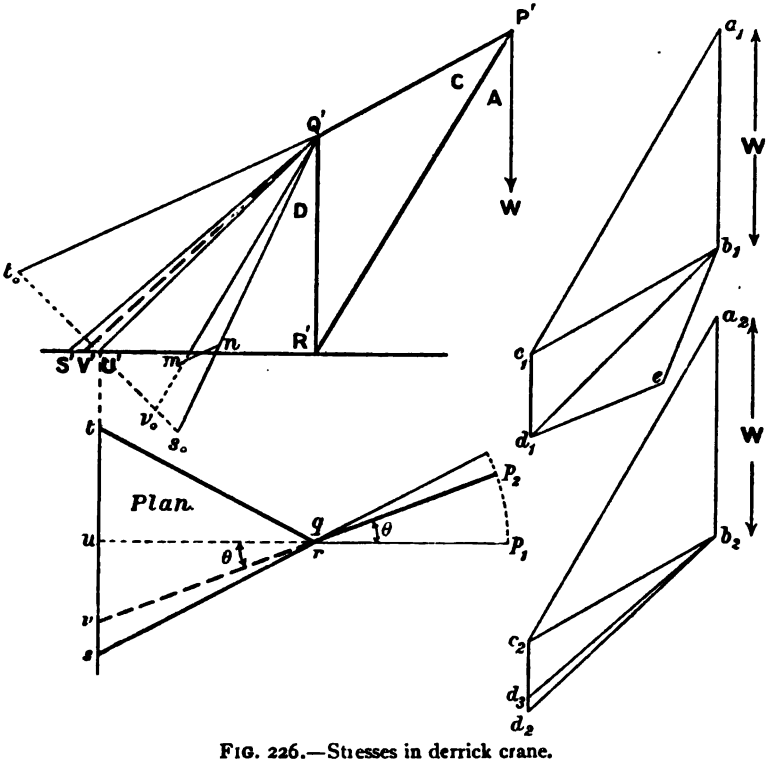



FIG. 226.—Stresses in derrick crane.

plane produced does not go outside the angle between the planes RQS and QTR no thrust will be imposed upon QS or QT.

Fig. 226 with letters corresponding to Fig. 225 shows the determination of the stresses for the central position, for the extreme position, and for an intermediate position. The plan shows the distance *st* between the feet and the elevation U'Q' gives the length of UQ, while

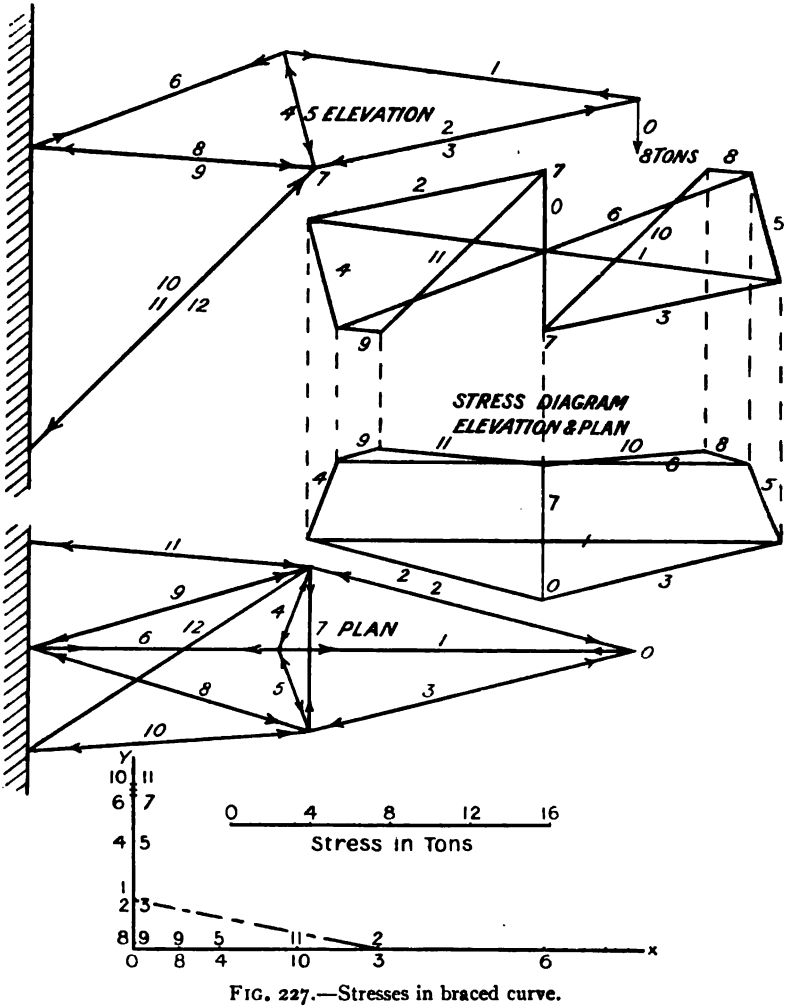



FIG. 227.—Stresses in braced curve.

the triangle $Q't_0s_0$ gives the real shape of the triangle QTS by making $U't_0 = ut$ and $U's_0 = us$. The stress diagram $a_1b_1c_1d_1$ is drawn for the central position, $b_1d_1$ being the pull in QU; this is resolved into components for the legs by drawing $b_1e$ parallel to $Q's_0$ and $ed_1$ parallel to $Q't_0$. For the position in which the vertical plane of P'Q'R' is inclined

$\theta$ to the central plane the stress diagram $a_2b_2c_2d_2$ is similarly drawn. The tensions in the legs are found by making $s_0v_0 = sv$ and $Q'm = b_2d_2$ and drawing $mn$ parallel to $t_0Q'$. Then $Q'n$ represents the tension in QS and $mn$ that in QT. For the extreme position when PQ, PR, and QS are in the same vertical plane, $a_2b_2c_2d_2$ is the stress diagram, $b_2d_2$ parallel to S'Q' is the tension in QS, and QT is not stressed. If $\theta$ is further increased QT suffers a thrust.

*Braced Crane.*—Fig. 227 shows a braced crane the members of which lie in various planes. The plan and elevation of the stress diagram are shown, corresponding lines in the space diagram being denoted by the same figure. The line 0 is drawn downwards in elevation to represent eight tons, and then lines 3, 1, and 2 are drawn, the line 1 being placed in the only symmetrical position with respect to line 0. 0, 3, 1, 2 give the order and direction of the forces at the crane head. The polygon for the joint where 1, 4, 5, and 6 meet is next drawn; the lines 4 and 5 being drawn in elevation of indefinite length, line 6 follows in the only symmetrical position, and the plan is projected, its sides being parallel to the plans of the members. It is always possible by the methods of solid geometry to complete the force polygon for a point if three sides are unknown in length but known in direction, the problem being simply to draw a line parallel to a given straight line to meet two given straight lines which is fully determinate unless all three lines lie in one plane. In this simple example the process is greatly facilitated by the symmetry of the frame. Any but the most symmetrical order of the lines in the stress diagram may involve duplication of certain sides in plan or elevation; in this case the (point) elevation of vector 7 is duplicated in elevation. The member 12 is not stress by the load; it is required for lateral stability to resist side forces such as wind. The feet of braces 10 and 11 being omitted, and any force in 10, 11, and 12 being treated as a reaction at the upper joint, the remainder of the structure is a perfect frame having 5 joints and 9 members in agreement with the formula $3n - 6$ given in Art. 124.

## EXAMPLES XIII.

1. The dimensions of a cantilever bridge, such as Fig. 218, being $AF_1 = 100$ ft., $F_1C = 50$ ft. $= F_2D$, $F_1F_2 = 200$ ft., $F_2B = 100$ ft., draw the bending-moment diagram and state the bending moment midway between A and $F_1$, at C, and midway between C and D, when the whole length carries a uniformly distributed load $w$ per ft.

2. Determine the extreme stresses in the top chord of the fourth bay from the end support of the anchor arm in Fig. 219 if the dead panel load is 5 tons and the live load is 15 tons per panel.

3. Which bays of the left anchor arm in Fig. 219 require counterbracing with the loads given in problem No. 2? Find the maximum and minimum tension in the diagonal of the fourth bay from the end support.

4. Which bays in the left anchor arm of Fig. 220 require counterbracing for a rolling load of 2 tons per ft. run if the dead load is $\frac{1}{2}$ ton per ft. and the diagonals are designed as ties only?

5. Find the maximum and minimum stresses in the diagonal of the fourth bay from the end support of the anchor arm of Fig. 220 with the loads given in problem No. 4.

6. Find the maximum and minimum stresses in GH in Fig. 220 with the loads given in problem No. 4.

7. Find the extreme stresses in the upper (loaded) chord of the fourth bay from the end support in the anchor arm of Fig. 220, the loads being as given in problem No. 4.

8. Find approximately the extreme stresses in the member EN (inclined 45°) of the centre bearing swingbridge, Fig. 221, the ends of which are simply supported when the bridge is closed, the dead load being $\frac{1}{2}$ ton per ft. and the live load 1·5 ton per ft. and the panel length being 15 ft.

9. Find the extreme stresses in the diagonal of the bay HJ, Fig. 222, with the loads and dimensions in the example at the end of Art. 152.

10. Find the thrust in each shear leg and in the guy rope for equal legs placed with their feet 10 ft. apart, the line joining them being 30 ft. from the foot of the guy rope on the same ground level. The guy rope from the ground to the head measures 50 ft., and 15 tons is suspended from the head with an overhang of 15 ft. from the base.

11. Solve problem No. 10 if the load hangs from a snatch block, one end of the chain going to the head and the other alongside the guy rope.

12. A derrick crane, Fig. 225, has the following dimensions, QR = 20 ft., UR = 20 ft., TS = 20 ft., the jib PR is inclined at 60° and the tie QR at 30°. A load of 1000 lbs. hangs from the crane head. Find the stresses in the members, (*a*) for the central position, (*b*) when the jib and tie lie in a plane inclined 20° to the central plane, (*c*) when the jib and tie lie in the plane RSQ. What is the minimum balance weight required at S in the last case?

13. A tripod is made up of poles AB, AC, and AD, each 9 ft. long, their feet forming a triangle BCD on horizontal ground such that BC = 8 ft., CD = 7 ft., BD = 9 ft. Find the thrust in each leg when 3000 lbs. hangs from A.

# CHAPTER XIV

## *DEFLECTION AND INDETERMINATE FRAMES*

**155. Deflection of Perfect Frames.**—When the various members of a perfect frame are subject to pull or thrust, strains of the individual members take place, causing rotation of the members about their pins and resulting in deflections at various parts of the structure. These deflections depend upon the strains of the members (which depend upon the loads and dimensions of the members) and also upon the geometrical form of the structure. The total deflection of a given point may depend upon the strains of all the members, and the effects of the several strains in producing deflection are separable.

*Notation.*—(Applicable to any perfect frame and illustrated in Figs. 228, 229, 230 and 231.) Denoting members by numbers 1, 2, 3, 4, etc., let $P_1$, $P_2$, $P_3$, $P_4$, etc., be the pulls in those members respectively due to any given system of loads. Let $e_1$, $e_2$, $e_3$, etc., be their respective stiffnesses or total pulls required per linear unit of stretch so that $e_1 = \frac{EA_1}{l_1}$, where $A_1$ = (constant) area of cross section and $l_1$ = length of member No. (1) and E = Young's modulus for the material, and let $k_1$, $k_2$, $k_3$, $k_4$, etc., be the respective pulls (positive or negative) produced in the respective members by a unit pull at a particular joint C in any specified direction in which the deflection $\Delta$ of that joint is required. Consider the effect of the stretch (positive or negative) of the member, (1), say (Figs. 228 to 231), if a force of 1 lb. alone is applied in the specified direction at the joint C (all other members being supposed quite rigid or non-elastic). Let $d_1$ be the deflection produced in that direction at C. Then the work done $\frac{1}{2} \times 1 \times d_1$ by the force of 1 lb. is equal to the internal or strain energy of member (1) since the other strains are zero. The strain energy, Art. 34, of (1) is half the product of the pull $k_1$ and the stretch $\frac{k_1}{e_1}$, hence

$$\tfrac{1}{2} \times 1 \times d_1 = \tfrac{1}{2} \cdot k_1 \cdot \frac{k_1}{e_1}$$

or $d = k_1 \times \frac{k_1}{e}$ or $k_1$ times the stretch of member (1) . . . . (1)

This is a geometrical relation, and it is evident that if a member (1) were to stretch any amount, $x$ say, from any cause, the consequent deflection of C would be $k_1 x$.

An important principle is thus established connecting the stretch (positive or negative) of any member and the consequent deflection of any joint in the structure, viz. *if unit pull at any joint in any specified direction would cause a pull k in any member, the deflection of that joint in the given direction due to any stretch of the member is k times the stretch of the member.* In other words, $k$ is the ratio of the resulting deflection at C to the stretch of member.

If any member, (1), say, sustains a pull $P_1$ (positive or negative) its stretch is $\frac{P_1}{e_1}$, and the consequent part of the deflection at C is

$$\delta_1 = k_1 \cdot \frac{P_1}{e_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, allowing for all the members of the structure

$$\Delta = \delta_1 + \delta_2 + \delta_3 + \text{etc.} = k_1 \cdot \frac{P_1}{e_1} + k_2 \frac{P_2}{e_2} + k_3 \cdot \frac{P_3}{e_3} + \text{etc.} \quad (3)$$

or $$\Delta = \Sigma\left(\frac{Pk}{e}\right) \text{ or } \Sigma\left(\frac{Pkl}{AE}\right) \text{ or } \Sigma\left(\frac{pkl}{E}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $p = \frac{P}{A}$ is the unit stress in the member. The portions of $\Delta$ resulting from the strains of different members of the structure are obviously separable, *e.g.* the deflection of a girder resulting from the elasticity of the web members may be separated from the deflection resulting from the strains of the chord members.

If the deflection in the direction of the load is required at a joint carrying a load W which is the sole load on the structure

$$P = kW \text{ and (4) becomes } \Delta = W\Sigma\left(\frac{k^2}{e}\right) \text{ or } W\Sigma\left(\frac{k^2 l}{AE}\right) \quad (5)$$

*Temperature Deflection.*—If any member extends due to increase of $t°$ in temperature its total stretch (see Art. 31) is $\alpha \cdot t \cdot l$, where $\alpha$ is the coefficient of expansion; the consequent deflection in the specified direction is $k$ times this, viz.

$$k \cdot \alpha \cdot t \cdot l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the deflection due to change of temperature of several members will be

$$\Delta = \Sigma(k\alpha t l) . \quad (7)$$

Reckoned on the whole of a structure this may frequently be zero for the particular direction required.

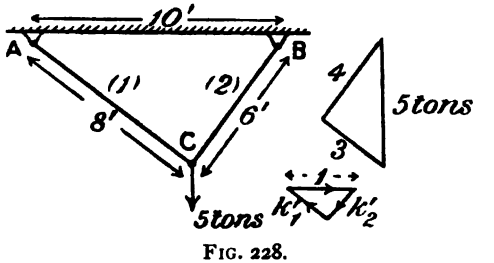



FIG. 228.

EXAMPLE 1.—Two pin-jointed rods AC and BC are hinged to a rigid ceiling at points A and B 10 feet apart. The piece AC is 8 feet long and forms a right angle

with BC; A, B, and C being in the same vertical plane. Find the elastic deflection of C vertically and horizontally when a load of 5 tons is suspended from that point, each rod being one square inch in cross-sectional area and E = 12,000 tons per sq. inch.

The frame and stress diagrams (triangles) are sketched in Fig. 228.

*Vertical Deflection.*—From the upper triangle of forces,

for AC, $P_1 = 3$ tons, and $k_1 = \frac{3}{5}$
for BC, $P_2 = 4$ tons, and $k_2 = \frac{4}{5}$

$$e_1 = \frac{A_1 E}{l_1} = \frac{1 \times 12{,}000}{8 \times 12} = 125 \text{ tons per inch deflection}$$

$$e_2 = \frac{12{,}000}{6 \times 12} = 166{\cdot}7 \text{ tons per inch}$$

Hence from (4)

$$\Delta = \sum\left(\frac{Pk}{e}\right) = (3 \times \tfrac{3}{5} \times \tfrac{1}{125} + 4 \times \tfrac{4}{5} \times \tfrac{6}{1000}) = \frac{72 + 96}{5000} = 0{\cdot}0336 \text{ inch}$$

*Horizontal Deflection.*—From the lower triangle (which is similar to ABC), for unit pull to the right $k_1' = \frac{4}{5}$, $k_2' = -\frac{3}{5}$ (the negative sign following from the fact that a pull to the right causes *thrust* in BC). Then from (4)

$$\Delta = (3 \times \tfrac{4}{5} \times \tfrac{1}{125} - 4 \times \tfrac{3}{5} \times \tfrac{6}{1000}) = \frac{96 - 72}{5000} = 0{\cdot}0048 \text{ inch}$$

The resultant deflection might be found by compounding by vector rules these two perpendicular component deflections.

EXAMPLE 2.—The jib of a crane is 15 feet long and is attached to a rigid support 7 feet vertically below the end of the tie rod, which is 10 feet long. If the jib and tie have uniform cross-sectional areas of 8 and 3 square inches respectively, find the elastic vertical and horizontal deflections of the crane head when a load of 5 tons is suspended from it. Take E for both as 13,000 tons per square inch.

The frame diagram and triangles of forces for 5 tons vertically and unit force horizontally are shown in Fig. 229.

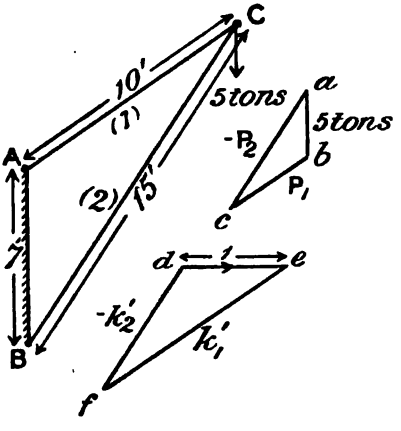



FIG. 229.—Deflection of jib crane.

*Vertical Deflection.*—From the triangle of forces *abc*, which is similar to ABC,

$$P_1 = 5 \times \tfrac{10}{7} = \tfrac{50}{7} \text{ tons.} \qquad p_1 = \frac{P_1}{A_1} = \frac{50}{7 \times 3} = \frac{50}{21} \text{ tons per sq. inch.}$$

$$P_2 = -5 \times \tfrac{15}{7} = -\tfrac{75}{7} \text{ tons (a thrust).} \qquad p_2 = \frac{P_2}{A_2} = -\frac{75}{7 \times 8} = -\frac{75}{56}$$

and writing unity instead of 5 and $k_1 = \frac{10}{7}$, $k_2 = -\frac{15}{7}$. Hence from (4)

$$\Delta = \Sigma\left(\frac{pkl}{E}\right) = \frac{1}{E}\Sigma(pkl) = \frac{1}{13000}\left(\frac{50}{91} \times \frac{10}{7} \times 120 + \frac{75}{56} \times \frac{15}{7} \times \frac{180}{1}\right)$$
$$= 0{\cdot}0712 \text{ inch.}$$

*Horizontal Deflection.*—In the triangle *def* the angle $\hat{dfe}$ = AĈB, and

$$\cos \text{ACB} = \frac{100 + 225 - 49}{2 \times 10 \times 15} = \frac{23}{25}, \text{ therefore sin } \hat{dfe} = \frac{4\sqrt{6}}{25}.$$

Also sin $\hat{def}$ = − cos CÂB = $\frac{225 - 100 - 49}{2 \times 10 \times 7} = \frac{19}{35}$, and sin $\hat{fde}$ = cos AB̂C = $\frac{49 + 225 - 100}{2 \times 7 \times 15} = \frac{29}{35}$.

Hence $k_1' = \frac{29}{35} \times \frac{25}{4\sqrt{6}} = 2{\cdot}115$. $k_2' = -\frac{19}{35} \times \frac{25}{4\sqrt{6}} = -1{\cdot}383$.

$$\Delta = \frac{1}{E}\Sigma(pk'l) = \frac{1}{13000}\left(\frac{50}{91} \times 2{\cdot}115 \times 120 + \frac{75}{56} \times 1{\cdot}383 \times 180\right)$$
$$= 0{\cdot}0720 \text{ inch.}$$

EXAMPLE 3.—The cantilever shown in Fig. 230 carries various loads at its joints, and the sections of the members are so proportioned that the unit stress in each tie rod is 5 tons per square inch and in each strut is 2 tons per square inch. The length of each member is 5 feet, except EB, which is 2·5 feet. Estimate the vertical deflections of the points C and D taking E = 12,500 tons per square inch.

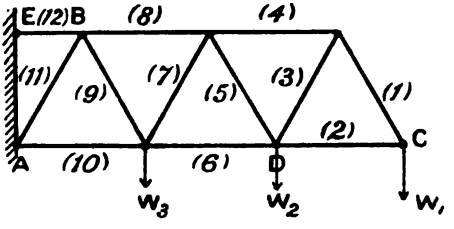



FIG. 230.—Deflection of braced cantilever.

*Deflection at* C. — The values of $k$ for the various members are very simply found by the method of sections; the various parts of the products $\Sigma(pkl)$ are tabulated below in inch units.

| Member. | $p$ | $k$ | $\frac{1}{13}pkl \times \sqrt{3}$ |
|---|---|---|---|
| 1 | +5 | $+\frac{2}{\sqrt{3}}$ | 50 |
| 2 | −2 | $-\frac{1}{\sqrt{3}}$ | 10 |
| 3 | −2 | $-\frac{2}{\sqrt{3}}$ | 20 |
| 4 | +5 | $+\frac{2}{\sqrt{3}}$ | 50 |
| 5 | +5 | $+\frac{2}{\sqrt{3}}$ | 50 |

| Member. | $p$ | $k$ | $\frac{1}{12}pkl \times \sqrt{3}$ |
|---|---|---|---|
| 6 | −2 | $-\frac{3}{\sqrt{3}}$ | 30 |
| 7 | −2 | $-\frac{2}{\sqrt{3}}$ | 20 |
| 8 | +5 | $+\frac{4}{\sqrt{3}}$ | 100 |
| 9 | +5 | $+\frac{2}{\sqrt{3}}$ | 50 |
| 10 | −2 | $-\frac{5}{\sqrt{3}}$ | 50 |
| 11 | −2 | $-\frac{2}{\sqrt{3}}$ | 20 |
| 12 | +5 | $+\frac{6}{\sqrt{3}}$ | 75 |
| | Total . . | $\frac{\sqrt{3}}{12}\Sigma(pkl) =$ | 525 |

Or $\Sigma(pkl) = \frac{12 \times 525}{\sqrt{3}}$, hence $\Delta = \frac{1}{E}\Sigma(pkl) = \frac{12 \times 525}{12{,}500 \times \sqrt{3}} = 0{\cdot}291$ inch.

*Deflection at D.*—For members 1, 2, 3, and 4, $k = 0$.

| Member . . . . | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|
| $p$ . . . . . | +5 | −2 | −2 | +5 | +5 | −2 | −2 | +5 |
| $k$ . . . . . | $+\frac{2}{\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ | $-\frac{2}{\sqrt{3}}$ | $+\frac{2}{\sqrt{3}}$ | $+\frac{2}{\sqrt{3}}$ | $-\frac{3}{\sqrt{3}}$ | $-\frac{2}{\sqrt{3}}$ | $+\frac{4}{\sqrt{3}}$ |
| $\frac{\sqrt{3}}{12}pkl$ . . . | 50 | 10 | 20 | 50 | 50 | 30 | 20 | 50 |

$\frac{\sqrt{3}}{12}\Sigma(pkl) = 280$, hence $\Delta = \frac{12 \times 280}{12{,}500 \times \sqrt{3}} = 0{\cdot}155$ inch.

EXAMPLE 4.—A Pratt truss (Fig. 231) has 6 bays each 6 feet long and 8 feet high. Taking the stress in each tie as 5 tons per square inch and in each vertical strut as 2 tons per square inch and in the end post and top chord as 3 tons per square inch, estimate the elastic deflection midway between the supports, taking E = 12,500 tons per square inch.

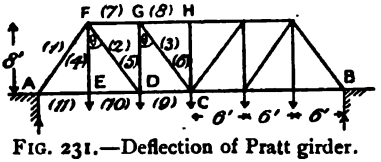



FIG. 231.—Deflection of Pratt girder.

If $\theta$ = inclination of the diagonals to the vertical, $\tan\theta = \frac{6}{8} = \frac{3}{4}$, $\sec\theta = \frac{5}{4}$. Taking half the structure and reference numbers given in Fig. 231, and finding the values of $k$ for deflection at C by the method of sections, we get

| Member. | $k$ | $p$ | $\frac{1}{12}l$ | $\frac{1}{12}pkl$ |
|---|---|---|---|---|
| 1 | $-\frac{5}{8}$ | −3 | 10 | 18·75 |
| 2 | $\frac{5}{8}$ | +5 | 10 | 31·25 |
| 3 | $\frac{5}{8}$ | +5 | 10 | 31·25 |
| 4 | 0 | +5 | 8 | 0 |
| 5 | $-\frac{1}{2}$ | −2 | 8 | 8·0 |
| 7 | $-\frac{3}{4}$ | −3 | 6 | 13·5 |
| 8 | $-\frac{9}{8}$ | −3 | 6 | 20·25 |
| 9 | $\frac{3}{4}$ | +5 | 6 | 22·5 |
| 10 | $\frac{9}{8}$ | +5 | 6 | 11·25 |
| 11 | $\frac{9}{8}$ | +5 | 6 | 11·25 |
| | | Total . . . . . | | 168·00 |
| | | For other half . . . . | | 168·00 |
| | | For the structure . . . | | 336·00 |

For member (6) $k = 0$, hence there is no further addition, and

$$\Sigma(pkl) = 12 \times 336 = 4032, \text{ and } \Delta = \frac{1}{E}\Sigma(pkl) = \frac{4032}{12{,}500} = 0{\cdot}323 \text{ inch.}$$

A glance at the last column of the above table shows how large a proportion of the total deflection results from strain of the web members. A fraction $\frac{2 \times 31{\cdot}25}{336} = 0{\cdot}186$ of the whole deflection results from stretch of the members (2) and (3) alone or twice this fraction from the four diagonal ties.

If it were desired to find the deflection of the joint D, say under the same loading, new values of $k$ would have to be calculated which will not be symmetrical for the two halves of the girder. The value for the member (3) will be negative, which with a positive value of $p$ will give a negative product, *i.e.* the effect of the stretching of this member is to diminish the deflection at D.

EXAMPLE 5.—Find the deflection of point C in Fig. 231 if there is a load of 10 tons at each joint of the lower chord, the sectional areas of the members being as given in the following table.

The values of $k$ and $l$ are as given in Example 4. The values of P are readily calculated by the method of sections.

| Member. | P(tons). | $k$ | $\frac{l}{12}$ = (feet). | A = sq. ins. | $\frac{1}{12}\frac{Pkl}{A}$ | Stretch $= \frac{Pl}{AE}$ (inches). |
|---|---|---|---|---|---|---|
| 1 | −31·25 | $-\frac{8}{8}$ | 10 | 10 | 19·5 | −0·0300 |
| 2 | +18·75 | $+\frac{8}{8}$ | 10 | 3 | 39·1 | +0·0600 |
| 3 | + 6·25 | $+\frac{8}{8}$ | 10 | 2 | 19·5 | +0·0300 |
| 4 | +10·0 | 0 | 8 | 2 | 0 | +0·0384 |
| 5 | − 5·0 | $-\frac{4}{8}$ | 8 | 1·5 | 13·3 | −0·0256 |
| 7 | −30·0 | $-\frac{6}{8}$ | 6 | 10 | 13·5 | −0·01728 |
| 8 | −33·75 | $-\frac{6}{8}$ | 6 | 10 | 22·8 | −0·01945 |
| 9 | +30·0 | $+\frac{6}{8}$ | 6 | 5 | 27·0 | +0·03458 |
| 10 | +18·75 | $+\frac{6}{8}$ | 6 | 4 | 10·5 | +0·0270 |
| 11 | +18·75 | $+\frac{6}{8}$ | 6 | 4 | 10·5 | +0·0270 |
| | | | | Total . . . | 175·7 | |

For the whole structure since P = 0 and $k$ = 0 for member 6

$\frac{1}{12}\sum\left(\frac{Pkl}{A}\right) = 175{\cdot}7 \times 2 = 351{\cdot}4$, hence $\Delta = \frac{12 \times 351{\cdot}4}{12{,}500} = 0{\cdot}337$ inch.

The last column of the table refers to the graphical solution given in Art. 157.

EXAMPLE 6.—Find the central vertical deflection for the structure of Example 5 due to the upper chord and end posts rising 10° F. above the remainder of the girder. Coefficient of expansion 0·0000062 per degree F.

Using the previous figure and expansion of each heated member by 0·0000062 of its length, we get

| Member. | Expansion in inches = 0·0000062 × $l$ | $k$ | $katl$ inches. |
|---|---|---|---|
| 1 | 0·00744 | $-\frac{8}{8}$ | −0·00465 |
| 7 | 0·004464 | $-\frac{6}{8}$ | −0·00335 |
| 8 | 0·004464 | $-\frac{6}{8}$ | −0·00502 |
| | | | $\Sigma(katl) = -0{\cdot}01302$ |

and allowing for the whole structure the deflection is $-2 \times 0{\cdot}013 = -0{\cdot}026$ inch, *i.e.* 0·026 inch *upwards*.

**156. Deflection from the Principle of Work.**—The formulæ of the previous article were based upon a simple geometrical principle which was established from an application of the equation of external work to internal work or resilience of a member of the structure. They may be based directly upon this principle of work; for, using the notation of Art. 155, the total resilience of member (1) is

$$\tfrac{1}{2}P_1\left(\frac{P_1}{e_1}\right)$$

Of this, the work due to a force of 1 lb. in the specified direction is

$$\tfrac{1}{2}P_1\left(\frac{k_1}{c_1}\right)$$

Hence,

$$\tfrac{1}{2}\times 1\times\Delta=\tfrac{1}{2}\Sigma\left(\frac{Pk}{c}\right),\text{ or }\Delta=\Sigma\left(\frac{Pk}{c}\right),\text{ or }\Sigma\left(\frac{Pkl}{AE}\right),\text{ or }\Sigma\left(\frac{pkl}{E}\right)\quad(1)$$

**157. Geometrical Method of Determining Deflections.**—It is easy to obtain the movement of one point of a perfect frame relative to another point by calculating the amount of stretching or shortening of the members of the frame. A simple example will illustrate the method, and for this purpose the problem given in Example 2, Art. 155, Fig. 229, may be chosen.

The unit stress in the tie rod was $\frac{50}{21}$ tons per square inch, hence the stretch is

$$\frac{50}{21}\times\frac{120}{13{,}000}=0{\cdot}0220\text{ inch.}$$

The unit stress in the jib was $\frac{75}{56}$ tons per square inch, hence the shortening is

$$\frac{75}{56}\times\frac{15\times 12}{13{,}000}=0{\cdot}01854\text{ inch.}$$

If we take the jib (Fig. 232) as shortened to $BC_2$ and the tie rod as extended to $AC_1$, then by striking arcs from centres A and B with radii $AC_1$ and $BC_2$ respectively the intersection gives C′ the new position of C, and the vertical and horizontal projections of CC′ give the vertical and horizontal deflections of C. But the alterations of length $CC_1$ and $CC_2$ are too small to be shown on the same diagram as ABC. For very small changes in length the angles $C\hat{C}_1C'$ and $C\hat{C}_2C'$ are right angles. We therefore set off the figure $CC_1C'C_2$ only, to a very much larger scale, as shown at $Pc_1c'c_2$, in which Pc′ gives the actual deflection of C, while Pn gives the vertical and nc′ the horizontal deflection.

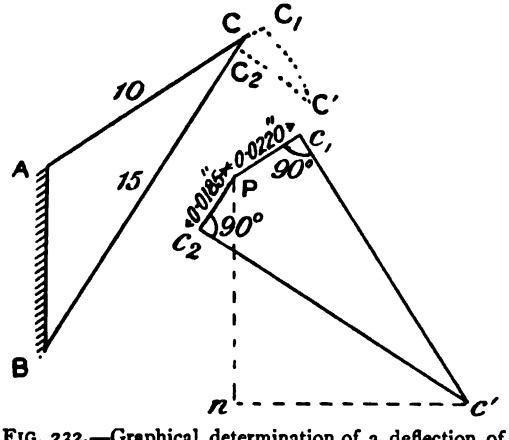



FIG. 232.—Graphical determination of a deflection of a jib crane.

The principle is further exemplified with suitable notation for the diagram in Fig. 233, a simple triangular roof truss ABC, in which A is hinged to a fixed point and B is free to slide horizontally; $ab$ = stretch of AB, $ac_2$ = compression of AC, $bc_1$ = compression of BC. Then $b$ and

$c$ give the new position of B and C. The deflection $ac$ of C may be split into horizontal and vertical components by projection.

EXAMPLE.—Find the deflection of point C (Fig. 231) under the loads in Example 5, Art. 155.

| Member . . . . . | CH | HG | GC | CD | GD | FG | FD | ED | FE | AF | AE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Stretch $\frac{Pl}{AE}$ . . . | ·0 | −0·01945 | 0·03 | 0·03458 | −0·0256 | −0·01728 | 0·06 | 0·0266 | 0·0384 | −0·03 | 0·0266 |
| Shown in Fig. 234 by the line . . | $ch(=o)$ | $hg_1$ | $cg_2$ | $cd_1$ | $gd_2$ | $g/_1$ | $d/_2$ | $de_1$ | $fe_2$ | $fa_2$ | $ea_1$ |

The extensions $\frac{Pl}{AE}$ are calculated from the stresses and sectional areas given in the above table of the example quoted, and are set off to scale in Fig. 234, starting from point C, which may be taken as fixed.

The vertical deflection of A *above* C equals the vertical projection of the line $ca$. If A and B remain at the same level this also gives the deflection of C below AB. The vertical deflections of E and D are given by the projections of $ae$ and $ad$.

In case of unsymmetrical loading, if HC is supposed to remain fixed the upward deflections of A and B can be found. A small rotation about C, the

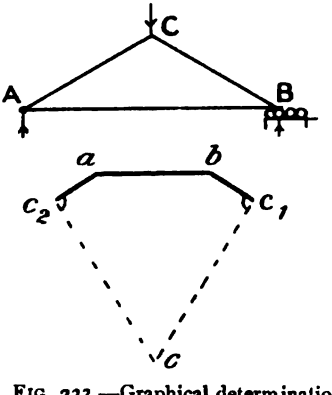
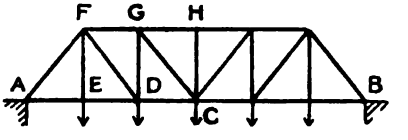



FIG. 233.—Graphical determination of roof deflections.



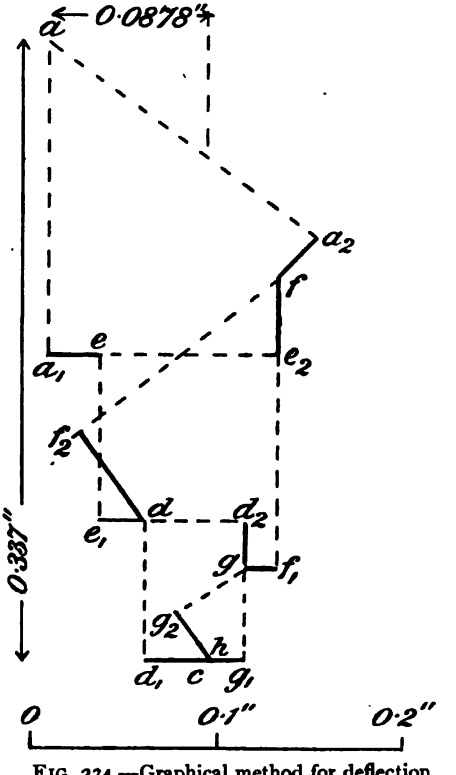




FIG. 234.—Graphical method for deflection of a Pratt truss.

amount of which can easily be calculated, will then bring A and B into a horizontal line. The correction of the deflections of other points can easily be estimated to allow for this rotation.

**158. Statically Indeterminate Structures.**—When a framed structure has more members (see Art. 124) than are required for a perfect frame, the distribution of internal stress depends upon the relative stiffness of the various members. The methods of finding the stress in frames having one or more redundant members are based upon the same principles as those applicable to the closely analogous problems of statically indeterminate systems already dealt with, such as the weight supported by two or more forces (Example 2, Art. 9) and the continuous beam resting on more than two supports (Art. 94 and Chap. VIII.). There are three ways of approaching a solution to such problems, and they may be called: (1) The Method of Deformations; (2) The Principle of Minimum Resilience; (3) The Principle of Work. The three ways lead of course to the same results. Before proceeding to the general methods, it may be well to illustrate the principles by a simple example.

EXAMPLE 1.—A weight W is held in equilibrium by two vertical elastic supports $a$ and $b$, either struts or ties (such as two parallel wires). The elastic stiffness or force per unit of deformation of the first is $e_a$ and that of the second is $e_b$. Find the proportion of the load borne by each support.

Let F be the load carried by the first support $a$.

(1) *Method of Deformation.*—Equating the deformation or alteration in length of the two supports

$$\frac{F}{e_a} = \frac{W - F}{e_b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a simple equation for F giving $F = W\frac{e_a}{e_a + e_b}$.

(2) *Principle of Minimum Resilience.*—The resilience U (Art. 34)

$$= \tfrac{1}{2}\left\{F \times \frac{F}{e_a} + (W - F)\frac{W - F}{e_b}\right\}$$

And if F is such as to make U a minimum,

$$\frac{dU}{dF} = \frac{F}{e_a} - \frac{W - F}{e_b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is evidently identical with (1).

(3) *Principle of Work.*—

$$\text{Resilience (U)} = \text{external work}$$

$$\tfrac{1}{2}\left\{\frac{F^2}{e_a} + \frac{(W - F)^2}{e_b}\right\} = \tfrac{1}{2}W \times \frac{F}{e_a}$$

which when simplified reduces to equation (1).

EXAMPLE 2.—In Chap. VII. the loads on props partially supporting beams were calculated by the method of deformations, *i.e.* by equating the upward deflection caused by the prop to the downward deflection

caused by the load minus the deflection of the prop (if any). The reader will find it instructive to solve for himself the same problems by writing U the resilience in terms of the prop reaction P (by the method given in Art. 108) and then applying the Principle of Minimum Resilience and the Principle of Work.

EXAMPLE 3.—If a vertical bar DC hinged to C and to the ceiling is added to the system in Example 1, Art. 155, find the stress in each bar, all three being 1 square inch in section.

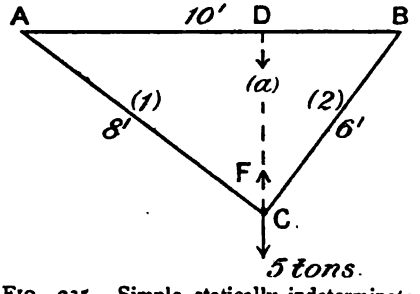


FIG. 235.—Simple statically indeterminate frame.

The system is shown in Fig. 235, the tension in DC being shown as forces F at D and C. The stress in member (1) due to the combined action of the 5 tons load and the tension F is $P_1 = \frac{3}{5}(5-F)$, and in member (2) is $P_2 = \frac{4}{5}(5-F)$, hence the vertical deflection of C or stretch of DC in inches is

$$\Delta = \frac{1}{E}\sum\left(\frac{Pkl}{A}\right) = \frac{1}{E}\{\tfrac{3}{5}(5-F)\tfrac{3}{5}\times 96 + \tfrac{4}{5}(5-F)\tfrac{4}{5}\times 72\} = \frac{288\times 7}{25E}(5-F)$$

But the length DC $= \frac{8\times 6}{10} = 4{\cdot}8$ feet $= 57{\cdot}6$ inches, and the stretch of DC is therefore $\frac{F}{1\times E}\times 57{\cdot}6$ inches. Hence equating this to the deflection of C

$$57{\cdot}6F = \frac{288\times 7}{25}(5-F) \quad \text{and} \quad F = \tfrac{35}{12} = 2\tfrac{11}{12} \text{ tons}$$

$5 - F = 2\frac{1}{12}$ tons

The tension in AC, $P_1 = \frac{3}{5}\times\frac{25}{12} = 1{\cdot}25$ ton

" " " BC, $P_2 = \frac{4}{5}\times\frac{25}{12} = 1{\cdot}6$ tons

The deflection of C would evidently be $\frac{35}{12} \div 5$ times, or $\frac{5}{12}$ of that found in Example 1, Art. 155.

**159. Method of Deformations applied to Redundant Frame Members.**—*Notation.*—Let the unknown tensile stresses in any superfluous members *a*, *b*, *c*, etc., be $F_a$, $F_b$, $F_c$, etc. The number of redundant members is the number in excess of $2n-3$ (see Art. 124), and the choice as to which are considered redundant is largely arbitrary.

The tensile stress in any member, number (1) say, is made up of a number of terms, being

$$P_1 = R_1 + {}_ak_1F_a + {}_bk_1F_b + {}_ck_1F_c +, \text{ etc.} \quad . \quad . \quad (1)$$

and in member (2) being

$$P_2 = R_2 + {}_ak_2F_a + {}_bk_2F_b + {}_ck_2F_c +, \text{ etc.} \quad . \quad . \quad (2)$$

where $R_1$, $R_2$, $R_3$, etc., are the tensile stresses in the members arising from the loads alone with the redundant members removed, and the

terms $k$. F are the tensions arising from the forces exerted by the various redundant members each acting alone with all the other redundant members removed. Let $e_1, e_2, e_3$, etc., be the tensile stiffnesses of the respective members. The tensions $P_1$, $P_2$, $P_3$, etc., may be found in terms of the known external loads and the unknown forces $F_a$, $F_b$, $F_c$, etc., by the ordinary rules dealt with in Chap. XI., either graphically or algebraically. The constants ${}_ak_1$, ${}_bk_1$, ${}_ck_1$, etc. (which may be positive or negative) are, as already used in Art. 155, numerically equal to the stress in lbs. produced in member (1) by pairs of forces of 1 lb. each pulling inwards at the pins at the ends of the redundant members $a$, $b$, $c$, etc., respectively. (Note that the suffix denotes the member and the prefix indicates the particular redundant member supposed replaced by inward forces at its ends.)

*Single Redundant Member.*—If there is but one redundant member, $a$, say, in a frame (Fig. 235 may be referred to in order to fix the ideas) the deflection of one end towards the other (taken as fixed) is by Art. 154 (4)

$$\Delta = \sum\left(\frac{P \cdot {}_ak}{e}\right) = \frac{P_1 \cdot {}_ak_1}{e_1} + \frac{P_2 \cdot {}_ak_2}{e_2} + \frac{P_3 \cdot {}_ak_3}{e_3} +, \text{ etc.} \quad . \quad . \quad . \quad (3)$$

$$\Delta = \frac{{}_ak_1}{e_1}(R_1 + {}_ak_1 \cdot F_a) + \frac{{}_ak_2}{e_2}(R_2 + {}_ak_2F_a) + \frac{{}_ak_3}{e_3}(R_3 + {}_ak_3 \cdot F_a) +, \text{ etc.}$$

$$\text{or } \Delta = \sum\left(\frac{{}_ak \cdot R}{e}\right) + F_a\sum\left(\frac{{}_ak^2}{e}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which represents the compression or *shortening* of member $a$. But due to the tension $F_a$ the member $a$ *extends* by an amount of $\frac{F_a}{e_a}$, where $e_a$ is the stiffness of member $a$, or $\Delta = -\frac{F_a}{e_a}$, hence

$$-\frac{F_a}{e_a} = \sum\left(\frac{{}_ak \cdot R}{e}\right) + F_a\sum\left(\frac{{}_ak^2}{e}\right) . \quad . \quad . \quad . \quad (5)$$

a simple equation for $F_a$, from which

$$F_a = -\frac{\sum\left(\frac{{}_ak \cdot R}{e}\right)}{\frac{1}{e_a} + \sum\left(\frac{{}_ak^2}{e}\right)} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the summations excluding the member $a$. In this case where there is only one redundant member the prefix $a$ to the constants $k$ may be omitted. Also the first term of the denominator may be omitted if the member $a$ is included in the summation of the second term, ${}_ak_a$ being unity.

*Any Number of Redundant Members.*—For any number, $n$, say, of redundant members $a$, $b$, $c$, etc., the equation arising from the deformation of the member $a$ is

$$\Delta = \sum\left(\frac{Pk}{e}\right) = -\frac{F_a}{e_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or written more fully,

$$-\frac{F_a}{e_a} = \frac{ak_1}{e_1}(R_1 + ak_1 \,.\, F_a + bk_1 \,.\, F_b + ck_1 \,.\, F_c +, \text{ etc.})$$
$$+ \frac{ak_2}{e_2}(R_2 + ak_2 F_a + bk_2 \,.\, F_b +, \text{ etc.})$$
$$+ \frac{ak_3}{e_3}(R_3 + ak_3 \,.\, F_a + bk_3 F_b + ck_3 \,.\, F_c +, \text{etc.}) \quad (8)$$

There are altogether $n$ similar simultaneous simple equations, one for each redundant member and each containing the $n$ unknown quantities $F_a$, $F_b$, $F_c$, etc., and from these equations each may be found. It may be noted that the solution of the case involving several redundant members is closely analogous to Wilson's solution (Art. 105) for continuous supported beams while that for a single redundant member corresponds to the case of a beam with a single prop.

EXAMPLE.—The crossed lattice girder shown in Fig. 236 is loaded as shown; the diagonals are inclined at 45°. The ratios of length to

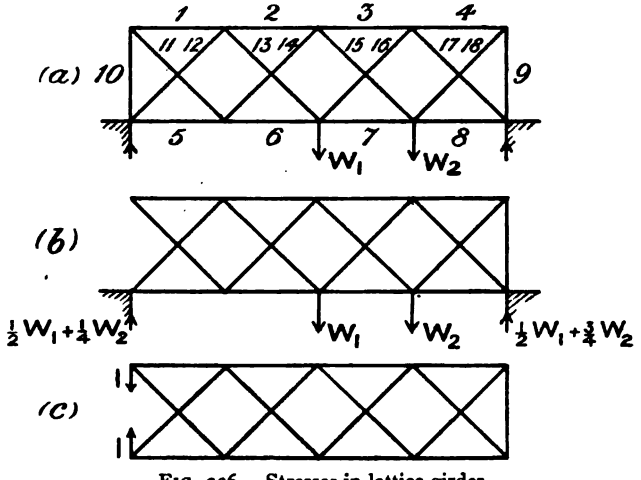



FIG. 236.—Stresses in lattice girder.

areas of cross-section in inch units is 20 for each diagonal number, 6 for each top chord member, 8 for each lower boom member and 10 for the two vertical members. Determine the stresses in all the members.

Select member 10 as the redundant one. For the diagonals $e = \frac{EA}{l} = \frac{E}{20}$, or $\frac{1}{e} = \frac{20}{E}$. For lower boom members $\frac{1}{e} = \frac{8}{E}$. For top chord members $\frac{1}{e} = \frac{6}{E}$. For verticals $\frac{1}{e} = \frac{10}{E}$.

Then from (6), since E is the same in each term,

$$F_{10} = -\frac{\Sigma\left(\frac{10k \cdot R}{e}\right)}{\frac{1}{e_{10}} + \Sigma\left(\frac{10k^2}{e}\right)} = -\frac{\Sigma\left(\frac{10k \cdot R \cdot l}{A}\right)}{\frac{l_{10}}{A_{10}} + \Sigma\left(\frac{10k^2 \cdot l}{A}\right)}$$

The values of R are readily found by the method of sections from inspection of (*b*), Fig. 236, with member 10 removed and those of $k$ from (*c*), Fig. 236, and are tabulated below.

TABLE A.

| Member. | R | $10k$ | $\frac{l}{A}$ | $\frac{10kRl}{A}$ | $\frac{10k^2l}{A}$ | $P = R - k \times 0{\cdot}2\dot{6}W_2$ |
|---|---|---|---|---|---|---|
| 1 | 0 | +1 | 6 | 0 | 6 | $-0{\cdot}2\dot{6}W_2$ |
| 2 | $-W_1 - \frac{1}{2}W_2$ | −1 | 6 | $6W_1 + 3W_2$ | 6 | $-W_1 - 0{\cdot}237W_2$ |
| 3 | $-W_1 - \frac{1}{2}W_2$ | +1 | 6 | $-6W_1 - 3W_2$ | 6 | $-W_1 - 0{\cdot}762W_2$ |
| 4 | $-W_2$ | −1 | 6 | $+6W_2$ | 6 | $-0{\cdot}73W_2$ |
| 5 | $\frac{1}{2}W_1 + \frac{1}{4}W_2$ | +1 | 8 | $4W_1 + 2W_2$ | 8 | $0{\cdot}5W_1 - 0{\cdot}0126W_2$ |
| 6 | $\frac{1}{2}W_1 + \frac{1}{4}W_2$ | −1 | 8 | $-4W_1 - 2W_2$ | 8 | $0{\cdot}5W_1 + 0{\cdot}5126W_2$ |
| 7 | $\frac{1}{2}W_1 + \frac{3}{4}W_2$ | +1 | 8 | $4W_1 + 6W_2$ | 8 | $0{\cdot}5W_1 + 0{\cdot}4873W_2$ |
| 8 | $\frac{1}{2}W_1 - \frac{1}{4}W_2$ | −1 | 8 | $-4W_1 + 2W_2$ | 8 | $0{\cdot}5W_1 + 0{\cdot}0126W_2$ |
| 9 | $-W_2$ | −1 | 10 | $+10W_2$ | 10 | $-0{\cdot}73W_2$ |
| 11 | 0 | $-\sqrt{2}$ | 20 | 0 | 40 | $+0{\cdot}3714W_2$ |
| 12 | $-\frac{\sqrt{2}}{2}W_1 - \frac{\sqrt{2}}{4}W_2$ | $-\sqrt{2}$ | 20 | $20W_1 + 10W_2$ | 40 | $-0{\cdot}7071W_1 + 0{\cdot}0179W_2$ |
| 13 | $\frac{\sqrt{2}}{2}W_1 + \frac{\sqrt{2}}{4}W_2$ | $+\sqrt{2}$ | 20 | $20W_1 + 10W_2$ | 40 | $0{\cdot}7071W_1 - 0{\cdot}0179W_2$ |
| 14 | 0 | $+\sqrt{2}$ | 20 | 0 | 40 | $-0{\cdot}3714W_2$ |
| 15 | 0 | $-\sqrt{2}$ | 20 | 0 | 40 | $+0{\cdot}3714W_2$ |
| 16 | $\frac{\sqrt{2}}{2}W_1 - \frac{\sqrt{2}}{4}W_2$ | $-\sqrt{2}$ | 20 | $-20W_1 + 10W_2$ | 40 | $0{\cdot}7071W_1 + 0{\cdot}0179W_2$ |
| 17 | $-\frac{\sqrt{2}}{2}W_1 + \frac{\sqrt{2}}{4}W_2$ | $+\sqrt{2}$ | 20 | $-20W_1 + 10W_2$ | 40 | $-0{\cdot}7071W_1 - 0{\cdot}0179W_2$ |
| 18 | $\sqrt{2}W_2$ | $+\sqrt{2}$ | 20 | $+40W_2$ | 40 | $+1{\cdot}0428W_2$ |
| | Totals are . . . . | | | $104W_2$ | 386 | and $\frac{l_{10}}{A_{10}} = 10$ |

$$F_{10} = -\frac{104W_2}{396} = -0{\cdot}2\dot{6}W_2$$

In the last column of Table A the resulting stresses are given. From the symmetry, the coefficients for a load $W_3$ on the remaining lower chord joint may be obtained and the stresses for this case have been entered in Table B. The coefficients of $W_2$ in, say, members 1, 12, and 5, give the coefficients of $W_3$ in 4, 18, and 8 respectively. For comparison the stresses according to the conventional method of superposition (see Art. 136) are shown in Table B with loads on each

lower chord joint. These are readily obtained by splitting the girder into two systems shown in Fig. 237 and adding algebraically the stresses in the members (2, 3, 6, 7) forming a part of each.

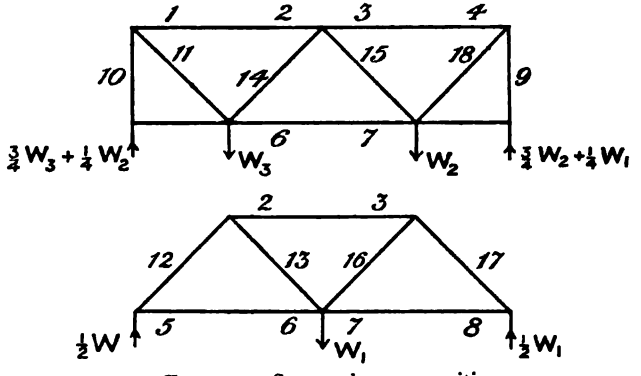



FIG. 237.—Stresses by superposition.

TABLE B.

| Member. | Calculated Stress. | Conventional stress by method of superposition. |
|---|---|---|
| 1 | $-0{\cdot}26W_2 - 0{\cdot}73W_3$ | $-0{\cdot}25W_2 - 0{\cdot}75W_3$ |
| 2 | $-W_1 - 0{\cdot}237W_2 - 0{\cdot}762W_3$ | $-W_1 - 0{\cdot}25W_2 - 0{\cdot}75W_3$ |
| 3 | $-W_1 - 0{\cdot}762W_2 - 0{\cdot}267W_3$ | $-W_1 - 0{\cdot}75W_2 - 0{\cdot}25W_3$ |
| 4 | $-0{\cdot}73W_2 - 0{\cdot}26W_3$ | $-0{\cdot}75W_2 - 0{\cdot}25W_3$ |
| 5 | $0{\cdot}5W_1 - 0{\cdot}0126W_2 + 0{\cdot}0126W_3$ | $0{\cdot}5W_1$ |
| 6 | $0{\cdot}5W_1 + 0{\cdot}5126W_2 + 0{\cdot}4873W_3$ | $0{\cdot}5W_1 + 0{\cdot}5W_2 + 0{\cdot}5W_3$ |
| 7 | $0{\cdot}5W_1 + 0{\cdot}4873W_2 + 0{\cdot}5126W_3$ | $0{\cdot}5W_1 + 0{\cdot}5W_2 + 0{\cdot}5W_3$ |
| 8 | $0{\cdot}5W_1 + 0{\cdot}0126W_2 - 0{\cdot}0126W_3$ | $0{\cdot}5W_1$ |
| 9 | $-0{\cdot}73W_2 - 0{\cdot}26W_3$ | $-0{\cdot}75W_2 - 0{\cdot}25W_3$ |
| 10 | $-0{\cdot}26W_2 - 0{\cdot}73W_3$ | $-0{\cdot}25W_2 - 0{\cdot}75W_3$ |
| 11 | $+0{\cdot}3714W_2 + 1{\cdot}0428W_3$ | $+0{\cdot}3535W_2 + 1{\cdot}0607W_3$ |
| 12 | $-0{\cdot}7071W_1 + 0{\cdot}0179W_2 - 0{\cdot}0179W_3$ | $-0{\cdot}7071W_1$ |
| 13 | $0{\cdot}7071W_1 - 0{\cdot}0179W_2 + 0{\cdot}0179W_3$ | $0{\cdot}7071W_1$ |
| 14 | $-0{\cdot}3714W_2 + 0{\cdot}3714W_3$ | $-0{\cdot}3535W_2 + 0{\cdot}3535W_3$ |
| 15 | $0{\cdot}3714W_2 - 0{\cdot}3714W_3$ | $+0{\cdot}3535W_2 - 0{\cdot}3535W_3$ |
| 16 | $0{\cdot}7071W_1 + 0{\cdot}0179W_2 - 0{\cdot}0179W_3$ | $0{\cdot}7071W_1$ |
| 17 | $-0{\cdot}7071W_1 - 0{\cdot}0179W_2 + 0{\cdot}0179W_3$ | $-0{\cdot}7071W_1$ |
| 18 | $1{\cdot}0428W_2 + 0{\cdot}3714W_3$ | $1{\cdot}0607W_2 + 0{\cdot}3535W_3$ |

A comparison of the results shows firstly that if the structure is symmetrically loaded, *i.e.* if $W_2 = W_3$, the simple conventional method gives exactly correct results; and, secondly, that if the loading is not symmetrical the results are still nearly correct.

**160. Other Methods for Redundant Members.**[1]—The equations of

[1] An excellent account with numerical examples of the application of the principle of minimum resilience to statically indeterminate problems will be found

the previous article may also be derived by the Principle of Minimum Resilience and by the Principle of Work, the methods being briefly as follows, using the notation of Art. 159.

*Principle of Minimum Resilience.*—Let U = total resilience of the frame.

$u_1, u_2, u_3, \ldots u_a, u_b, u_c$, etc., be the resilience of members indicated by the suffixes, so that

$$U = u_1 + u_2 + u_3 + \text{etc.} + u_a + u_b + u_c + \text{etc.}$$

$$u_1 = \tfrac{1}{2}P_1 \times \frac{P_1}{e_1} = \tfrac{1}{2}\frac{P_1^2}{e_1}$$

where $P_1$ has the value (1), Art. 159. Differentiating (partially) with respect to $F_a$

$$\frac{du_1}{dF_a} = \frac{1}{2}\cdot\frac{2P_1}{e_1}\cdot\frac{dP_1}{dF_a} = \frac{P_1}{e_1}\cdot\frac{dP_1}{dF_a} = \frac{P_1}{e_1}\times {}_a k_1$$

$$= \frac{{}_a k_1}{e_1}(R_1 + {}_a k_1 \cdot F_a + {}_b k_1 \cdot F_b + {}_c k_1 F_c + \text{etc.})$$

Similarly $$\frac{du_2}{dF_a} = \frac{{}_a k_2}{e_2}(R_2 + {}_a k_2 \cdot F_a + {}_b k_2 \cdot F_b + {}_c k_2 \cdot F_c + \text{etc.})$$

Also $$\frac{du_a}{dF_a} = \frac{F_a}{e_a} \qquad \frac{du_b}{dF_a} = 0 \qquad \frac{du_c}{dF_a} = 0$$

Hence if $F_a$ is such that U is a minimum, $\frac{dU}{dF_a} = 0$ and

$$0 = \frac{dU}{dF_a} = \frac{du_1}{dF_a} + \frac{du_2}{dF_a} + \frac{du_3}{dF_a}, \text{ etc.} + \frac{du_a}{dF_a} + \frac{du_b}{dF_b} + \text{etc.}$$

which with the above values for the terms on the right-hand side gives equation (8), Art. 159, and for $n$ redundant members there are $n$ such equations.

*Principle of Work.*—By the principle of work, if a redundant member $a$ be replaced by opposite pulls $F_a$ at its ends the algebraic sum of the work done by these forces and the resilience caused in the structure is zero. Hence

$$\tfrac{1}{2}F_a \times \frac{F_a}{e_a} + \tfrac{1}{2}P_1 \times \frac{{}_a k_1 \cdot F_a}{e_1} + \tfrac{1}{2}P_2\frac{{}_a k_2 \cdot F_a}{e_2} + \text{etc.} = 0$$

And dividing each term by $\frac{1}{2}F_a$ gives

$$\frac{F_a}{e_a} + \frac{P_1 \cdot {}_a k_1}{e_1} + \frac{P_2 \cdot {}_a k_2}{e_2} + \text{etc.} = 0$$

and when the values such as (1), Art. 159, are substituted for $P_1$, $P_2$, etc. this also gives equation (8), Art. 159.

**161. Stress due to Errors or Changes in Length.**—If a frame having a redundant member has one member made too short or shortened by a fall in temperature that member will exert inward pulls

in "Statically Indeterminate Structures and the Principle of Least Work," by H. M. Martin, reprinted from *Engineering*. Also see articles on "Statically Indeterminate Frames," by Max Am Ende in the *Engineer*, March 13, 1885; Nov. 19, 1888; Nov. 9, 1890; Sept. 21, 1894; Feb. 1 and 15, 1895.

at its ends and the frame will be *self-strained*. With the notation of Art. 159, suppose member $a$ is an amount $x$ too short. Then when the member $a$ is forced into its place the approach of its end connections toward one another, plus the stretch of the member, is equal to $x$, or

$\Delta + \frac{F_a}{e_a} = x$, and since R = 0, from (4), Art. 159, $\Delta = F_a \Sigma\left(\frac{ak^2}{e}\right)$

hence

$$F_a = \frac{x}{\frac{1}{e_a} + \Sigma\left(\frac{ak^2}{e}\right)} \quad \ldots\ldots (1)$$

EXAMPLE.—Six bars each 1 square inch in section are to form a square of 30-inch sides with two diagonals, pin-jointed at the corners. If one side bar is the last to be added and is too short by 0·01 inch, find the stress in all the bars if the short bar is forced into its position. E = 30,000,000 lbs. per square inch.

For each 30-inch side bar $e = \frac{EA}{l}$ = 1,000,000 lbs. per inch deflection and $k = +1$.

For each $30\sqrt{2}$" diagonal $e = \frac{1}{\sqrt{2}} \times 1{,}000{,}000$ lbs. per inch deflection, $k = -\sqrt{2}$, hence from (1) the tension in the bar is

$$F = \frac{0{\cdot}01}{\frac{1}{10^6} + \frac{1}{10^6}(1 + 1 + 1 + 2\sqrt{2} + 2\sqrt{2})} = \frac{10{,}000}{4 + 4\sqrt{2}} = 1036 \text{ lbs.}$$

The other sides of the square have the same tension, and the diagonals have each a thrust $1036\sqrt{2} = 1465$ lbs.

**162. Continuous Framed Girders.**—The principles used for finding the stresses in redundant members of frames are also applicable to finding the value of a redundant supporting force such as a prop to a framed girder. For example, the reaction $R_C$ at C in Fig. 221 may be found by finding the deflection of C as if the supporting force $R_C$ were absent, and equating it to the upward deflection of a force $R_C$ at C. Similarly if Fig. 222 represents a continuous girder the reactions at C and D may be found from two simultaneous equations, the dimensions of all the members being known. If the panel KK'DC is unbraced the stresses in this structure may best be found by treating KK' and CD as redundant members and replacing them by four equal forces at their ends; the structure then falls into two simply supported trusses.

EXAMPLE.—The ratios of length to uniform cross-sectional area in inch units being as given in the table following, find the reactions in Fig. 221 when unit loads are carried at each of the joints D, E, F.

Firstly, assume the support at C to be removed. Then $R_A = 2\frac{1}{2}$, $R_B = \frac{3}{4}$, hence by the method of sections find the values of the stress P in each bar as tabulated below. Then take unit downward force at C and find the values of $k$ as tabulated. Multiply the terms P, $k$, and $\frac{l}{A}$

and find the sum $\Sigma\left(\frac{Pkl}{A}\right)$ which by (4), Art. 155, is E times the deflection at C. Next calculate $\frac{k^2l}{A}$ for each member and find the sum $\Sigma\left(\frac{k^2l}{A}\right)$, which by (5), Art. 155, is E times the deflection for unit force at C. Then

$$R_C \times \Sigma\left(\frac{k^2l}{A}\right) = \Sigma\left(\frac{Pkl}{A}\right)$$

and

$$R_C = \Sigma\left(\frac{Pkl}{A}\right) \div \Sigma\left(\frac{k^2l}{A}\right)$$

| Member. | $\frac{l}{A}$ | P | $k$ | $\frac{Pkl}{A}$ positive terms | $\frac{Pkl}{A}$ negative terms | $\frac{k^2l}{A}$ |
|---|---|---|---|---|---|---|
| AM | 30 | $-2{\cdot}25\sqrt{2}$ | $-0{\cdot}5\sqrt{2}$ | 67·50 | | 15·0 |
| ME | 20 | $+1{\cdot}25\sqrt{2}$ | $+0{\cdot}5\sqrt{2}$ | 25·00 | | 10·0 |
| EN | 15 | $-0{\cdot}25\sqrt{2}$ | $-0{\cdot}5\sqrt{2}$ | 3·75 | | 7·5 |
| NC | 10 | $-0{\cdot}75\sqrt{2}$ | $+0{\cdot}5\sqrt{2}$ | — | −7·50 | 5·0 |
| CP | 10 | $+0{\cdot}75\sqrt{2}$ | $+0{\cdot}5\sqrt{2}$ | 7·50 | | 5·0 |
| PH | 15 | $-0{\cdot}75\sqrt{2}$ | $-0{\cdot}5\sqrt{2}$ | 11·25 | | 7·5 |
| HQ | 20 | $+0{\cdot}75\sqrt{2}$ | $+0{\cdot}5\sqrt{2}$ | 15·00 | | 10·0 |
| QB | 30 | $-0{\cdot}75\sqrt{2}$ | $-0{\cdot}5\sqrt{2}$ | 22·50 | | 15·0 |
| MD | — | 1 | 0 | 0 | | — |
| NF | — | 1 | 0 | 0 | | — |
| PG | — | 0 | 0 | 0 | | — |
| QJ | — | 0 | 0 | 0 | | — |
| AD | 12 | +2·25 | +0·5 | 13·50 | | 3·0 |
| DE | 12 | +2·25 | +0·5 | 13·50 | | 3·0 |
| EF | 10 | +3·75 | +1·5 | 56·25 | | 22·5 |
| FC | 10 | +3·75 | +1·5 | 56·25 | | 22·5 |
| CG | 10 | +2·25 | +1·5 | 33·75 | | 22·5 |
| GH | 10 | +2·25 | +1·5 | 33·75 | | 22·5 |
| HJ | 12 | +0·75 | +0·5 | 4·50 | | 3·0 |
| JB | 12 | +0·75 | +0·5 | 4·50 | | 3·0 |
| QP | 10 | −1·5 | −1 | 15·00 | | 10·0 |
| PN | 10 | −3·0 | −2 | 60·00 | | 40·0 |
| NM | 10 | −3·5 | −1 | 35·00 | | 10·0 |
| | | | | 478·50 | | $\Sigma\left(\frac{k^2l}{A}\right) = 237{\cdot}0$ |
| | | | | −7·50 | | |
| | | | $\Sigma\left(\frac{Pkl}{A}\right) =$ | 471·00 | | |

$R_C = \frac{471}{237} = 1{\cdot}9873$, hence by moments $R_B = -\frac{1}{8}(4 \times 1{\cdot}9873 - 6)$
$= -0{\cdot}2437$. $R_A = 3{\cdot}2437 - 1{\cdot}9873 = 1{\cdot}2564$

A reference to the table of reactions in the example of Art. 151 gives the approximate results as $R_A = 1{\cdot}2657$, $R_B = -0{\cdot}2344$, thus justifying the approximation involved of using for this case the rules applicable to a solid continuous girder.

**163. Simple Principles applicable to Indeterminate Structures.**—The analogy between quite different cases of statically indeterminate stresses may have been noticed, and for convenience a general rule may be stated. Suppose that two elements, $a$ and $b$, jointly or "in parallel," resist a load, and in consequence exert upon each other a force F, tending to deform $a$ and restore $b$, say; let $x_0$ be the deflection of $b$, say, due to the load if $a$ were removed. Let the stiffnesses or forces per foot of elastic deflection, in the direction of F of $a$ and $b$, be $e_a$ and $e_b$ respectively. Let $\frac{1}{e_a}$ and $\frac{1}{e_b}$ be called their respective "elasticities." Then the actual deflection

$$\frac{F}{e_a} = x_0 - \frac{F}{e_b} \qquad (1)$$

hence

$$F = \frac{x_0}{\frac{1}{e_a} + \frac{1}{e_b}} \qquad (2)$$

$$\text{or action of } a \text{ on } b = \frac{\text{strain for } b \text{ acting alone}}{\text{sum of the elasticities}} \qquad (3)$$

and if the load can be reduced to a force W in the direction of F,

$$x_0 = \frac{W}{e_b} \text{ and } F = \frac{e_a}{e_a + e_b} W \text{ and } W - F = \frac{e_b}{e_a + e_b} W \qquad (4)$$

which are the same results as in Art. 158, but not limited to struts or ties or to any one type of elastic constraint. Putting equation (4) in words, the two elements $a$ and $b$ divide the load W in proportion to their stiffnesses. The actual deflection is

$$\delta = \frac{W}{\text{total stiffness}} = \frac{W}{e_a + e_b} \qquad (5)$$

EXAMPLES.—We have had examples in Art. 95 for the uniformly loaded rigidly propped cantilever in which $a$ represents the prop and $b$ the cantilever,

$$F = P, \quad x_0 = \frac{1}{8} \cdot \frac{wl^4}{EI}, \quad \frac{1}{e_b} = \frac{l^3}{3EI}, \quad e_a = e, \quad W = \tfrac{3}{8}wl = F(e_a = \infty).$$

Also in the uniformly loaded beam ($b$) on elastic end supports and a central elastic prop ($a$) in Art. 94, $e_a = e$ and $\frac{1}{e_b} = \frac{1}{2e} + \frac{l^3}{48EI}$, while $x_0 = \frac{5}{384} \cdot \frac{wl^4}{EI}$, and $W = \frac{5}{8}wl$.

In the present chapter the important formula (7), Art. 159, is but another example of the same principle, for $-\sum\left(\frac{ak}{e} \cdot \frac{R}{e}\right) = x_0$, and $\sum\left(\frac{ak^2}{e}\right) = \frac{1}{e_b}$, where $b$ represents the frame with the member $a$ removed, and $\sum\left(\frac{ak^2}{e}\right)$ is the deflection per unit of force.

If two elements $a$ and $b$ resist a force "in series" so that each

bears the whole force, the elasticity $\frac{I}{e}$ of the two is the sum of the elasticities

or $$\frac{I}{e} = \frac{I}{e_a} + \frac{I}{e_b} \quad \ldots\ldots\ldots (6)$$

or the stiffness $$e = \frac{e_a e_b}{e_a + e_b} \quad \ldots\ldots\ldots (7)$$

which is evidently less than either $e_a$ or $e_b$.

## EXAMPLES XIV.

1. Two pin-jointed rods AC and BC in the same vertical plane are hinged to a rigid support at A and B, 8 feet apart in the same horizontal line. Find the vertical and horizontal deflections of C when a load of 7 tons hangs from that point if AC and BC are inclined 30° and 45° respectively to the horizontal (AĈB being an obtuse angle) and the sectional areas are 1·5 and 2 square inches respectively. E = 12,500 tons per square inch.

2. An N girder of four bays has vertical posts at its ends and carries 16 tons at each joint of the lower chord. The bays are each 6 feet long and 8 feet high. Taking the tensile stress in the diagonals and the bottom chord at 5 tons per square inch and the compressive stress in the verticals and top chord as $2\frac{1}{2}$ tons per square inch, find the central deflection if E = 12,500 tons per square inch.

3. A Warren girder made up of members of equal lengths has four bays in the lower boom and three in the upper boom, and rests on supports which are 24 feet apart. If under a central load the stresses in the ties are 6 tons per square inch, and in the struts 3 tons per square inch, estimate the central elastic deflection, taking E = 12,500 tons per square inch.

4. If the point C in Problem 1 is joined to a fixed point D midway between A and B by a bar DC 1 square inch in cross-section, find the pull in each bar if 10 tons is suspended from the point C.

5. A frame consisting of 6 bars each 1 square inch in section and hinged together to form a square of 20-inch side with two diagonals, is suspended from one corner. The opposite corner supports a load of 1000 lbs. Find the stress in each bar.

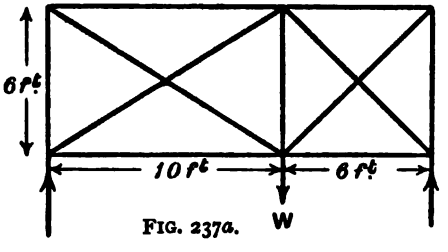


FIG. 237a.

6. The diagram (Fig. 237a) represents a freely jointed frame supported at its ends and carrying a load W as shown. Find the stress in the two diagonal members meeting at the loaded joint if the ratio of length to area of cross-section is the same for every member.

7. If one of the diagonals of the steel frame in Problem 5 is heated 40° F. above the remaining bars, find the resulting stresses in the sides and diagonals of the frame.

8. If the girder of Problem 2 under the same loading were propped at the centre to the same level as the ends, find the reaction on the central prop. What would the reaction be for a continuous solid girder of uniform section with the load (*a*) uniformly distributed at 16 + 6 or $2\frac{2}{3}$ tons per ft. directly applied, (*b*) concentrated as 16 tons at panel points with 8 tons carried directly at each end support?

# CHAPTER XV[1]

## *SOME INDETERMINATE COMBINATIONS*[1]

**164. Trussed Beams.**—Trussed beams consisting of a combination of beams with ties and struts form an important structural element. The distribution of stress cannot be determined by the ordinary principles of statics, but may be determined by those given in Chapter XIV.

The simplest form of a trussed beam is shown in Fig. 238. AB is

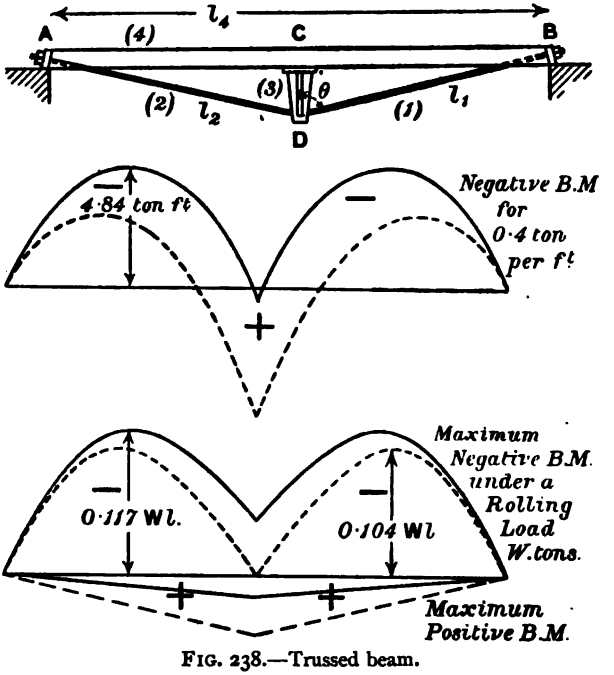



FIG. 238.—Trussed beam.

a continuous beam; CD is a strut braced to the beam ends A and B by tie rods AD and DB. The stresses in the various members are of course dependent upon the initial stresses due to tightening up the

[1] Arts. 165–173 inclusive may be omitted on a first reading of the subject.

rods, and are liable to alteration by change of temperature. The stresses due to loads carried on AB may be conveniently found by the methods of Art. 163, but if the proportions are such that the deformation of the bracing is negligible compared to the bending deflections of the beam, the stresses in the beam and reactions at C are practically those for a continuous beam AB rigidly propped at C to the level of AB, and this has been dealt with in Chapters VII. and VIII.[1] The example at the end of the article with proportions usual in practice shows how much in error such treatment may be in some cases.

Allowing for the elasticity of the ties and strut, let $e_1$ and $e_2$ be the stiffnesses $\left(\frac{EA}{l}\right)$ of the ties, $e_3$ that of the strut, and $e_4$ that of the beam in axial thrust. Then unit downward pressure of the beam on the strut brings a tension $\frac{1}{2}\sec\theta$ in each tie, and an axial thrust $\frac{1}{2}\tan\theta$ in the beam where $\theta$ is the angle CDB = angle CDA. And unit stretch of the tie allows a vertical deflection $\frac{1}{2}\sec\theta$ of D, while unit compression of the beam allows a vertical deflection $\frac{1}{2}\tan\theta$ of D. Hence remembering that if 1 ton vertically at D produces a tension $k$ in any member then the deflection of D per unit stretch of that member is $k$, the elasticity (vertically) of the truss system $\frac{1}{e_a}$, plus that of the beam $\frac{1}{e_b}$ is

$$\frac{1}{e_a}+\frac{1}{e_b}=\frac{1}{e_3}+\frac{(\frac{1}{2}\sec\theta)^2}{e_1}+\frac{(\frac{1}{2}\sec\theta)^2}{e_2}+\frac{(\frac{1}{2}\tan\theta)^2}{e_4}+\frac{l_4^3}{48EI}=\frac{1}{e}\text{ say}\quad(1)$$

where usually $e_1 = e_2$, and hence

$$\frac{1}{e}=\frac{1}{e_3}+\frac{l_1^2}{2l_3^2e_1}+\frac{1}{16}\cdot\frac{l_4^2}{l_1^2e_4}+\frac{l_4^3}{48E_4I_4}\quad(1A)$$

or

$$\frac{1}{e}=\frac{l_3}{A_3E_3}+\frac{l_1^3}{2l_3^2E_1A_1}+\frac{1}{16}\cdot\frac{l_4^3}{l_1^2A_4E_4}+\frac{l_4^3}{48E_4I_4}\quad(1B)$$

Hence if $\delta_c$ is the central deflection which the load would cause in the beam if simply supported at A and B, the thrust F in the post CD, by (2), Art. 163, is

$$F=\delta_c\div\frac{1}{e}=e.\delta_c.\quad(2)$$

The calculation of $\delta_c$ for any load is dealt with in Chapter VII., Art. 94, and when F is known the resulting bending moment on AB for F and the load is easily calculated, and hence the bending stresses may be found.

The pull in the tie rods is $\frac{1}{2}F\sec\theta$ which induces a thrust $\frac{1}{2}F\tan\theta$, which may be taken as uniformly distributed in the beam, AB to be added to the bending stresses, the increment of bending stress due to the thrust acting on the deflected beam being neglected. If

[1] Solutions of this kind for several types of trussed beams are given in a paper on "Trussed Beams," by Mr. George Higgins of Melbourne University. *Proceedings Australian Association for Advancement of Science.*

$e_1$, $e_2$, and $e_3$ are great compared to $\frac{48EI}{l^3}$, the stiffness of the beam (1) becomes $e = \frac{48EI}{l^3}$, and then (2) reduces to $F = \frac{48EI\delta c}{l^3}$, the reaction of a rigid prop at the centre of a continuous beam of two equal spans.

EXAMPLE 1.—A trussed timber beam 20 feet long is square in section 9″ × 9″, and has a central cast-iron strut 2 feet long, and 24 square inches in sectional area; the wrought-iron tie rods which are each 10·2 feet long are 1 inch diameter. Take E for wrought iron as 12,000 tons per square inch, for cast iron 6000 tons per square inch, and for timber as 600 tons per square inch. Find the thrust in the strut, the pull in the tie rods, the maximum bending moment in the beam and the extreme stresses, when the beam carries a uniformly distributed load of 0·4 ton per foot.

The elasticity in inches per ton of load at C for the several elements are

$$\text{for strut } \frac{24}{6000 \times 24} = 0{\cdot}0001\dot{6}$$

$$\text{for two ties } \frac{2 \times 10{\cdot}2 \times 12}{12{,}000 \times 0{\cdot}7854} \times \frac{1}{4} \times \left(\frac{10{\cdot}2}{2}\right)^2 = 0{\cdot}169$$

$$\text{for beam in compression } \frac{240}{600 \times 81} \times \left(\frac{10}{2}\right)^2 \times \frac{1}{4} = 0{\cdot}0308$$

$$\text{for bending } \frac{(240)^3 \times 12}{48 \times 600 \times 9^4} = 0{\cdot}8780$$

or the total elasticity

$$\frac{1}{e} = 0{\cdot}0002 + 0{\cdot}169 + 0{\cdot}0308 + 0{\cdot}8780 = 1{\cdot}0780 \text{ inches per ton}$$

which is considerably in excess of the value (0·878) for flexure alone, the difference being mainly in the ties. The deflection at C for the unbraced beam is

$$\frac{5}{384} \cdot \frac{W}{EI} = \frac{5 \times 8 \times 240^3 \times 12}{384 \times 600 \times 9^4} = 4{\cdot}49 \text{ inches}$$

hence the thrust in the strut C from (2) is

$$4{\cdot}39 \div 1{\cdot}078 = 4{\cdot}07 \text{ tons}$$

instead of $\frac{4{\cdot}39}{0{\cdot}878}$ or $\frac{5}{8}$ of $8 = 5$ tons if the flexibility of the beam alone were allowed for.

The pull in each tie rod is $4{\cdot}07 \times \frac{1}{2} \times \frac{10{\cdot}2}{2} = 10{\cdot}38$ tons

The thrust in the beam is

$\frac{1}{2} \times 4{\cdot}07 \times \frac{10}{2} = 10{\cdot}175$ tons, or $\frac{10{\cdot}175}{81} = 0{\cdot}1255$ ton per sq. in.

The bending moment at $x$ feet from the end is

$$-\frac{8 - 4{\cdot}07}{2}x + \frac{0{\cdot}4x^2}{2} \text{ ton-feet}$$

which is represented by a parabola which may easily be plotted. The maximum negative bending moment occurs with zero shearing force at $x$ feet from the end where

$$x = \frac{8 - 4{\cdot}07}{2} \div 0{\cdot}4 = 4{\cdot}9125 \text{ feet}$$

The maximum negative bending moment is

$$-1{\cdot}965 \times 4{\cdot}9125 + \tfrac{1}{2} \times 0{\cdot}4 \times (4{\cdot}9125)^2 = -4{\cdot}84 \text{ ton-feet}$$

The bending moment at the post is

$$4 \times 5 - 1{\cdot}965 \times 10 = +0{\cdot}35 \text{ ton-foot}$$

the thrust at the post (4·07) being just greater than that required (4 tons) to change the sign of the bending moment; there are points of inflexion just on either side of the centre, viz. at $2 \times 4{\cdot}9125 = 9{\cdot}825$ feet from the ends. The curves of bending moment are shown in Fig. 238.

The modulus of section is

$$\frac{81 \times 9}{6} = 121{\cdot}5 \text{ (inches)}$$

hence the extreme bending stress is

$$\frac{4{\cdot}84 \times 12}{121{\cdot}5} = 0{\cdot}478 \text{ ton per square inch}$$

the maximum compressive stress is

$$0{\cdot}478 + 0{\cdot}126 = 0{\cdot}604 \text{ ton per square inch}$$

and the maximum tensile stress is

$$0{\cdot}478 - 0{\cdot}126 = 0{\cdot}352 \text{ ton per square inch}$$

The vertical end reactions are each 4 tons, made up of $\frac{1}{2}(8 - 4{\cdot}07) = 1{\cdot}965$ tons from the beam, and $\frac{1}{2} \times 4{\cdot}07 = 2{\cdot}035$ tons from the tie rods. If the elasticity of the bracing were neglected the maximum negative bending moment would have been

$$-\frac{9}{512} \times \frac{0{\cdot}4 \times 400}{1} = -2{\cdot}81 \text{ ton-feet}$$

while the maximum positive bending moment (at the post) would have been

$$\frac{1}{32} \times \frac{0{\cdot}4 \times 400}{1} = +5 \text{ ton-feet}$$

The curve for this assumption is shown dotted for comparison in Fig. 238. On the other hand, the stresses in the bracing would be exaggerated by the supposition that the bracing was perfectly rigid.

An empirical rule[1] for estimating the maximum bending stress is to take the beam as if separated into two parts at C. This would give a maximum bending moment

$$= \tfrac{1}{8} \times 0{\cdot}4 \times 100 = -5 \text{ ton-feet}$$

[1] See "Notes on Building Construction," Part IV., Chapter XI. (Longmans).

which seems to be justified by the result $-4{\cdot}84$, for this more elaborate calculation in an example with typical proportions.

EXAMPLE 2.—If the beam in Example 1 is traversed by a concentrated load W tons, find the position and amount of the maximum bending moment.

If the load is a distance $nl$ from the near end of the beam, length $l$, say, supposed simply supported at its ends, the central deflection, writing $b = nl$, $a = (1 - n)l$, and $x = \frac{1}{2}l$ in (7), Art. 96, is

$$\delta_c = \frac{Wl^3n\,(3 - 4n^2)}{48EI}$$

Hence from (2), using the previous value $1{\cdot}078$,

$$F = \frac{1}{1{\cdot}078} \cdot \frac{Wn\,(3-4n^2) \times 240^3 \times 12}{48 \times 600 \times 9^4} = 0{\cdot}814n\,(3 - 4n^2)\,W$$

The upward reaction on the beam at the near end is $W(1 - n) - \frac{1}{2}F$, and the bending moment under the load (which exceeds in magnitude that at the centre) is $-nl$ times this or

$$M = -Wln\{1 - n - 0{\cdot}407n(3 - 4n^2)\} = -Wln(1 - 2{\cdot}221n + 1{\cdot}628n^3)$$

This is plotted on the lower part of Fig. 238. The maximum value for any position, found by writing $\frac{dM}{dx} = 0$, is about $M = -0{\cdot}1175Wl$, for a value of $n$, a trifle under $0{\cdot}25$. The maximum positive bending moments occur at the centre, and elsewhere, when the reaction on the far end of the beam (from the load) is a downward maximum value. Even at the centre of the beam the greatest maximum positive bending moment only amounts to $0{\cdot}0156Wl$, falling off uniformly to zero at the ends.

The maximum negative and positive bending moment curves for the rigidly propped beam are shown dotted, and may be obtained by using $0{\cdot}878$ in place of $1{\cdot}078$ above, or

$$F = n(3 - 4n^2)W, \qquad M = -Wln(1 - 2{\cdot}5n + 2n^2)$$

the greatest values being about

$$-0{\cdot}1038Wl \text{ under the load for } n = 0{\cdot}216,$$
$$\text{and } +0{\cdot}048Wl \text{ at C for } n = 0{\cdot}289$$

The empirical calculation taking the beam as discontinuous at C gives the maximum bending moment at $\frac{1}{4}l$ as

$$-\tfrac{1}{4}W \times \tfrac{1}{2}l = -0{\cdot}125Wl$$

a fair estimate on the safe side.

**165. Simple Braced Shed Frames and Portals.**—An important type of statically indeterminate building frame containing continuous members resisting flexure and others acting as ties and struts only is introduced by the simple framework shown by Fig. 239. Members AB and CD are similar and represent vertical stanchions hinged at each end and their caps connected by a cross beam, knee braces inclined $\theta$ to the horizontal connecting K to G and E to F. The sole load is a horizontal force W, such as a wind load, applied at B.

*Reactions.*—By taking moments about A and D it is evident that the vertical component V of the reactions at D and A are equal and opposite as shown and of magnitude $WL/l$. The horizontal components $H_1$ and $H_2$ are such that $H_1 + H_2 = W$, but their magnitudes depend upon the compressibility of the cross beam in the direction of its length. If $R_1$ and $R_2$ are the resultant reactions they must meet at

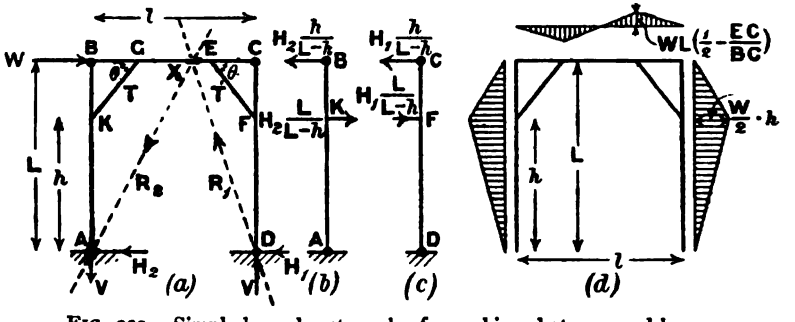



FIG. 239.—Simple braced rectangular frame, hinged at caps and bases.

some point X in the line of action of W, and X will be a point of inflection of the beam BC, for the resultant force on the structure either side of X passes through X and has therefore zero moment about X. By taking a section through the hinge C cutting EF, and moments about C of forces on the structure to the right of the section, if T' is the thrust in EF,

$$T' \cdot CF \cos\theta = L \cdot H_1 \quad \text{or } T' = H_1 \frac{L}{L-h} \sec\theta \quad . \quad . \quad (1)$$

And similarly if T is the tension in KG,

$$T = H_2 \cdot \frac{L}{L-h} \sec\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And if $t'$ = tension in EC, by moments about D of the forces on the stanchion CD,

$$t' \cdot L = T' \cos\theta \,.\, h, \text{ hence } t' = H_1 \cdot \frac{h}{L-h} \quad . \quad . \quad . \quad (3)$$

and similarly if

$$t = \text{thrust in BG}, t - W = H_2 \frac{h}{L-h} \quad . \quad . \quad . \quad (4)$$

hence the transverse or bending forces on the stanchions are as shown at (*b*) and (*c*), Fig. 239, and it is evident that the forces being proportional to $H_2$ and $H_1$ respectively the type of deflection is the same in each case. Hence the deflections of B and C are proportional to $H_2$ and $H_1$ respectively, and if we treat the compressibility from B to C as negligible compared to the flexibility of the stanchions the deflections are equal and

$$H_1 = H_2 = \tfrac{1}{2}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The point of inflection X is then midway between B and C. Further,

if we know the dimensions of the members and express all the forces in terms of say $H_1$ (writing $H_2 = W - H_1$), using the principle of deformations (Art. 158), we can find $H_1$ in terms of W by equating say the deflection of AB at B to the deflection of CD at C plus the compression of BC. This principle will be applicable to the cases which follow, whether B is united to C by some form of bracing or by a roof, and also to the case of stanchions fixed at their bases, for the fixing couples will then be proportional to $H_1$ and $H_2$. For stanchions of different lengths or moments of inertia (I) the principle of deformation may be applied to find $H_1$ and $H_2$.

*Stresses in Members.*—Assuming a rigid connection of B to C, from (5), (1) and (2), we get,

$$T = T' = \tfrac{1}{2}W \frac{L}{L - h} \sec\theta \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and from (3) and (4),

$$t - W = t' = \tfrac{1}{2}W \frac{h}{L - h} \text{ and } t = \tfrac{1}{2}W \frac{2L - h}{L - h} \quad . \quad . \quad (7)$$

And from a vertical section through GE

$$\text{the thrust in GE} = H_1 = \tfrac{1}{2}W \quad . \quad . \quad . \quad . \quad (8)$$

which is accompanied by a bending moment which at a distance $x$ from C is equal to $H_1L - Vx = WL\left(\frac{1}{2} - \frac{x}{l}\right)$. The shearing force from B to G is $V - T \sin\theta = WL\left(\frac{1}{l} - \frac{1}{2BG}\right)$ which is negative, and from G to E is $V = W \cdot \frac{L}{l}$. The bending moment diagrams are shown at (*d*), Fig. 239.

*Flexible Braces.*—If the braces KG and EF are of such small section (or I) in proportion to their lengths that they will not carry any appreciable thrust the brace on the leeward side (EF) may be neglected. The structure is now statically determinate, $H_2 = W$, and the brace KG carries twice the former tension, *i.e.*

$$T = W \frac{L}{L - h} \sec\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The reaction $R_2$ then passes through C, while $R_1 = V$ along DC. The bending moment at $G = -V \times GC = -W \frac{L}{l} \cdot GC$ and at K is $Wh$.

**166. Columns fixed at Bases.**—The effect of fixed ends to the column at the bases in the foregoing and other types of portal bracing may easily be seen if we find the point of inflexion I of the column, for above this the structure is under precisely the same condition as that with hinged bases in Art. 165. The point I may be found as follows: Let P, Fig. 240, be the horizontal component of say the tension in the brace KG on say the windward stanchion of (*a*) Fig. 239, but supposed fixed at the base. Then the transverse forces are as shown in Fig. 240. And taking the bracing B to C as rigid we assume that B and K deflect

equally, an assumption which overcomes the indeterminate condition arising from the fixture of the stanchion ends. The deflection of K may be written from (2) Art. 95, and the general expression of which (2) is a particular value, while the deflection of B may be written from (2) and (5) of Art. 95. Equating these deflections,

$$\frac{Ph^3}{3EI} - \frac{(P-H)h^2}{2EI}(L - \tfrac{1}{3}h) = \frac{Ph^3}{3EI} + \frac{Ph^2(L-h)}{2EI} - \frac{(P-H)L^3}{3EI} \quad (1)$$

From which

$$P = \frac{H}{2} \cdot \frac{2L^2 + 2Lh - h^2}{L^2 + Lh - 2h^2} \text{ and } P - H = \frac{H}{2} \cdot \frac{3h^2}{L^2 + Lh - 2h^2} \quad (2)$$

Then at a distance $x$ from A the bending moment

$$M_x = P(h - x) - (P - H)(L - x)$$

and substituting the above values of P and P − H, $M_x$ vanishes for

$$x = x = \frac{h}{2} \cdot \frac{2L + h}{L + 2h} \quad \ldots \ldots \quad (3)$$

This is always greater than $\frac{1}{2}h$ or $KI_1$ is always less than $\frac{1}{2}h$ By symmetry $DI_2 = AI_1 = x_0$. It is usual to take $x_0 = \frac{1}{2}h$, which is on the

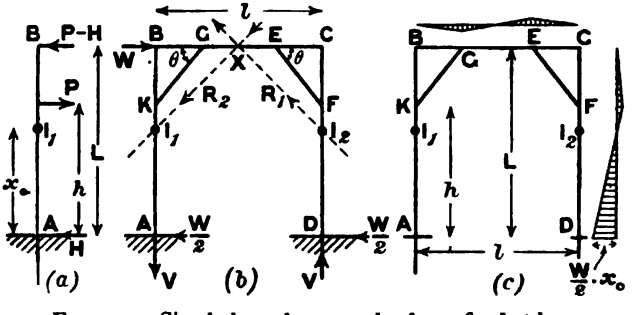



FIG. 240.—Simple braced rectangular frame fixed at bases.

safe side for calculating stresses in the bracing. When the points of inflection $I_1$ and $I_2$ have been determined it is only necessary to write $L - x_0$ for L and $h - x_0$ for $h$ in the formulæ (6) and (7) of Art. 165 to obtain the stresses in the members. The bending moment diagrams are shown in Fig. 240 (*c*). There is the same algebraic change of bending moment between F and D as for the hinged posts, viz. from $\frac{1}{2}W(h - x_0)$ at F to $\frac{1}{2}Wx_0$ of opposite sign at D. The bending moment at E is $W\left(\frac{1}{2} - \frac{EC}{BC}\right)(L - x_0)$. The vertical components of the reactions are $\frac{W}{l}(L - x_0)$.

**167. Other Forms.**—Fig. 241 shows a special case of bracing, such as Fig. 239, where the braces meet at X, the point of inflection midway between B and C. Considering forces on the part to the right of say

a vertical section through X, the only external force is $R_1$ which passes through X, hence since the force in TY is the only one not passing through X, the stress in TY = 0. It follows that the stresses in TB and CY are zero, and the remaining stresses follow from the formulæ in Art. 165. From the half structure shown to the right of Fig. 241 it is evident that the stresses in XC may be found by a triangle of forces for the point X, XC having no bending stress CF will have no thrust, but will have a shearing force equal to the tension in CX. Also the vertical component of the stress in FX is equal to V.

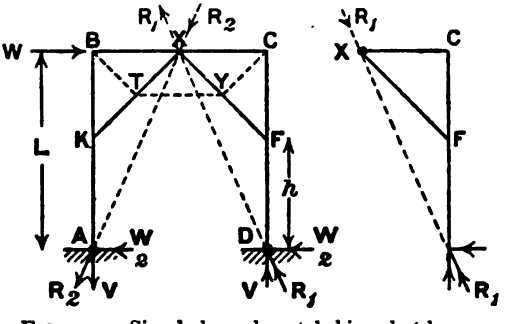



FIG. 241.—Simple braced portal, hinged at bases.

If the stanchions are fixed in direction at their lower ends it is only necessary to calculate the position of the points of contraflexure as for $I_1$ and $I_2$ in Art. 166, and use these points in place of A and D.

*Graphical Solution.*—A simple method of finding the stress in the internal bracings of portals for which the author is indebted to Ketchum's "Steel Mill Buildings" is illustrated in Fig. 242. The

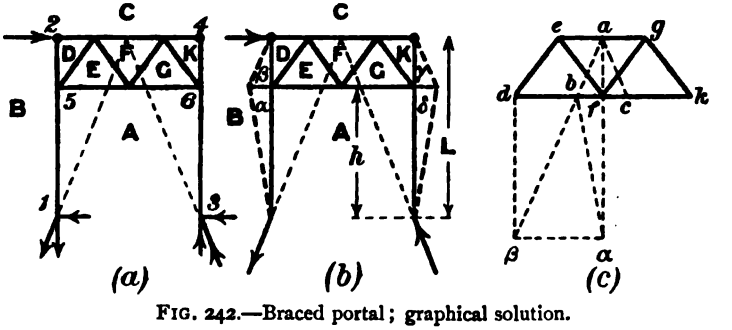



FIG. 242.—Braced portal; graphical solution.

bracing DEFGK at (*a*) unites the stanchions 1, 2, and 3, 4, which at (*b*) are replaced, 1, 2, by pin-ended members Aα, αB, αβ, Bβ, and 3, 4 by γC, γδ, δC, δA, the lengths of βα and γδ being immaterial. The stress diagram (*c*) can be drawn starting from say joint ABα, after putting in *abc* for the external forces so that *bc* = W. The dotted portion relates to imaginary members and may be dispensed with, leaving only the portion drawn in continuous lines; the points γ, δ symmetrical with βα have not been added, as they are not necessary to the work. The portion relating to bracing members which are struts

or ties only needs no comment, but with regard to the stanchions the portion 1, 5 withstands the force *ab*, viz. the vertical projection (which is *af*) as a thrust and a transverse shearing force equal to the horizontal projection of *ab* (*i.e.* to the length *bf*); the portion 5, 2 carries the force *bd*, which represents zero thrust and a shearing force *bd*. The method is applicable to the form shown in Fig. 241 and to other simple forms. In case the stanchions are "fixed" in direction at their bases it is only necessary to calculate the positions of the points of inflection by (3), Art. 166, and to use these points as the feet of hinged stanchions.

**168. Stanchions with Cross Beam.**—(*a*) *Bases hinged.*—If the bracing in Art. 165 and Fig. 239 is replaced by a horizontal beam *rigidly* connected to the stanchion caps, the solution of the stresses depends upon the fact that the beam ends bend through the same angles ($i_B$ and $i_C$) as the stanchion ends. Thus in Fig. 243, since each stanchion is acted upon by a transverse force (H) and a couple (HL) at

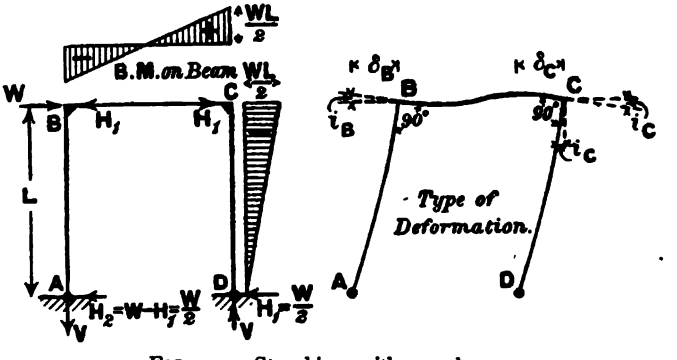



FIG. 243.—Stanchions with cross beam.

the top, the deflections are of similar type, and if we neglect the compressibility of the beam BC, since the deflections of B and C are equal, the two horizontal reactions are equal, *i.e.* $H_1 = W - H_1 = \frac{1}{2}W$.

By symmetry the slopes are equal at B and C, *i.e.* $i_B = i_C$, and from (12), Art. 103, and using suffixes B and C instead of A and B, since for the beam $M_C = H_1 . L$ or $\frac{1}{2}WL$,

$$i_C = \frac{H_1 L l}{6EI_b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $I_b$ = moment of inertia of the cross-section of the beam.

Hence for $\delta_C = \delta_B$ we have—

$$\delta_C = \frac{H_1 L^3}{3EI} + i_C L = \frac{H_1 L^3}{3EI} + \frac{H_1 L^2 l}{6EI_b} = \frac{WL^3}{6EI}\left(1 + \frac{1}{2\alpha}\right) \quad . \quad (2)$$

where I = moment of inertia of the stanchion and $\frac{I_b}{l} \div \frac{I}{L} = \alpha$. If the crossbeam is very stiff, *i.e.* if $\alpha$ is great, this becomes—

$$\delta_C = \delta_B = \frac{WL^3}{6EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The bending-moment diagrams for the beam and one stanchion are shown in Fig. 243.

If the stanchions are of different lengths $L_1$ and $L_2$ or sections (I), the values of $H_1$ and $H_2$ may be found by equating $\delta_C$ and $\delta_B$ found as in (2), but the values of $i_C$ and $i_B$ will not be equal; they may be found from (12), Art. 103, by writing $M_C = H_1 . L_1$, $M_B = -H_2 . L_2$.

(*b*) *Bases fixed in Direction.*—If A and D, Fig. 243, are fixed in direction instead of hinged as shown, by symmetry again $H_1 = H_2 = \frac{1}{2}W$, and for the beam $M_B = -M_C$, $M_D = M_A$, $M_C = H_1 . L - M_D = \frac{1}{2}WL - M_D$, hence from (12), Art. 103,

$$E . I_b . i_C = \tfrac{1}{6}(\tfrac{1}{2}WL - M_D)l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and by integration or from (1), Art. 97—

$$E . I . i_C = M_D . L - \tfrac{1}{2}HL^2 = M_D . L - \tfrac{1}{4}WL^2 \quad . \quad (5)$$

hence, equating the values of $i_C$ from (4) and (5)—

$$M_D = M_A = \tfrac{1}{2}WL . \frac{3a + 1}{6a + 1}, \quad M_C = -M_B = \tfrac{1}{2}WL\left(\frac{3a}{6a + 1}\right) \quad (6)$$

At a distance $x$ from D,

$$M = M_D - \tfrac{1}{2}Wx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which vanishes for

$$x_0 = M_D \div \tfrac{1}{2}W = L(3a + 1) \div (6a + 1) \quad . \quad . \quad . \quad (8)$$

The bending moment on the stanchion will vary uniformly from $M_D$ at D to $M_D - \frac{1}{2}WL$ at C, passing through zero at $x_0$ from D. On CB it will vary uniformly from $\frac{1}{2}WL - M_D$ at C to the same value with opposite sign at B. The reader may as an exercise sketch the bending-moment diagrams and deformed shape of the structure.

If $a$ is great, *i.e.* the flexibility of the cross beam is negligible in comparison with that of the stanchions, (8) becomes $x_0 = \frac{1}{2}L$, and the bending moments at A, B, C, and D are each of magnitude $\frac{1}{4}WL$.

If $a$ is small $M_D = \frac{1}{3}WL$, $M_C = 0$, the case of hinges at the caps. If we write $L - x_0$ in this case in place of L in the previous one, the moments at the tops of the stanchions may be found; also the vertical reactions (equal and opposite) found by moments about the points of inflection will be

$$\frac{W}{l}(L - x_0) = \frac{WL}{l} \cdot \frac{3a}{6a + 1}$$

**169. Effect of Distributed Side Loads.**—In Arts. 165–168 the side loads were supposed taken at the ends of the windward stanchion. They may more frequently be more or less distributed along its length. To examine the effect of this, suppose the total load W uniformly distributed along the length of the windward stanchion.

*Stiff Cross Girder: Hinged Bases.*—Fig. 244 represents the same structure as Fig. 243, with a uniformly distributed load W on AB, but with a stiff girder, *i.e.* $\frac{I_b}{l} \div \frac{I}{L}$ or $a$ is great.

From (2), Art. 95, the deflection at C $= \frac{HL^3}{3EI}$, and from (7) and (2), Art. 95, the deflection at B $= \frac{(W-H)L^3}{3EI} - \frac{WL^3}{8EI}$, and equating these two we find—

$$H = \tfrac{5}{16}W, \qquad W - H = \tfrac{11}{16}W \quad . \quad . \quad . \quad (1)$$

The bending moments throughout the structure are then simply found, and are indicated by diagrams in Fig. 244.

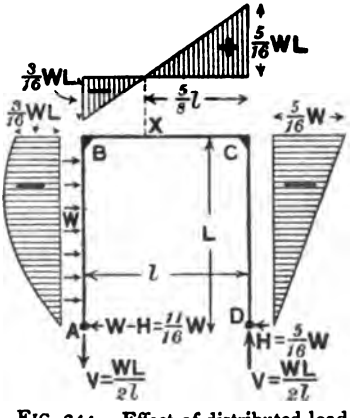



FIG. 244.—Effect of distributed load.

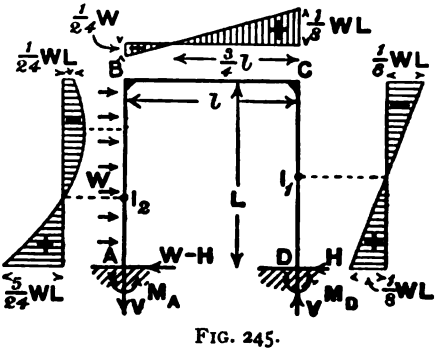



FIG. 245.

*Flexible Cross Girder: Hinged Bases.*—In this case, still using Fig. 244 from (12), Art. 103, with altered suffixes, writing for the beam

$$M = HL \text{ and } M_B = \tfrac{1}{2}WL - (W-H)L = L(H - \tfrac{1}{2}W)$$

we find—

$$i_B = Ll(W - 3H) \div 6EI_b, \quad i_C = Ll(6H - W) \div 12E.I_b \quad (2)$$

Then writing deflection at $C = \frac{HL^3}{3ET} + Li_C$ . . . . . (3)

Deflection at $B = \frac{(W-H)L^3}{3EI} - \frac{WL^3}{8EI} + Li_B$ . . (4)

and equating these deflections we find, putting as before $\frac{I_bL}{Il} = a$—

$$H = \frac{W}{8} \cdot \frac{5a+6}{(2a+3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which approaches the value (1) when $a$ is great, *i.e.* when the top girder is stiff, and approaches $\frac{1}{4}W$ when the stanchions are very stiff compared to the cross girder.

*Stiff Cross Girder: "Fixed" Bases* (Fig. 245).—Since DC remains vertical at D and C, from (1) and (2), Art. 95 (or from (1), Art. 97), $M_D = \frac{1}{2}HL$, hence $I_1D$ the distance from D to the point of inflection $= \frac{1}{2}L$. And similarly $M_A = \frac{1}{2}WL - \frac{1}{2}HL$.

The deflection at C from (2) and (11), Art. 95, is—

$$\frac{HL^3}{3EI} - \frac{M_DL^2}{2EI} = \frac{HL^3}{12EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

And the deflection at D is—

$$\frac{(W - H)L^3}{3EI} - \frac{M_AL^2}{2EI} - \frac{WL^3}{8EI} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

And equating (6) to (7)—

$$H = \tfrac{1}{4}W, \quad W - H = \tfrac{3}{4}W \quad . \quad . \quad . \quad . \quad . \quad (8)$$

At $x$ from A $$M = M_A + \tfrac{1}{2}\frac{W}{L} \cdot x^2 - \tfrac{3}{4}Wx,$$

which vanishes for $I_2 . A = x = 0.368L$ . . . . . (9)

The bending-moment diagrams are shown in Fig. 245.

*Flexible Cross Girder: "Fixed" Bases.*—Taking $M_A$ and $M_D$ or $M_B$ and $M_C$ as unknown quantities in addition to H, we can state the end slopes $i_C$ and $i_B$ for the stanchions from Art. 95 or Art. 97. The same slopes can also be deduced from (12), Art. 103, and equated to the previous values, thus eliminating $i_C$ and $i_B$. A third relation is obtained by equating the deflections of B and C and solving; we then obtain—

$$H = W(2a + 3) \div 8(a + 2) \quad . \quad . \quad . \quad . \quad (10)$$

If $a$ is great this reduces to $\frac{1}{4}W$ in agreement with (8), while if $a$ is small $H = \frac{3}{16} \cdot W$, the value for a beam hinged at B and C, as is shown by equating cap deflections $\frac{HL^3}{3EI}$ to $\frac{WL^3}{8EI} - \frac{HL^3}{3EI}$. Thus for all relative stiffnesses of the stanchions and beams H will lie between $\frac{1}{4}W$ and $\frac{3}{16}W$. The general expressions for $M_A$ and $M_D$ obtained as described are—

$$M_A = \frac{WL}{24} \cdot \frac{30a^2 + 73a + 15}{(1 + 6a)(a + 2)} \quad . \quad . \quad . \quad . \quad (11)$$

$$M_D = \frac{WL}{24} \cdot \frac{18a^2 + 35a + 9}{(1 + 6a)(a + 2)} \quad . \quad . \quad . \quad . \quad (12)$$

When $a$ is great $M_D$ becomes $\frac{1}{8}WL$ and $M_D$ becomes $\frac{5}{24}WL$ in accordance with the previous case and Fig. 245. The types of bending-moment diagrams will be as in Fig. 245, but the proportions will depend upon the relative stiffnesses of the parts.

The foregoing results with those of the previous article are summarised for comparison in the following table:—

TABLE OF BENDING MOMENTS AND REACTIONS ON STANCHIONS CONNECTED BY CAP BEAMS AND SUBJECT TO A SIDE LOAD W.

| Side load W | Condition of stanchion bases. | Cap beam. | $M_A$ | $M_D$ | $M_B$ | $M_C$ | H |
|---|---|---|---|---|---|---|---|
| Concentrated at B. | Fixed | Stiff | $\frac{1}{4}WL$ | $\frac{1}{4}WL$ | $-\frac{1}{4}WL$ | $-\frac{1}{4}WL$ | $\frac{1}{2}W$ |
| | Fixed | Flexible | $\frac{1}{2}WL\cdot\frac{3\alpha+1}{6\alpha+1}$ | $\frac{1}{2}WL\cdot\frac{3\alpha+1}{6\alpha+1}$ | $-\frac{1}{2}WL\cdot\frac{3\alpha}{6\alpha+1}$ | $-\frac{1}{2}WL\cdot\frac{3\alpha}{6\alpha+1}$ | $\frac{1}{2}W$ |
| | Hinged | Stiff | 0 | 0 | $-\frac{1}{2}WL$ | $-\frac{1}{2}WL$ | $\frac{1}{2}W$ |
| | Hinged | Flexible | 0 | 0 | $-\frac{1}{2}WL$ | $-\frac{1}{2}WL$ | $\frac{1}{2}W$ |
| | Fixed | Hinged | $\frac{1}{2}WL$ | $\frac{1}{2}WL$ | 0 | 0 | $\frac{1}{2}W$ |
| Distributed uniformly along AB. | Fixed | Stiff | $\frac{5}{24}WL$ | $\frac{1}{8}WL$ | $-\frac{1}{24}WL$ | $-\frac{1}{8}WL$ | $\frac{1}{4}W$ |
| | Fixed | Flexible | $\frac{WL}{24}\cdot\frac{30\alpha^2+73\alpha+15}{(1+6\alpha)(2+\alpha)}$ | $\frac{WL}{24}\cdot\frac{18\alpha^2+35\alpha+9}{(1+6\alpha)(2+\alpha)}$ | $-\frac{WL}{24}\cdot\frac{\alpha(6\alpha+23)}{(1+6\alpha)(2+\alpha)}$ | $-\frac{WL}{24}\cdot\frac{\alpha(18\alpha+25)}{(1+6\alpha)(2+\alpha)}$ | $W\cdot\frac{2\alpha+3}{8(\alpha+2)}$ |
| | Hinged | Stiff | 0 | 0 | $-\frac{3}{16}WL$ | $-\frac{5}{16}WL$ | $\frac{5}{16}WL$ |
| | Hinged | Flexible | 0 | 0 | $-\frac{3}{8}WL\cdot\frac{\alpha+2}{2\alpha+3}$ | $-\frac{WL}{8}\cdot\frac{5\alpha+6}{2\alpha+3}$ | $\frac{W}{8}\cdot\frac{5\alpha+6}{2\alpha+3}$ |
| | Fixed | Hinged | $\frac{5}{16}WL$ | $\frac{3}{16}WL$ | 0 | 0 | $\frac{3}{16}W$ |

The cases for stiff cap beams are derived from the cases for flexible beams by taking the limit for $\alpha = \infty$, and those for hinged cap beams by writing $\alpha = 0$, but these cases may be solved independently more easily than the general cases. The points A, B, C, D are those indicated in Figs. 243–245. H is the horizontal reaction on the leeward stanchion; $\alpha = \frac{I_b L}{I l'}$. In the case of a wind load P, say, write $W = \frac{1}{2}P$ in the first five cases, and $W = P$ in the last five cases.

*Partial Distribution.*—When the total load is transferred to the windward stanchion at a number of isolated points, as when a side wind load on the wall or sheeting is carried by rails attached to the stanchions, the results will be intermediate between those of the first five lines of the above table taking $W = \frac{1}{2}$ total load, and those of the last five lines taking W for the whole load. For example, taking the structure in the last line of the table, if the load is carried at the base, the top, and a point midway between, the total load W would be divided (see Art. 131) into $\frac{1}{4}W$ at the top, $\frac{1}{4}W$ at the base, and $\frac{1}{2}W$ halfway along the stanchion. Hence, equating the deflections of the stanchion tops, using Art. 95—

$$\frac{1}{3}\cdot\frac{HL^3}{EI} = \frac{W}{4}\cdot\frac{L^3}{3EI} + \frac{W}{2}\cdot\left(\frac{L}{2}\right)^3\frac{1}{3EI} + \frac{W}{2}\left(\frac{L}{2}\right)^3\frac{1}{2EI} - \frac{1}{3}\cdot\frac{HL^3}{EI}$$

hence $H = \frac{13}{64}W$, whereas for uniform distribution we had $H = \frac{3}{16}W$. Thus for a single intermediate wall support the value of H falls from $\frac{16}{64}W$ to $\frac{13}{64}W$ where the limit for a large number of rails is only $\frac{12}{64}W$. For several concentrated horizontal forces $F_1$, $F_2$, $F_3$, etc., on the windward stanchion at distances $n_1L$, $n_2L$, $n_3L$, etc., from the base we should have, from Art. 95—

$$\frac{1}{3}\cdot\frac{HL^3}{EI} = \frac{L^3}{3EI}\Sigma(Fn^2) + \frac{L^3}{2EI}\Sigma\{Fn^2(1-n)\} - \frac{HL^3}{3EI}$$

or
$$H = \tfrac{1}{2}\Sigma(Fn^2) + \tfrac{3}{4}\Sigma\{Fn^2(1-n)\}$$

of which the above is a particular case in which $F_1 = \frac{1}{4}W$, $n_1 = 1$, $F_2 = \frac{1}{2}W$, $n_2 = \frac{1}{2}$. Similar rules may be framed for the other types of support, but the extreme cases of concentration at the two ends and complete distribution will be sufficient guide for estimation of H in any case.

*Stiff Open Bracing: Hinged Bases.*—The effect of uniform distribution of the load on the structure of the type shown in Figs. 239 to 242 may be briefly noticed. Using the notation of Figs. 239, 241, and 242, we assume that BC is a braced girder so stiff as to resist flexure so that points B and K remain in the same vertical, and likewise C and F, although the portions BK and CF bend. This enables the slopes at K and F to be written in terms of H and W, where H is the horizontal reaction on the leeward side. Then equating the deflections of K and F say, *i.e.* neglecting the strains in the bracing in comparison with flexure, we find—

$$H = \frac{W}{16}\cdot\frac{L^2 + 5Lh - h^2}{Lh} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The transverse pull of the bracing at the foot of the knee bracing (point K, Fig. 239) on the windward stanchion is found by moments about the hinged top, and is $(\frac{1}{2}W - H)\frac{L}{L-h}$, and subtracting this from H gives the thrust at the top. The transverse forces on the leeward stanchion are as in Fig. 239 (*c*), writing H for $H_1$. If *h* approaches L, H approaches its lowest value, which is $\frac{5}{16}W$, in agreement with (1).

*Stiff Open Bracing: "Fixed" Bases.*—By similar methods for uniformly distributed loads, using Art. 95 for the transverse forces on the stanchions, taking H the horizontal reaction on the leeward stanchion and P′ the horizontal pull of the foot of the bracing on the windward stanchion as unknown quantities, and equating all the deflections at B, K, C, and F, we find

$$H = \frac{W}{8} \cdot \frac{3L^3 + 7L^2h - 5Lh^2 + h^3}{Lh(4L - h)} \quad \ldots\ldots\ldots \quad (14)$$

which of course approaches the value $H = \frac{1}{4}W$ in agreement with (8) when $h$ approaches L.

$$P' = \frac{W}{16} \cdot \frac{6L^5 - 4L^4h - 17L^3h^2 + 15L^2h^3 - 7Lh^4 + h^5}{Lh(L + 2h)(4L - h)(L - h)} \quad . \quad (15)$$

The thrust of the foot of the bracing on the leeward stanchion will be as given by P in (2) Art. 166, the value (14) being used for H. With H, P, and P′ known, all the other stresses in the bracings and stanchions are easily determined. Perhaps the best method for the bracing would be to find the point of inflection and then use the graphical solution given in Art. 167.

The assumption of a rigid bracing in comparison with the flexibility of the stanchions would of course only be reasonable for moderately large values of $h$ compared to L. For ordinary values of $h$/L, H is much less than $\frac{1}{2}W$, *i.e.* the windward stanchion carries much more than half the load.

**170. Wind Stresses in more Complex Structures.**—The methods of Arts. 168 and 169 may be applied to the analysis of the stresses in more complex frames, but where more members are introduced the solution becomes more complex and practically too difficult. An example will suffice to illustrate the application to more complex forms,

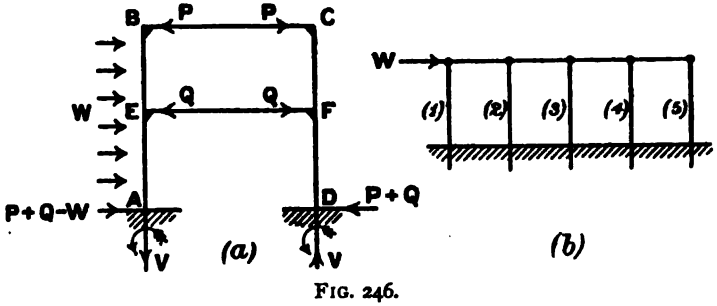



FIG. 246.

which by different combinations of hinged and rigid attachments might be greatly extended. Fig. 246 (*a*) represents two stanchions AB and CD fixed in direction at their bases A and D, and connected by two cross-beams BC and EF rigidly attached to the stanchions. Suppose a load W either concentrated or distributed to act on AB. Let P and Q represent the thrust (positive or negative) in BC and EF respectively.

Then selecting as unknown quantities P, Q, and the bending moments at B, C, F, and E, we can form six equations to find these unknown quantities as follows: $i_B$ and $i_E$ for the stanchion may be written from Art. 95. Assuming rigid connections, these may be equated to $i_B$ and $i_E$ for the beam written from (13), Art. 103. Two similar equations may be written for C and F, and finally two more may be formed by equating the deflection at B to that at C, and the deflection at E to that at F, all the four deflections being written from Art. 95.

Steel building frames of several stories are too complex for exact calculation, and the connections do not warrant the assumption of rigid joints, and various empirical approximations have to be used. Fig. 246 (*b*) represents say $(n-1)$ portals, consisting of $n$ stanchions connected by beams across their caps. If F is the transverse horizontal force taken by any stanchion, then its deflection is $\frac{FL^3}{3EI}$, which must be the same for each if the load W is concentrated at the windward stanchion cap; hence for equal lengths $\frac{F}{I}$ must be the same for each and the sum of the thrusts F balances W; hence for the $m$th stanchion

$$F = \frac{WI_m}{I_1 + I_2 + I_3 + \ldots + I_n} \quad \ldots \quad (1)$$

where the suffixes denote the various stanchions. If all the stanchions are similar $F = \frac{1}{n}W$.

If the load W is distributed uniformly over the outer stanchion and all the stanchions are similar, the remaining stanchions all take a horizontal thrust H, and equating the deflections of the windward and any other stanchion

$$\frac{WL^3}{8EI} - \frac{(n-1)HL^3}{3EI} = \frac{HL^3}{3EI} \quad \text{hence } H = \frac{3}{8n}.W. \quad (2)$$

or $\frac{3}{4}\cdot\frac{W}{2n}$, *i.e.* $\frac{3}{4}$ of the load which would be borne by the stanchions if $\frac{1}{2}W$ is transferred to the windward stanchion cap.

If the load W is uniformly distributed let H be the thrust in the first beam, then

$$\frac{WL^3}{8EI_1} - \frac{HL^3}{3EI_1} = \frac{HL^3}{3E(I_2 + I_3 + I_4 + \ldots + I_n)} \quad (3)$$

Hence

$$H = \tfrac{3}{8}W.\frac{I_2 + I_3 + I_4 + \ldots + I_n}{I_1 + I_2 + I_3 + \ldots + I_n} \text{ or } \tfrac{3}{8}W\Sigma_n^2(I) \div \Sigma_n^1(I) \quad (4)$$

the horizontal force taken by any, say the $m$th, stanchion from (1) being

$$F = H \times I_m \div \Sigma_n^2(I) = \tfrac{3}{8}WI_m \div \Sigma_n^1(I) \quad \ldots \quad (5)$$

which gives the value (2) in the case of similar stanchions.

**171. Applications to Steel Buildings.**— In applying the analysis in the foregoing articles to the estimation of wind stresses in the

stanchions and roofs of steel buildings, it is difficult to know how far any particular condition of fixture or rigidity will be realized. For the purpose of design some rough estimate is made upon the safest assumption, but without a thorough understanding of the problem it is dangerous to make simple assumptions which, while being on the side of safety for one element of the structure, may be quite unsafe for another. Thus the assumption that the roof is rigid to compression between its supports is on the safe side for the estimation of the leeward stanchion stresses (which are often greater than those on the windward stanchion); it is on the wrong side for estimation of the windward stanchion bending moments; for to take the other extreme, if the roof offers little resistance to horizontal compression, lacking other connections, the windward stanchion will carry nearly the whole wind load. Again, the condition of "fixed" direction at the base of a stanchion depends upon anchorage or fastening at the foot (see Art. 185), and upon an absolutely rigid foundation, probably it is never *fully* attained. This fact is equivalent to a lowering of the point of inflection and consequent increase of the bending moment at the foot of a knee brace (see (c), Fig. 240), or at the rigid cap connection (see Fig. 245); but to assume it would be to presume a diminished moment at the base (see (c), Fig. 240) and might be unsafe. Thus to assume in Fig. 240 that $I_2$ is lowered to halfway between F and D (a common recommendation) is for the purpose of computing the maximum moment on the stanchion to assume the most favourable possible conditions, *i.e.* to make the *least* safe assumption. To assume the base hinged ($I_2$ lowered to D) is the safest assumption (see Fig. 239 (d)), but may be unnecessarily wasteful in material.

*Roof attached to Stanchion Caps only.*—If there is nothing amounting to a knee brace the maximum bending moment on the stanchions due to wind load, assuming a stiff roof, will be given by the table in Art. 169, half the horizontal roof load being assumed to be taken at the end of each stanchion and the transverse loads on the windward stanchion according to circumstances. The most severe condition is the hinged base and rigid top, the maximum value occurring at the top of the leeward stanchion $M_C = -L \times (\frac{1}{2}$ horizontal roof load $+ \frac{5}{16}$ wall load). With regard to the stresses in the roof bracing, these may be determined as in Arts. 133, 134, 138. The effect of the wind load on the walls is to bring a thrust H horizontally (transmitted to the leeward stanchion) upon the roof; the effect of this is evidently opposite to that of the normal wind and the vertical loads upon the roof, *i.e.* it will reduce the wind and dead load stresses. Unless the conditions were such as to cause a possible reversal of stress in the ties this effect would generally be neglected. If taken into account with the wind load the *least* value of H should be chosen as external horizontal thrusts at B and C. There will of course, with rigid attachments, be some bending moment on the main rafters.

*Kneebraced Roof.*—The wind stresses on the kneebraces may be estimated as explained in Arts. 165 and 166, taking the horizontal component of the wind load as divided between the two stanchions. In

addition there are the stresses similarly determined from the distributed wind load on the stanchions by using the values of H, P′, and P given by (13), (14), and (15) in Art. 169. It will generally be nearly correct to assume that half the horizontal wind force on the roof is concentrated at the top of the windward stanchion, and to apportion the side load as explained in Art. 131 at the top of the stanchion and the foot of the kneebrace. The vertical components of the reactions at the feet or at the tops of the stanchions are easily calculated by moments after the points of inflection have been determined. When the stress in a kneebrace has been calculated it is easy to draw the complete stress diagram, remembering that the stress in the stanchion may be split into a vertical thrust and a horizontal shearing force.

*Graphical Method.*—The method given in Art. 167 may be applied for finding the wind stresses in a kneebraced roof if the stanchion bases are assumed to be hinged or if the points of inflection are calculated or assumed. Thus if the loads in Fig. 247 are known the external force diagram 1234567 may be set out. The reactions 7, 18 and 18, 1 at the base of the stanchions may be found in various ways subject to the condition that their horizontal components are equal, *e.g.* one vertical component may be calculated by moments about the hinged base of the other stanchion. Then the addition of the fictitious dotted members 1, 27, 27, 26 and 2, 26, etc., enables the complete stress diagram to be drawn and the stresses in the internal bracing to be found. The lower stress diagram in Fig. 247 represents the case of flexible braces so that the stress in 18–10 is zero, that in 18–25 being thereby greatly increased. Member 7–18 the leeward stanchion of course then exerts only a vertical force 7–18 at its cap.

Fig. 248 shows the process for the same roof and loads when the stanchions are "fixed" at their bases, the points of inflection being found by (3) Art. 166, or assumed to be at half the height of the caps.

In Fig. 247 the reactions are determined graphically by setting off the resultant wind pressure at QS to intersect the frame centre line at P, joining P to A, meeting a vertical through Q in N; then a horizontal projection of N on the centre line gives T such that TQ gives the reaction at D and ST that at A. By resolving QS into vertical and horizontal components at P, the proof of the construction will become obvious, for the former gives equal components at A and D, and the latter gives the equal horizontal plus equal and opposite vertical components or reactions directed through the hinges toward P.

EXAMPLE 1.—A French roof truss of 60 feet span, 15 feet rise and no camber of the ties (Fig. 247), is attached to stanchions 20 feet high and braced as shown, the kneebraces meeting the stanchions 4·75 feet below the caps, and the principals are 10 feet apart. Find the wind stresses in the kneebraces and remaining members, and the maximum bending moment in the leeward stanchion, with a horizontal wind pressure estimated at 50 lbs. per square foot.

Wind pressure normal to the slope ((7) Art. 128) is $50 \times \frac{26{\cdot}5}{45}$ = 29·41 lbs. per square foot.

Wind load per node

$$= \tfrac{1}{4} \times \frac{30 \times \sqrt{5} \times 10}{2} \times 29{\cdot}4 = 2475 \text{ lbs.} = 1{\cdot}10 \text{ ton, say.}$$

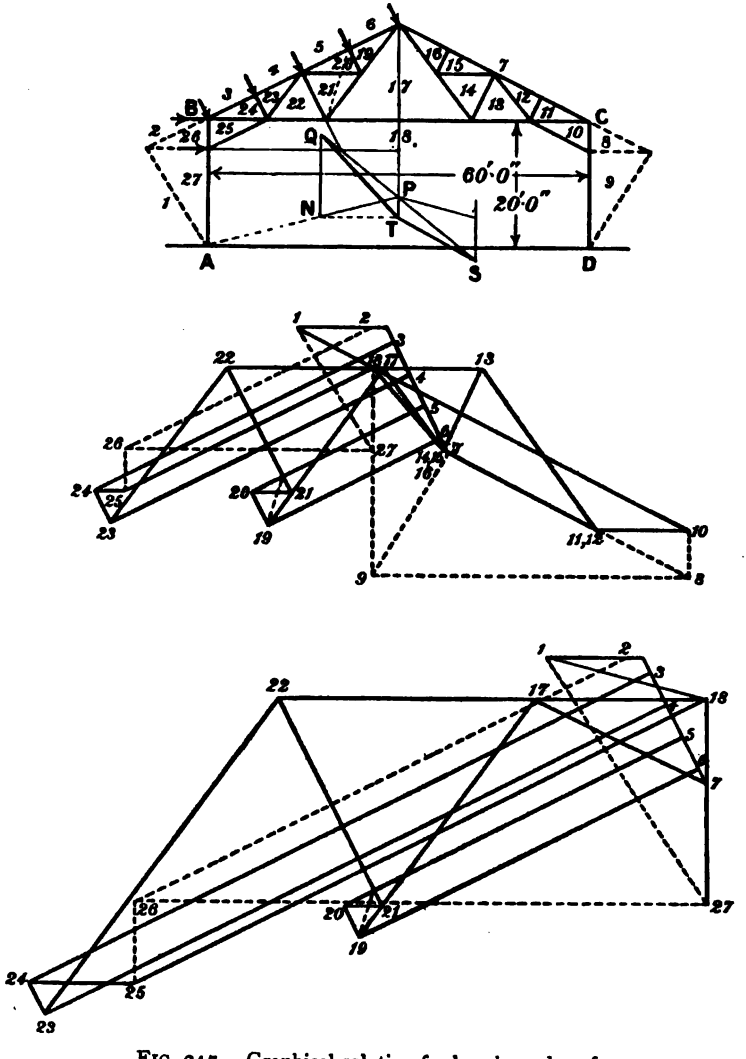



FIG. 247.—Graphical solution for kneebraced roof.

Horizontal pressure on wall at eaves

$$= \frac{4{\cdot}75}{2} \times 10 \times 50 = 1187 \text{ lbs.} = 0{\cdot}53 \text{ ton.}$$

Horizontal pressure on wall at foot of brace

$$= 10 \times 10 \times 50 = 5000 \text{ lbs.} = 2{\cdot}24 \text{ tons.}$$

By moments about A, vertical reaction at D,

$$= \tfrac{1}{60}\left\{(4{\cdot}48 \times 10) + \left(4{\cdot}40 \times \frac{2}{\sqrt{5}} \times 15\right) + \left(4{\cdot}40 \times \frac{1}{\sqrt{5}} \times 27{\cdot}5\right)\right\}$$

$$= 2{\cdot}632 \text{ tons.}$$

Hence the point 18 is determined after 1, 2, 3, 4, 5, 6, 7, have been set off in Fig. 247, for it lies 2·632 tons above the point 7 and halfway horizontally to 1, and the remainder of the diagram is drawn as shown. To check the kneebrace stresses, the distance of the brace 18–25 from B is measured or calculated as 4·28 feet. Omitting the force carried directly at A, total horizontal force $= \frac{4{\cdot}40}{\sqrt{5}} + 0{\cdot}53 + 2{\cdot}24 = 4{\cdot}74$ tons, and taking half this or 2·37 tons as horizontal reaction at A and D, by a section through B and moments about B, tension in the windward brace (18–25)

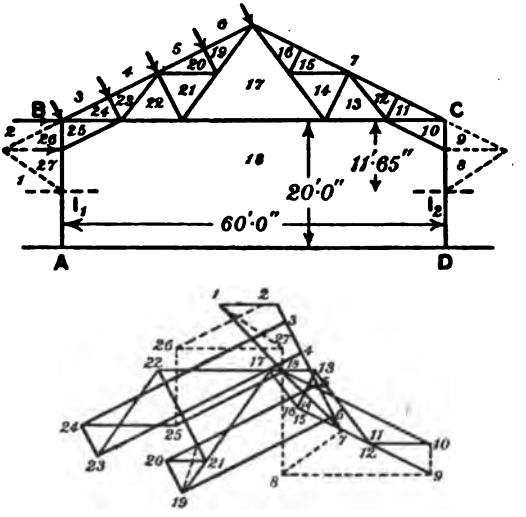



FIG. 248.—Case of "fixed" bases.

$$= \frac{1}{4{\cdot}28}(2{\cdot}37 \times 20 - 2{\cdot}24 \times 4{\cdot}75) = 8{\cdot}6 \text{ tons.}$$

And similarly by moments about C, thrust in the leeward brace (18–10)

$$= \frac{1}{4{\cdot}28}(2{\cdot}37 \times 20) = 11{\cdot}1 \text{ tons.}$$

Checking the centre tie 17–18, by moments about the vertex, of forces on the right-hand half of the roof, the tension in the tie is

$$\frac{1}{15}(2{\cdot}632 \times 30 - 2{\cdot}37 \times 35) = -0{\cdot}27 \text{ ton,}$$

*i.e.* 0·27 ton thrust as in the diagram. This result is partly due of course to the flexibility of the standards, diminishing the effect of the normal wind pressure on the slope which produces tension; for wind stresses in the roof bracing it would be safer to assume rigid supports. Actually the structure is indeterminate, and computations taking account of the

elasticity of the roof truss and stanchions would give an intermediate result. The maximum bending moment on the leeward stanchion is $2{\cdot}37 \times 15{\cdot}25 = 36{\cdot}1$ tons-feet

We may briefly examine the effect of complete distribution (uniform) of the horizontal wind pressure. In (13) Art. 169 we have $h = 15{\cdot}25'$ $L = 20'$, for the horizontal wind pressure $W = 4{\cdot}48$ tons, hence $H = 0{\cdot}56 \times 4{\cdot}48 = 2{\cdot}51$ tons; adding half the horizontal roof pressure $\frac{1}{2} \times \frac{4{\cdot}40}{\sqrt{5}}$, *i.e.* 0·99 ton gives a leeward reaction of $2{\cdot}51 + 0{\cdot}99 = 3{\cdot}5$ tons instead of 2·37 tons. The maximum bending moment on the leeward stanchion will be proportionally increased to $3{\cdot}5 \times 15{\cdot}25 = 53{\cdot}5$ ton-feet, and the thrust in the leeward kneebrace would be

$$\frac{1}{4{\cdot}28}(3{\cdot}5 \times 20) = 16{\cdot}3 \text{ tons or nearly 50 per cent. greater.}$$

The tension in the windward kneebrace will be reduced by distribution of the pressure to

$$\frac{1}{4{\cdot}28}\{(4{\cdot}48 - 2{\cdot}51 + 0{\cdot}99) \times 20 - (4{\cdot}48 \times 10)\} = 3{\cdot}4 \text{ tons.}$$

With flexible braces incapable of taking thrust this would be

$$\frac{1}{4{\cdot}28}\left\{\left(4{\cdot}48 + \frac{4{\cdot}40}{\sqrt{5}}\right)20 - 4{\cdot}48 \times 10\right\} = 19{\cdot}7 \text{ tons.}$$

EXAMPLE 2.—Find the stresses in Ex. 1 if the stanchions are fixed in direction at their bases.

For an approximate graphical solution it is assumed that the heights of the points of inflection are in both stanchions given by (3), Art. 166, viz.

$$\frac{15{\cdot}25}{2} \times \frac{40 + 15{\cdot}25}{50{\cdot}5} = 8{\cdot}35 \text{ ft.} = A{\cdot}I_1$$

Pressure at eaves = 0·53 ton as before.

Pressure at foot of kneebrace = $10 \times \frac{1}{2}(20 - 8{\cdot}35)\,50 = 2915$ lbs. = 1·3 tons.

By moments about $I_1$ the vertical leeward reaction is

$$\frac{1}{60}\left\{\left(2{\cdot}6 \times \frac{11{\cdot}65}{2}\right) + \left(4{\cdot}40 \times \frac{2}{\sqrt{5}} \times 15\right) + \left(4{\cdot}40 \times \frac{1}{\sqrt{5}} \times 19{\cdot}15\right)\right\} = \begin{cases} 1{\cdot}86 \\ \text{tons.} \end{cases}$$

from which the point 18 is found, allowing half the effective horizontal forces, *i.e.* loads above the level $I_1I_2$ in each reaction, viz.

$$\frac{1}{2}\left(1{\cdot}3 + 0{\cdot}53 + \frac{4{\cdot}40}{\sqrt{5}}\right) = 1{\cdot}9 \text{ ton.}$$

Checking the thrust in the leeward kneebrace 18–10, by moments about the cap, thrust

$$= \frac{1}{4{\cdot}28}(11{\cdot}65 \times 1{\cdot}9) = 5{\cdot}3 \text{ tons.}$$

Examining the effect of uniform distribution of the load on the

stanchion from (14) Art. 169, the horizontal wind load transmitted to the leeward stanchion base is, $0{\cdot}2935 \times 4{\cdot}48 = 1{\cdot}315$ ton, and adding half the horizontal roof pressure gives $1{\cdot}315 + 0{\cdot}99 = 2{\cdot}305$ tons. Hence from (1) and (2) Art. 166 the height of $I_2$ is as before 8·35 ft. and the thrust in the leeward kneebrace is proportionally increased to

$$\frac{1}{4{\cdot}28}(11{\cdot}65 \times 2{\cdot}305) = 6{\cdot}28 \text{ tons.}$$

The height of $I_2$ is slightly modified by considering the load as distributed.

**172. Vertical Loads on Rectangular Frames.**—The bending stresses in a stanchion due to vertical loads on horizontal cross beams can be found by the same principles as were used in Arts. 168 and 169. The maximum stresses may be estimated in accordance with Art. 120, combining all the bending moments and the total thrust. Sometimes the bending stresses in stanchions are so great that the direct thrust may not be of very great importance.

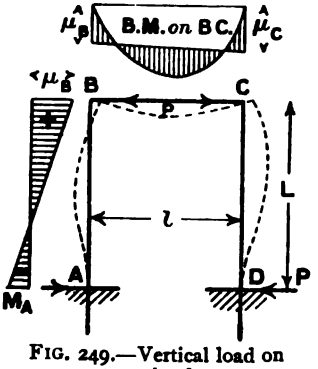



FIG. 249.—Vertical load on rectangular frame.

*Stanchions with Caps connected by Cross Beam. Bases "Fixed"* (Fig. 249). —Let the load on the horizontal beam BC of length $l$ rigidly attached to the stanchion caps B and C be such that it would give a bending moment diagram of area A with centroid distant $\bar{x}$ from B (see Art. 97, section (*c*)) if BC were freely supported at its ends. Let $\mu$ stand for the positive bending moments on the beam, then the clockwise couples applied to the left and right of the stanchions are $\mu_B$ and $-\mu_C$ respectively. Neglecting the longitudinal compressibility of the beam, the deflections of B and C are equal, or if P = thrust in the beam and I = moment of inertia of the stanchion section, from Art. 95,

$$\frac{\mu_B L^2}{2EI} - \frac{PL^3}{3EI} = -\frac{\mu_C L^2}{2EI} + \frac{PL^3}{3EI} \quad \ldots \quad (1)$$

hence

$$\mu_B + \mu_C = \tfrac{2}{3}PL \quad \ldots \quad (2)$$

Also

$$i_B = \frac{\mu_B L}{EI} - \frac{PL^2}{2EI} \qquad i_C = -\frac{\mu_C L}{EI} + \frac{PL^2}{2EI} \quad \ldots \quad (3)$$

$$i_B - i_C = (\mu_B + \mu_C)\frac{L}{EI} - \frac{PL^2}{EI} \quad \ldots \quad (4)$$

But if $I_b$ = moment of inertia of the beam section from (13*a*) Art. 103,

$$i_B - i_C = -\frac{A}{EI_b} - (\mu_B + \mu_C)\frac{l}{2EI_b} \quad \ldots \quad (5)$$

and equating (4) and (5),

$$P = -\frac{A}{Ll} \cdot \frac{3}{\alpha + 2} \quad \ldots \quad (6)$$

where $\alpha$ is the ratio $\frac{I_b}{l}$ to $\frac{I}{L}$ or $\frac{I_b L}{Il}$. Hence from (3) above, and (13) Art. 103,

$$\mu_B = -\frac{A}{l}\left\{\frac{8 + 15\alpha - \frac{6\bar{x}}{l}(\alpha + 2)}{(6\alpha + 1)(\alpha + 2)}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

$$\mu_C = -\frac{A}{l}\left\{\frac{\frac{6\bar{x}}{l}(\alpha + 2) - 4 + 9\alpha}{(6\alpha + 1)(\alpha + 2)}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The bending moment diagrams are shown in Fig. 249.

If $\alpha$ is small, *i.e.* the stanchions are very rigid compared to the beam, putting $\alpha = 0$, these reduce to the values (6) and (7) of Art. 103 for the built-in beam. If the loading is symmetrical

$$\bar{x} = \tfrac{1}{2}l, \ \mu_B = \mu_C = -\frac{2A}{l(\alpha + 2)} \quad . \quad . \quad . \quad . \quad (9)$$

$M_A = \mu_B - PL$, which is equal to $\frac{1}{2}\mu_B$ for symmetrical loading, the point of inflection being $\frac{1}{3}$ L from A as in (34) Art. 121, to which the stanchion reduces for symmetrical loading.

*Hinged Bases.*—Using the letters of Fig. 249, in this case $\mu_A = \mu_D = 0$, $\mu_B = PL = \mu_C$; the deflections at B and C are

$$L \cdot i_B - \frac{PL^3}{3EI} \text{ and } L \cdot i_C + \frac{PL^3}{3EI} \quad . \quad . \quad . \quad . \quad (10)$$

hence,
$$i_B - i_C = \frac{2PL^2}{3EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and equating this to (5) gives
$$P = -\frac{A}{lL} \cdot \frac{3}{3 + 2\alpha} \quad . \quad . \quad . \quad . \quad (12)$$

and
$$\mu_B = \mu_C = -\frac{A}{l} \cdot \frac{3}{3 + 2\alpha} \quad . \quad . \quad . \quad . \quad (13)$$

This might alternatively be solved by integrating equation (2) Art. 93 for each stanchion under the conditions that the deflections are zero at the bases, and equal at the caps, and that the slopes at the caps are given by equation (3) when $\mu_B = \mu_C = PL$. These five conditions determine the four constants of integration and give an equation to find P.

EXAMPLE 1.—Uniformly distributed load W on the cross girder, bases fixed in direction; $\frac{I_b \cdot L}{I \cdot l} = \alpha = 1$, say.

$A = -\frac{2}{3} \cdot \frac{Wl^2}{8} = -\frac{Wl^2}{12}$; hence from (9) $\mu_B = \mu_C = \frac{WL}{18}$. If $\alpha = 0$ the value is $\frac{WL}{12}$ (see Fig. 152).

EXAMPLE 2.—On the same structure a load W at $\frac{1}{4}l$ from the left-hand stanchion (B).

$A = -\frac{3}{32}Wl^2$, $\bar{x} = \frac{5}{12}l$, hence from (7) and (8)

$$\mu_B = 0{\cdot}0692Wl. \qquad \mu_C = 0{\cdot}0558Wl.$$

The bending moment under the load is $\frac{3}{16}Wl - \mu_C - \frac{3}{4}(\mu_B - \mu_C)$ $= 0{\cdot}1216Wl.$

If $a = 0$ (*i.e.* very stiff stanchions), $\mu_B = \frac{9}{64}Wl$, $\mu_C = \frac{3}{64}Wl$, as in Ex. 1, Art. 103.

**173. More Complex Rectangular Frames.**—More complex structures such as shown in Fig. 250 may be solved by the principles of the previous article, but the unknown quantities and equations to find them become more numerous. We may briefly indicate the methods for Fig. 250 and similar structures with rigidly attached cross beams BC and KF.

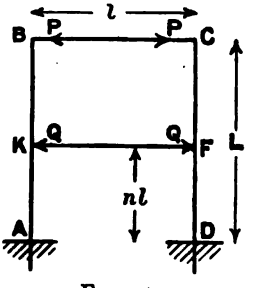


FIG. 250.

*Case I.*—A and D "fixed," B and C free except for rigid attachment to the girder. Unknown quantities, moments $\mu_B$, $\mu_C$, $\mu_K$, and $\mu_F$, and thrusts P and Q; $i_B$ for stanchion (see Art. 95) $= i_B$ for beam BC (see Art. 103). Three similar equations for points K, C, and F. Deflections at K and B = deflections at F and C (see Art. 95 for the values).

In case of symmetry, only the equations of slopes at B and K together with deflections at B and K equated to zero are required.

In this case the equations reduce to

$$\mu_B = -\frac{A_1}{l} - \frac{2a_1(1-n)}{4-n}(\mu_B - \tfrac{1}{2}n\mu_K)$$

$$\mu_K = -\frac{A_2}{l} - \frac{2a_2(1-n)}{4-n}(\mu_K - \tfrac{1}{2}\mu_B)$$

where $A_2$ and $a_2$ refer to the bending-moment diagram, and ratio $\frac{I_b L}{Il}$ for the girder KF.

EXAMPLE.—If $a_1 = 1 = a_2$ and $n = \frac{1}{2}$ with a central load W on the girder BC only,

$$\mu_B = \tfrac{112}{1148}Wl. \qquad \mu_K = \tfrac{126}{1148}Wl.$$

With a central load on the girder KF only,

$$\mu_B = \tfrac{7}{1148}Wl. \qquad \mu_K = \tfrac{126}{1148}Wl.$$

*Case II.*—A and D hinged, B and C as before. The horizontal reactions being P and Q, and equation of moments about B and C gives $\mu_B$ and $\mu_C$ in terms of the four unknown quantities $\mu_K$, $\mu_F$, P, and Q. Then equating the deflections at B and C and at K and F as found from $i_B$ and $i_C$ for the girder BC, (13), Art. 103, and again from $i_K$ and $i_F$ for the girder KF, give the equations required. For symmetrical loading

it is only necessary to write the deflections at B and K equal to zero. In this case the equations reduce to—

$$\mu_B = -\frac{A_1}{l} - \frac{(1-n)a_1}{6}\{(3+n)\mu_B - 2n\mu_K\}$$

$$\mu_K = -\frac{A_2}{l} - \frac{n(1-n)a_2}{3}(2\mu_K - \mu_B)$$

EXAMPLE.—If $a_1 = a_2 = 1$ and $n = \frac{1}{2}$, for a central load W on BC only—

$$\mu_B = \tfrac{7}{36}WL. \qquad \mu_K = \tfrac{1}{72}WL.$$

For a central load W on KF only—

$$\mu_B = \tfrac{1}{72}WL. \qquad \mu_K = \tfrac{31}{144}WL.$$

*Other Cases.*—A, B, C, and D all fixed vertically, A, B, C, and D all hinged, A and D fixed vertically, B and C hinged, all form interesting cases with possible applications, and may be worked out on similar lines to those given.

In frames consisting of two stanchions with several equally spaced cross girders symmetrically loaded, points of inflection in the stanchions would fall approximately midway between the girders, and a single storey would reduce to the case of A, B, C, D all hinged and $n = \frac{1}{2}$, where L is the distance between successive girders. If only a single cross girder of the series were loaded, the length L might be taken as twice the distance between successive cross girders.

**174. Secondary Stresses.**[1]—Stresses calculated upon the supposition that frame joints are frictionless pins the axes of which are situated exactly at the intersection of all the elastic lines of the members meeting in each joint, may be called primary stresses, and are first approximations to the stresses in the members of a frame. Actual frames differ materially from ideal conditions (1) in having either riveted joints, or pin joints which are far from frictionless; (2) in having members the elastic lines of which at some particular joints do not meet in a point, the members being thereby subjected to eccentric pulls or thrusts. Frames with riveted joints are really statically indeterminate, but second approximations to the stresses in such frames may be calculated after the primary stresses, and the sections are known by estimating approximately the *secondary stresses*, *i.e.* the stresses produced by deviations from the above ideal conditions. Rigidity of the joints will also considerably modify the deflections (see Art. 155) calculated on the assumption of frictionless joints. Any full treatment of the computation of secondary stresses is necessarily lengthy and beyond the scope of this volume, but secondary stresses are receiving increased attention, and an elementary insight into the principles involved in their estimation may be instructive.

*Stresses arising from Rigidity of the Joints.*—This is perhaps the most important type of secondary stress. If we assume that, instead of being free to turn at their ends, frame members are rigidly held in the same *relative* angular positions at the joints although the joints may have small

[1] For a much fuller treatment of secondary stresses see "Secondary Stresses in Bridge Trusses," by C. R. Grimme.

angular movement due to the strain of the frame, we can estimate the fixing couples at the ends of the members, and hence the secondary bending stresses resulting from the lack of free angular movement.

Thus, for example, a triangular frame ABC (Fig. 251) supports a vertical load at A, the joints at A, B, and C being rigid. To simplify

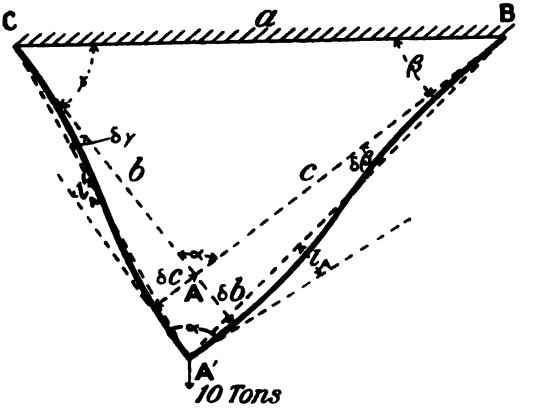



FIG. 251.

the problem, suppose BC is infinitely stiff or that B and C are rigidly fixed to rigid supports. Let AB = $c$, BC = $a$, AC = $b$, and let the primary unit tensile stresses in $b$ and $c$ as found by simple statics be $p_b$ and $p_c$ respectively, so that the extension of $b$ and $c$ are

$$\delta b = b \times p_b \div \mathrm{E}, \quad \delta c = c \times \frac{p_c}{\mathrm{E}}$$

where E is Young's modulus for the material.

Let A' found by arcs of radii $b + \delta b$ and $c + \delta c$ about C and B respectively be the new positions of A after strain shown much exaggerated in Fig. 251. The angles $\alpha$, $\beta$, $\gamma$ at A, B, and C remain unchanged, and in consequence the members are bent, the tangents to the members at A', B, and C being inclined to the sides of the triangle A'BC. Let $\alpha + \delta\alpha$, $\beta + \delta\beta$, and $\gamma + \delta\gamma$ be the angles at A', B, and C of the triangle A'BC. Then, knowing $\delta b$ and $\delta c$, the total increases, $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ are easily calculated geometrically from a diagram or by differentiation of the relation

$$\cos\beta = (a^2 + c^2 - b^2) \div 2ac \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For if only $b$ varies, differentiating with respect to $b$,

$$-\sin\beta \frac{d\beta}{db} = -\frac{b}{ac}$$

hence partially,

$$\delta\beta = \frac{b\delta b}{ac \sin\beta} = \frac{\delta b}{b} \cdot \frac{\sin\beta}{\sin\alpha \sin\gamma} = \frac{p^b}{\mathrm{E}}(\cot\alpha + \cot\gamma) \quad . \quad (2)$$

And if $c$ varies alone, differentiating (1) with respect to $c$—

$$-\sin\beta\frac{d\beta}{dc}=\frac{c^2-a^2+b^2}{2ac^2}=\frac{b}{ac}\cos\alpha$$

hence partially

$$\delta\beta=-\frac{\delta c}{c}\cot\alpha=-\frac{p_c}{E}\cot\alpha \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And from (2) and (3) the total variation is

$$\delta\beta=\frac{1}{E}\{p_b(\cot\alpha+\cot\gamma)-p_c\cot\alpha\} \quad . \quad . \quad . \quad (4)$$

a similar value holding for $\delta\gamma$—

$$\delta\gamma=\frac{1}{E}\{p_c(\cot\alpha+\cot\beta)-p_b\cot\alpha\} \quad . \quad . \quad . \quad (5)$$

while from (3), with the necessary modification for $\alpha$—

$$\delta\alpha=-\frac{1}{E}(p_c\cot\beta+p_b\cot\gamma) \quad . \quad . \quad . \quad . \quad (6)$$

If $a$ varies, the modifications in (4), (5), and (6) are easily made, *e.g.*

$$\delta\alpha=\frac{1}{E}\{(p_a-p_c)\cot\beta+(p_a-p_b)\cot\gamma\}$$

similar values holding for $\delta\beta$ and $\delta\gamma$.

We may now write the angle which the strained member makes with the line joining its ends, following as far as possible the convention of signs in Art. 93; thus in Fig. 251.

$$i_B=+\delta\beta \quad . \quad (7) \qquad i_C=-\delta\gamma \quad . \quad (8) \qquad i_A'=+\delta\alpha+i_A \quad . \quad (9)$$

$i_A$ being unknown.

Then if M with suffixes stands for bending moments at the joints, from (10) and (11), Art. 103 (putting A = 0)—

$$M_B=\frac{2}{c}(2i_B+i_A)EI, \qquad M_C=-\frac{2}{b}(i'_A+2i_C)EI \quad . \quad (10)$$

$$M_A=-\frac{2}{c}(i_B+2i_A)EI=\frac{2}{b}(2i'_A+i_C)EI \quad . \quad . \quad . \quad (11)$$

which gives four equations to find the four unknown quantities $M_A$, $M_B$, $M_C$, and $i_A$. Reducing (11) by substituting the values (9), we get—

$$i_A=(-b\delta\beta+c\delta\gamma-2c\delta\alpha)\div 2(b+c) \quad . \quad . \quad . \quad (12)$$

$$i'_A=(-b\delta\beta+c\delta\gamma+2b\delta\alpha)\div 2(b+c) \quad . \quad . \quad . \quad (13)$$

Whether secondary stresses of this amount will exist in pin-connected frames depends upon whether the friction moment exerted by the pin is capable of withstanding the moments calculated above for the various joints.

EXAMPLE.—Take $a=5$ ft., $b=3$ ft., $c=4$ ft. Load = 10 tons, sections of AB and AC rectangular $2''\times 1''$, the shorter side being perpendicular to the figure.

From a simple triangle of forces, as in Fig. 228, the primary unit stresses are $p_b=\frac{8}{2}=4$ tons per sq. in., $p_c=\frac{6}{2}=3$ tons per sq. in.

And from the triangle ABC, which is right angled at A—

$$\cot\alpha = 0 \qquad \cot\beta = \tfrac{4}{3} \qquad \cot\gamma = \tfrac{3}{4}.$$

Substituting in (4)—

$$\delta\beta = \frac{I}{E}(4 \times \tfrac{3}{4} - 3 \times 0) = \frac{3}{E}$$

and from (5)—

$$\delta\gamma = \frac{I}{E}(3 \times \tfrac{4}{3} - 4 \times 0) = \frac{4}{E}$$

and from (6)—

$$\delta\alpha = -\frac{I}{E}(3 \times \tfrac{4}{3} + 4 \times \tfrac{3}{4}) = -\frac{7}{E}$$

Hence from (7), (8), (12), and (13)—

$$i_B = \frac{3}{E} \qquad i_C = -\frac{4}{E} \qquad i_A = \frac{9}{2E} \qquad i_A' = -\frac{5}{2E}$$

And from (10), since $I = \frac{1}{12} \times 8 = \frac{2}{3}$,

$$M_B = \tfrac{2}{48}(6 + \tfrac{9}{2})I = \tfrac{21}{48} \times \tfrac{2}{3} = \tfrac{7}{24} \text{ ton-inch}$$
$$M_A = -\tfrac{2}{48}(3 + 9)I = -\tfrac{1}{3} \text{ ton-inch}$$
$$M_C = \tfrac{2}{36}(\tfrac{5}{2} + 8)I = \tfrac{7}{18} \text{ ton-inch}$$

Hence, since $f$ the tensile secondary bending stress per sq. in. on the fibres $= M \div Z$, where $Z = \frac{1}{6} \times 4 = \frac{2}{3}$—

at B $f = \frac{7}{24} \times \frac{3}{2} = \frac{7}{16} = 0{\cdot}4375$ ton per square inch
at C $f = \frac{7}{18} \times \frac{3}{2} = \frac{7}{12} = 0{\cdot}583$ " "
at A $f = -\frac{1}{3} \times \frac{3}{2} = -\frac{1}{2} = -0{\cdot}50$ " "

If instead of being 2″ × 1″ the member had been 4″ × ½″, these bending stresses would have been twice the above values, for I and consequently M would have been four times greater, and Z being doubled $f$ would have been doubled. The more slender the members, *i.e.* the smaller the ratio of breadth in the plane of bending to the length, the smaller the secondary stresses due to rigid joints.

*Stresses arising from Eccentric Connections.*—Fig. 252 illustrates a case of secondary stress arising from non-concurrency of the centre lines of members at a joint. Let the axes of the two slender diagonals intersect at a distance $h$ from the axis of a substantial upper boom of a girder. Then, if F is the resultant of the pull and the thrust in the two diagonals, the action is equivalent to a change of direct stress F in the boom together with a moment $F.h$ divided equally between the two adjacent panel lengths of the upper boom (see cases I and II, Art. 121, putting

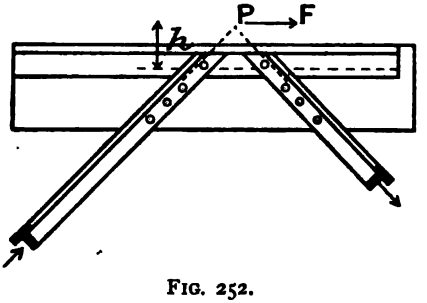


FIG. 252.

$n = \frac{1}{2}$), and therefore exerting a bending stress such as arises from a bending moment $\frac{1}{2}Fh$. Such stresses may reduce or may augment these due to rigid joints, and may be estimated separately, neglecting the rigidity of the joints. Actually the eccentricity will modify the angles at the joints and affect the secondary stress, but we assume in Fig. 252 that the boom section is so great compared to that of the web members that it withstands practically all the bending moment, which is equivalent to taking the case of web members freely jointed to the boom. If the web members are to be taken into account, we should divide the total moment $Fh$ among the members meeting at the joint in the ratio of their values of $I/l$, where I represents the moment of inertia of their cross-sectional areas, and $l$ their lengths, taking the conditions of end fixture the same for each.

There is very frequently a secondary stress of this kind in light members, such as angles in which the line of rivet centres does not coincide with the elastic line or "gravity axis" of the member, *e.g.* the angles in their attachments in Plate I.

## EXAMPLES XV.

1. A trussed purlin 18′ span is made of a British Standard Tee 4″ × 4″ × $\frac{3}{8}$″ with centre steel strut 1 inch diameter and 12 inches long. The tie rods are 1 inch diameter round steel; estimate the uniform load per foot length if the unit stress in the Tee is to be limited to 6 tons per sq. in. What is then the unit stress in the ties and in the strut?

2. Estimate the maximum bending moment on the stanchions of a shed carrying a roof, due to a wind pressure which is 40 lbs. per sq. ft. on the wall and 24 lbs. normal to the roof. Length of stanchions 15 ft., rise 8 ft., span 32 ft., distance between principals 20 ft. The roof is hinged to the stanchions, which are firmly anchored at their bases. The distributed horizontal wind load is carried directly by the stanchion.

3. Solve Problem No. 2 if the wind load on the walls is transferred to the stanchions at the cap and the base, and an intermediate point midway between the cap and the base.

4. Two vertical steel posts 15 ft. apart and 15 ft. long made of 5″ by 3″ British Standard beam sections (see Appendix, Table I.) are hinged at their bases, and their caps are connected by a beam of the same section rigidly attached to each. If this beam carries a central vertical load of 1 ton, estimate the maximum bending moment on the beam and on the posts.

5. Estimate the deflection of the beam in Problem No. 4. $E = 12{,}500$ tons per sq. in.

6. Solve Problem No. 4 if the bases of the posts are firmly fixed.

7. Solve Problem No. 5 if the bases of the posts are firmly fixed.

8. Find the maximum bending moment on each of the posts and the beam in Problem No. 6 if the load of one ton is placed 3 ft. 9 ins. from one post.

9. Find the deflection of the posts from the vertical in Problem No. 8. $E = 12{,}500$ tons per sq. in.

# CHAPTER XVI

## *FRAME MEMBERS AND STRUCTURAL CONNECTIONS*

**175. Determination of Sectional Areas.**—Chapters XI. to XV. are mainly devoted to the determination of the gross pull or thrust in the members of frames. When this pull or thrust has been determined the area is found (see Art. 41) by dividing this total force by the working unit stress. Any corrections for bending due to the weight of members between their ends or other secondary stresses[1] must then be made. The working stress under various conditions of loading causing fluctuations may be fixed by a specification. Some idea of its usual values and its variation with circumstances has been given in Chapter II., but we are now in a position to more fully understand the significance of the various methods of allowing for fluctuation in the load, and Art. 41 may with advantage be again referred to, and further illustrated. The simplest method is to use an equivalent dead load (see (8), Art. 41) equal to the maximum load plus $k$ times the variation of load, whether $k$ is unity or some other factor, such as those given in Art. 41. In this case a working unit stress independent of variation with fluctuation of the load is then employed.

If a member is subjected to both tension and compression the area necessary as a tie and as a strut should be calculated, and the greater value used.

EXAMPLE 1.—Find the sectional areas required for the member CR, Fig. 204, with the loads given in the example at the end of Art. 143, the unit stress being 7·5 tons per square inch. (*a*) Using the dynamic stress formula. (*b*) Using an impact coefficient of, (range of load) ÷ (maximum load).

(*a*) From the example quoted, maximum tension = 88 tons

Range = 88·0 − 16·3 = 71·7 tons

Equivalent dead load stress = 159·7 tons

Area required = 21·3 sq. in.

Which may be provided by say 4 angles 6″ × 4″ × $\frac{5}{8}$″ placed back to back in pairs as in the girder of Plate II.

(*b*) Impact allowance $= \frac{71{\cdot}7}{88} \times 71{\cdot}7 = 58{\cdot}4$ tons

Equivalent load $= 88 + 58{\cdot}4 = 146{\cdot}4$ tons

Area required $= \frac{146{\cdot}4}{7{\cdot}5} = 19{\cdot}5$ sq. in.

[1] See Arts. 174 and 184.

EXAMPLE 2.—Find the sectional area required for the member EP, Fig. 204, with loads as in the example of Art. 143. Unit stress 7·5 tons per square inch in tension, $7{\cdot}5 - 0{\cdot}025\,\frac{l}{k}$ in compression. Take (*a*) Dynamic method or impact coefficient unity. (*b*) Impact coefficient = range of load ÷ maximum load.

(*a*) Range of load = 36·5 + 15·6 = 52·1 tons

Equivalent dead tensile load = 36·5 + 52·1 = 88·6 tons

$$\text{Area required } \frac{88{\cdot}6}{7{\cdot}5} = 11{\cdot}8 \text{ sq. in.}$$

Equivalent dead load thrust 15·6 + 52·1 = 67·7 tons

Assuming that $\frac{l}{k}$ will be about 80 (never exceeding 100), the working unit stress is

$$7{\cdot}5 - 80 \times 0{\cdot}025 = 5{\cdot}5 \text{ tons per sq. in.}$$

$$\text{Area required } \frac{67{\cdot}7}{5{\cdot}5} = 12{\cdot}3 \text{ sq. in.}$$

Thus the section will probably be determined for the thrust. When the member section has been settled, and $\frac{l}{k}$ is definitely known, a check is required to ascertain whether the unit stress is within the required limit $7{\cdot}5 - 0{\cdot}025\frac{l}{k}$.

(*b*) Equivalent dead tensile load $= 36{\cdot}5 + \frac{(52{\cdot}1)^2}{36{\cdot}5} = 110{\cdot}9$ tons

$$\text{Area required } \frac{110{\cdot}9}{7{\cdot}5} = 14{\cdot}8 \text{ sq. in.}$$

Equivalent dead load thrust = 15·6 + 74·4 = 90 tons.

Probable area required $\frac{90}{5{\cdot}5} = 16{\cdot}4$ square inches, the thrust again deciding the area.

This example also illustrates the fact that the design from the above impact coefficient will in the case of reversed stresses give a larger area than the use of the impact factor unity, *i.e.* than the dynamic method (see Fig. 39).

**176. Riveted and Pin-jointed Frames.**—In Great Britain riveted girders for bridge work are used in almost all cases to the exclusion of pin-jointed frames. In America pin-jointed bridge girders have been much in favour for all comparatively short spans for a variety of reasons, such as cheapness, facility of rapid erection with little work at the site, and limitation of the secondary bending stresses (see Arts. 174 and 184) in the chord members. Recently, however, riveted trusses have been employed more freely; this may be accounted for by many reasons, such as means of handling larger completed pieces, and use of shorter spans owing to diminished cost of piers. Frequently some riveted web members are used in bridge trusses containing pin-connected

eyebar members; such eyebars may sometimes be used to avoid the difficulty of a splice in a very large section. It is noteworthy that the new Quebec Bridge is to have riveted trusses.

**177. Form of Sections for Members.**—The forms of section most suitable for the different members of a frame are necessarily determined largely by experience, and the types and ranges of section available. Practical design of structures involves not only a knowledge of structures, but of methods of manufacture and erection and of costs, matters of prime importance which are outside the scope of this volume. A small selection of important types of braced girder members is, however, given to illustrate many of the points already dealt with in the previous chapters. Typical examples of pin and riveted connections for roofs have been given in connection with Plate I., Chapter XI.

*Boom Sections.*—Fig. 253 shows typical sections for booms, (*a*) being applicable to rather small girders, (*b*) and (*c*) with many modifications to larger ones. The side plates and angles of (*b*) are often replaced

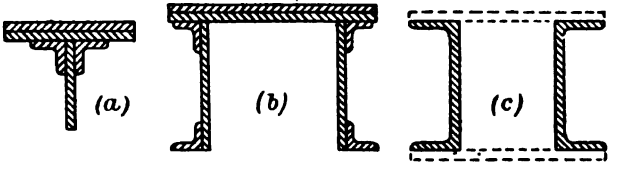



FIG. 253.—Typical boom sections.

by channel sections, as in the girder of Plate II. shown in section in Fig. 262. As shown at (*c*), Fig. 253, latticed channels are also used. The top chord is usually closed by plates as at (*b*), while the bottom chord is generally latticed, or quite open to prevent the accumulation of water, being stiffened by transverse diaphragm plates at intervals, attached by angles to the side of the vertical channels or plates. The minimum depth of a boom section is often limited to $\frac{1}{20}$ of a panel length.

*Web Member Sections.*—The ties are usually flats or angles in riveted trusses, while in pin-connected frames they are eyebars (see Art. 184). The struts are of very varied design, including several of those shown in Fig. 177. The attachment of web members to the booms is sometimes direct, but more frequently by means of gusset plates as in Plate II.

**178. Riveted Joints.**—Figs. 254 to 257 inclusive show the commonest forms of riveted joints. Lap joints are seldom used in structural steel frames, as they obviously involve eccentricity of the stress in the members. Fig. 258 shows four ways in which rupture may take place, the illustration being drawn from a single riveted lap joint: (*a*) shows shear of the rivet, in this case exposed to single shear, *i.e.* shear across one section only, (*b*) illustrates tearing of the plate, (*c*) crushing of the plate (or the rivet) due to too great a *bearing* pressure, while (*d*) shows bursting of the plate due to too small an overlap, which is easily avoided.

It should be recognized at the outset in dealing with riveted joints that the distribution of shearing stress over the cross section of a rivet is not known, nor is the distribution among a group always uniform, and that any quoted stress refers to the average over the whole area. A similar remark refers to the direct stress in the plates between rivet holes, and the resistance of the remaining plate is reduced by making the rivet holes. Another factor in the resistance of riveted joints is the frictional resistance of the parts to relative movement. This always strengthens the joint, but is never taken into account. It makes the stress calculations in an additional degree conventional. It is often specified that rivet holes shall be drilled, or punched so much below the required diameter, and then drilled out. The argument in favour of punching is that the increased cost of drilling if put into extra metal in the members more than makes good any loss of strength resulting from punching, except in very long spans where dead load becomes increasingly prominent.

*Single and Double Shear.*—In Figs. 254 and 255, and the single cover butt joint of Fig. 256, the rivets are in single shear, *i.e.* the stress

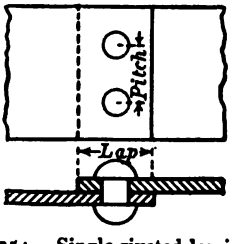



FIG. 254.—Single-riveted lap joint.

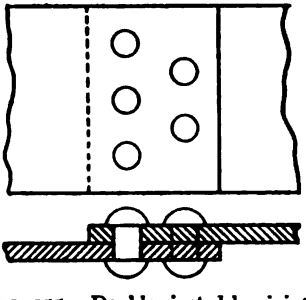


FIG. 255.—Double-riveted lap joint.

per rivet on either side of the joint is resisted only by one section of a rivet, while in Fig. 257 and in the double cover butt joint of Fig. 256 the rivets are in double shear, *i.e.* the pull or thrust per rivet on either side of the joint is resisted by two sections. It is usual to allow in cases of double shear a total stress of from 1·5 times to twice that allowed for single shear, specifications varying upon the allowed ratio; 1·75 may be taken as a usual value.

*Resistance of a Riveted Joint.*—(*a*) *To Tearing.*

$$b \cdot t \cdot f_t \quad \ldots\ldots\ldots\ldots \quad (1)$$

where $f_t$ = working tensile stress in a perforated plate, $t$ = thickness of plate, and $b$ = available resisting breadth of plate, *i.e.* the whole breadth minus the diameter of each rivet hole which is effective in reducing the tearing resistance. It is usual to take the diameter of the hole as $\frac{1}{16}''$ greater than that of the rivet before driving.

(*b*) *To Shearing.*—In single shear,

$$n \cdot \frac{\pi}{4} \cdot d^2 \cdot f_s \quad \ldots\ldots\ldots\ldots \quad (2)$$

where $n$ = number of rivets on each side of the joint, $d$ = diameter of rivets, $f_s$ = working unit stress in shear allowed in rivets.

In double shear,—conventionally taken to be about

$$1{\cdot}75\, n \cdot \frac{\pi}{4} \cdot d^2 \cdot f_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is usual to add a proportion, say 10 to 25 per cent., of rivet area for all rivets driven during erection called "field" rivets.

(*c*) *To Crushing.*

$$n \cdot d \cdot t f_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $f_b$ = the allowable unit stress for bearing reckoned on the longitudinal or axial section of the rivet. The stress $f_b$ is usually taken as $2f_s$. The ratio of $f_t$ to $f_s$ is often specified for riveted joints, and may be taken as about 1·2 to 1·4.[1]

The number of rivets required in a given case of direct pull or thrust may be found by equating the smaller of the two resistances, to

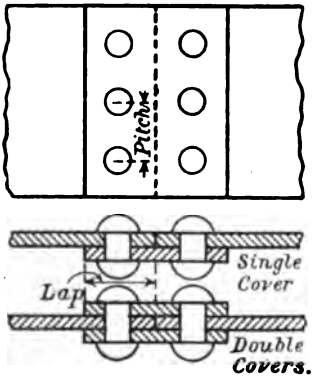



FIG. 256.—Single-riveted butt joint.

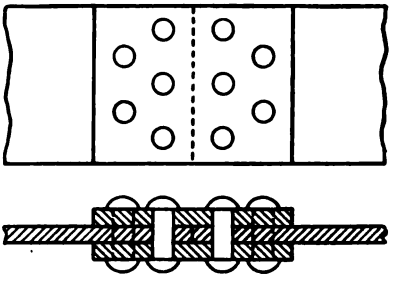


FIG. 257.—Double-riveted butt joint.

crushing (4), or shearing (2) or (3), to the resistance (1) to tearing. With the above values a double-covered butt joint will be weaker in shearing so long as $\frac{\pi}{4} \cdot 1{\cdot}75d$ is less than $2t$, *i.e.* so long as

$$d \text{ is less than } 1{\cdot}46t, \text{ or } t \text{ is greater than } 0{\cdot}685d.$$

Usual dead load values for $f_t$ for mild steel are 6 to 7·5 tons per square inch, and for $f_s$ 4·5 to 5·5 tons per square inch.

*Size of Rivets.*—There is no invariable rule as to the size of rivets for structural work. The diameters vary from $\frac{5}{8}$-inch to $1\frac{1}{8}$ inch according to the thickness of plate and convenience. The rule $d = 1{\cdot}2\sqrt{t}$

[1] Lower apparent values are deduced from tests to destruction of riveted joints in which the rivet resistance is augmented by friction.

may be taken as some indication of a suitable diameter, but this may be considerably varied to suit circumstances such as a convenient pitch.

*Limits of Pitch.*—3-inch pitch or possibly three times the rivet diameters is about the minimum space into which $\frac{7}{8}$-inch rivets can conveniently be placed. And to avoid opening or bulging of the plates a maximum of about 16 times the thickness of the thinnest plate in the joint is frequently specified

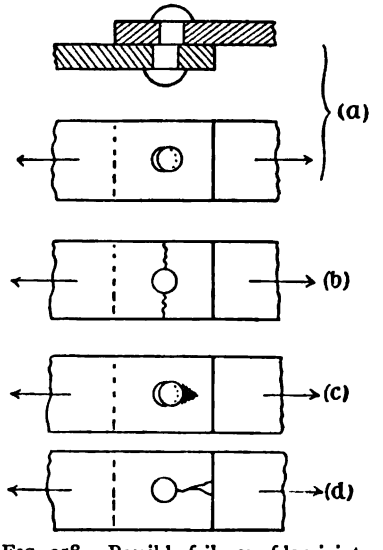



FIG. 258.—Possible failures of lap joints.

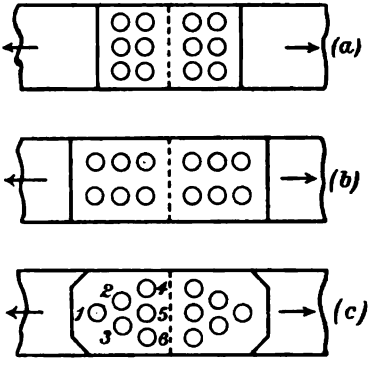



FIG. 259.

**179. Grouping of Rivets.**—The tearing resistance of the plates of a joint containing a given number of rivets depends upon the arrangement of the rivets. For example, in Fig. 259 the form (*a*) reduces the effective breadth of the plate to the whole breadth, minus three rivet hole diameters. The form (*b*) is stronger and only makes a reduction of two holes in the breadth, while (*c*) is the strongest form and only reduces the strength to the extent of one hole. For before the plates can pull asunder at a section through rivets 2 and 3 the rivet 1 must be sheared, and this offers a resistance at least equal to the tensile resistance of the corresponding breadth of plate. Similarly fracture of the plate across the diameters of rivets 4, 5, and 6 is resisted by the rivets 1, 2, and 3. The cover plates, on the other hand, are not so assisted, and their combined thickness must be such that after perforation their area exposed to tension is at least equal to that of the plate section through the rivet hole 1. Their combined thickness generally exceeds that of the plates by 50 per cent.

EXAMPLE 1.—Arrange a suitable double-covered butt joint to splice a $\frac{5}{8}$-inch tie plate 10 inches wide. Use $\frac{7}{8}$-inch rivets.

Taking the tensile load at 7·5 tons per square inch, equivalent tensile dead load, allowing one rivet hole $\frac{15}{16}$-inch diameter is

$$(10 - \tfrac{15}{16}) \times \tfrac{5}{8} \times 7{\cdot}5 = 42{\cdot}5 \text{ tons.}$$

Stress per $\frac{7}{8}$-inch rivet in double shear say

$$5 \times 1{\cdot}75 \times 0{\cdot}7854 \times (\tfrac{7}{8})^2 = 5{\cdot}26 \text{ tons.}$$

Number of rivets required $= 42{\cdot}5 \div 5{\cdot}26 = 8{\cdot}1$, say 9 rivets. The joint is shown in Fig. 260.

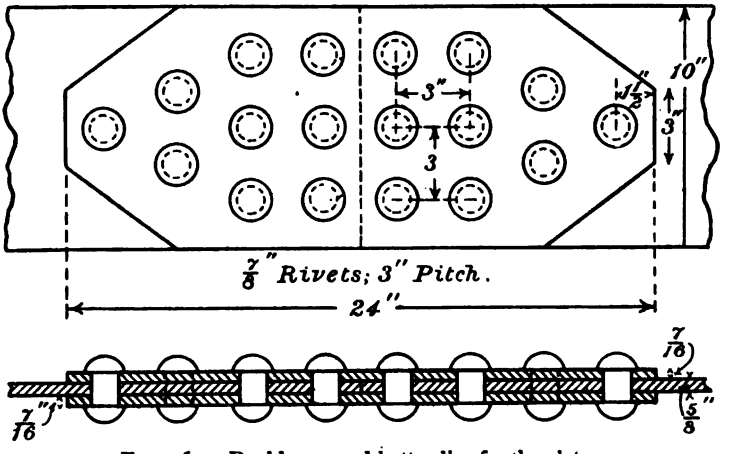



FIG. 260.—Double-covered butt splice for the plate.

EXAMPLE 2.—The member CR (Fig. 204) is subject to an equivalent dead load of 159·7 tons as in Example 1, Art. 175. Allowing 7·5 tons per square inch in tension, and 5·5 tons per square inch in shear stress in $\frac{7}{8}$-inch rivets, arrange the section and joints to the booms.

Net area required $159{\cdot}7 \div 7{\cdot}5 = 21{\cdot}3$ sq. ins.

Using four standard unequal angles 6″ × 4″ × $\frac{5}{8}$″ Table IV. Appendix, gives with one rivet hole in each a net area of $4(5{\cdot}86 - \frac{15}{16} \times \frac{5}{8}) = 21{\cdot}12$ sq. ins. If 6″ × 4″ × $\frac{11}{16}$″ angles are used there will be a margin above the specification. Using two angles attached to each side of the boom, each joint must withstand $159{\cdot}7 \div 2 = 79{\cdot}9$ tons. In single shear the resistance of each rivet is $(\frac{7}{8})^2 \times 0{\cdot}7854 \times 5{\cdot}5 = 3{\cdot}31$ tons.

Number required if in single shear $79{\cdot}9 \div 3{\cdot}31 = 24{\cdot}1$, say 25.

Resistance in bearing per rivet, say $\frac{7}{8} \times \frac{5}{8} \times 11 = 6{\cdot}01$ tons. Number required $79{\cdot}9 \div 6{\cdot}01 =$ say 14 or with $\frac{11}{16}$″ angles say 13. The details of a joint are shown in Plate II., member S3, which is of somewhat similar strength to this calculation, having at the upper joint passing through the gusset plate 12 rivets in double shear and two in single shear. But the allowable tension in the member for bearing stress in this joint would be only $\frac{1}{2} \times 14 \times \frac{7}{8} \times 11 = 67{\cdot}4$ tons, the gusset plate being $\frac{1}{2}$″ thick; $\frac{5}{8}$″ plates would allow the full 79·9 tons.

**180. Oblique Attachments.**—The centroid of the cross-sectional areas of a group of rivets should so far as possible lie on the centre line or gravity axis of the member which the group attaches to another member. Otherwise the resistance of the rivets will exert an eccentric force on the member, thereby subjecting it to bending, as in Art. 112. Fig. 261 shows at (*a*) an undesirable arrangement; assuming that all the rivets resist equally, the members are jointly subjected to a moment $Ph = P' . h$ when the pull passes at a distance $h$ from the centroid $G_1$ of the group of rivets. (See also Art. 174.) The grouping shown at (*b*) is frequently adopted as being the only possible plan to get in the necessary rivets, but across the section XX. which is reduced by one

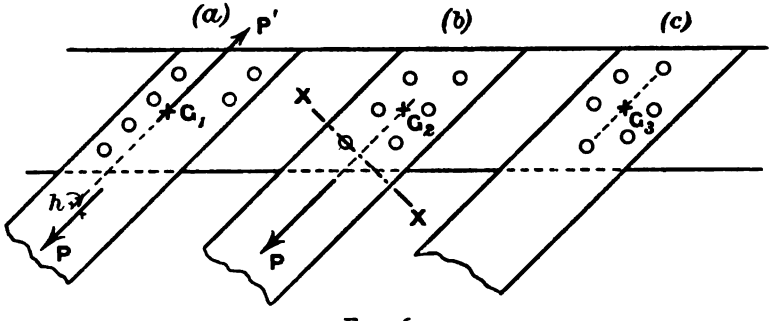



FIG. 261.

rivet hole the pull is not through the centroid of the section of the member, and bending stress accordingly arises. The symmetrical form of grouping (about the centre line of the sloping member) shown at (*c*) will obviate bending stresses, except such as arise due to rigidity of the joint, as explained in Art. 174.

**181. Flange or Boom Splices.**—In making a splice in a boom consisting of several pieces it is desirable while making the joint in each piece to come under one set of covers, to arrange that no two pieces lying next to each other should have their joints at the same section. A typical boom joint is shown in Fig. 262 (which represents the joint in the top chord of the girder shown in Plate II., but is placed on Plate IV.). The channels and the intermediate $\frac{1}{2}''$ flange plate make joint at the same section, but joints of the $\frac{3}{4}''$ mainplate and the outer $\frac{5}{16}''$ plate lie on either side of this section.

**182. Torsional Resistance of Rivet Groups.**—It frequently happens that the attachment of one member of a structure to another by a group of rivets is subjected to a direct pull or thrust, and in addition to a moment in the plane of the rivet cross-sections. In other words, the resultant force transmitted does not pass through the centroid of the rivet cross-sectional areas, but is eccentric. Examples occur in members of a riveted truss the joints of which being rigid cannot turn as if on frictionless pins, and are subject to moments which cause secondary stresses (Art. 174.) More obvious cases occur in the attachment of a

bracket to a stanchion or in the attachment of a cross-beam to the stanchions or to beams at its ends. Take Fig. 263 to represent a bracket attached to some support by, say, five rivets, A, B, C, D, and E, the centre of gravity or centroid of their cross-sectional areas being at G, two-fifths of the distance between the two centre lines from B. Let $h$ be the eccentricity of a load P from G. Then the rivets jointly resist (1) a direct force P, and (2) a moment P$h$. The exact distribution of these actions among the rivets cannot be calculated with any great certainty for many reasons, such as the inexactness of fitting and filling the holes. Taking probable conditions with good fitting, each rivet will exert on the bracket (1) a force equal to one-fifth P and parallel to the direction of P, (2) a force perpendicular to the line joining its centre to G and proportional to the distance from G, the sum of the moments of such forces being equal to and balancing the torsional moment P$h$. Let R be the distance from G to the centre of the most distant rivet A, and let F be the force exerted by the rivet A in consequence of the moment P$h$ or M. Then the force exerted by any rivet such as E, say distant $r$ from G, is

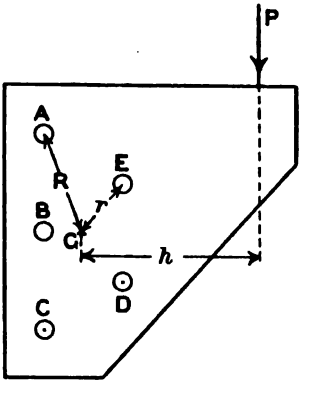



FIG. 263.—Eccentric thrust on group of rivets.

$$F \times \frac{r}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the total moment $M = \frac{F}{R} \Sigma(r^2)$ for all the rivets, hence

$$F = MR \div \Sigma(r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

while the force exerted by any rivet E is—

$$M \,.\, r \div \Sigma(r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in such a direction as to oppose the moment M = P$h$ of P, *i.e.* so as to have, as in Fig. 263, a contra-clockwise moment about G. The total force exerted by any rivet such as E is found by adding geometrically the above force (3) to the force one-fifth P opposing P. This addition may be made graphically or by trigonometrical calculation.

More generally the forces acting on any rivet of $n$ in a group will be $\frac{1}{n}$ of P and $M \,.\, r \div \Sigma(r^2)$.

*Approximation for a Large Group.*—The process of finding $\Sigma(r^2)$ for a large group will be tedious. It may be taken as $nk^2$ where $k$ is the radius of gyration of the circumscribing area about an axis through the centroid G and perpendicular to the cross-section of the group, the area being extended by half a pitch beyond the centre lines in each

direction, *e.g.* for nine rows of rivets, 3″ pitch, with nine rivets in each row. The true value of $\Sigma(r^2)$ (taking the sum of the squares of the distances from two perpendicular axes) is

$$2 \times 2 \times 9(12^2 + 9^2 + 6^2 + 3^2) = 9720.$$

For the area 27″ × 27″, $k^2 = \frac{1}{6} \times 27^2 = 121{\cdot}5$, $nk^2 = 81 \times 121{\cdot}5 = 9841$, which differs from 9720 by just over 1 per cent.

EXAMPLE 1.—Fig. 264 shows one of a pair of angle cleats 6″ × $3\frac{1}{2}$″ × $\frac{3}{8}$″ by which each end of the webs of an I beam carrying a uniformly distributed load of 26 tons is attached by bolts to supports at its ends.

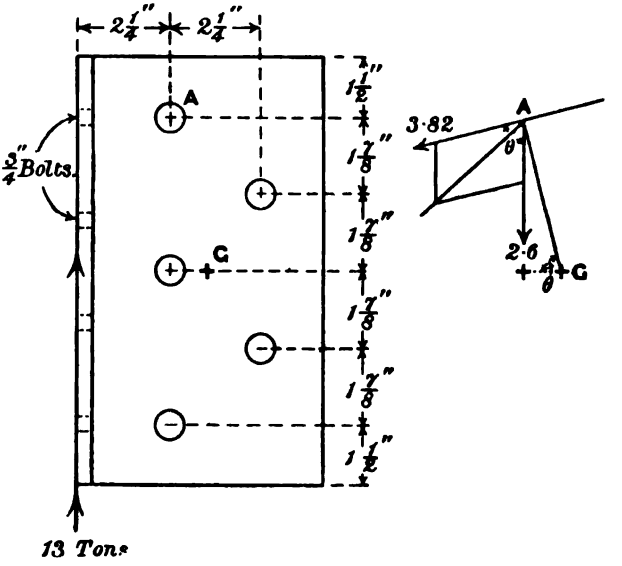



FIG. 264.—Maximum stress in rivets of angle cleat connection.

Find the maximum stress taken by any of the five rivets attaching the two cleats to either side of the beam.

Assuming a purely vertical reaction of 13 tons (half the load) at the back of the angle,[1] exerted by the bolts, the eccentricity is $2\frac{1}{4} + \frac{2}{5} \times 2\frac{1}{4} = 3{\cdot}15$″, hence the moment is $13 \times 3{\cdot}15 = 40{\cdot}95$ ton-inches. Any term $r^2$ in $\Sigma(r^2)$ may be estimated by the sum of the squares of horizontal and vertical components, hence taking these component values,

$$\Sigma(r^2) = \{3 \times (0{\cdot}9)^2\} + \{2 \times (1{\cdot}35)^2\} + \{2 \times (3\tfrac{3}{4})^2\} + \{2 \times (1\tfrac{7}{8})^2\}$$
$$= 41{\cdot}25 \text{ square inches}$$

[1] Generally there will be a considerable clockwise moment exerted on the angle cleat by the supports, consisting of a thrust at the bottom and tension in the upper bolts; this will reduce the moment exerted on the rivets and the resisting moment exerted by them. Unless there is sufficient play in the connection or the supports to allow the end tilt given by (6) and (7), Art. 97, there would be a large contra-clockwise moment exerted by the cleats on the rivets.

Distance R of rivet A from G (the centroid) $= \sqrt{3{\cdot}75^2 + 0{\cdot}9^2} = 3{\cdot}855''$. Hence from the relation (2) the resistance of rivet A to torsion about G is

$$40{\cdot}95 \times 3{\cdot}855 \div 41{\cdot}25 = 3{\cdot}82 \text{ tons},$$

which is inclined $\theta$ to the horizontal where

$$\tan\theta = \frac{3{\cdot}75}{0{\cdot}9} = 4{\cdot}16, \textit{ i.e. } \theta = 76{\cdot}5°.$$

In addition the rivet exerts on the two cleats a vertical downward force $13 \div 5 = 2{\cdot}6$ tons. Hence the resultant force exerted by the rivets on the pair of cleats, found graphically or as follows, is

$$\sqrt{(3{\cdot}82^2 + 2{\cdot}6^2 + 2 \times 3{\cdot}82 \times 2{\cdot}6 \times \cos 76{\cdot}5°)} = 5{\cdot}10 \text{ tons}$$

which even taken in double shear is a full allowance for a $\frac{3}{4}''$ rivet, and is a high rate of bearing pressure in passing through a $\frac{1}{2}''$ web, viz. $4{\cdot}87 \div (\frac{1}{2} \times \frac{3}{4}) = 13$ tons per square inch.

If the resulting force for the remaining rivets be similarly determined and a polygon of forces be drawn, the resultant will be found to be 13 tons vertically downwards, and if by a funicular polygon its position be determined it will be found to act in the same straight line as the 13 tons upward force at the back of the cleat, thus checking the calculations.

EXAMPLE 2.—One hundred $\frac{7}{8}''$ rivets arranged in the form of a square at 3″ pitch, in directions parallel to the sides of the square, resist a moment of 2000 ton-inches. Find the load on the four rivets at the corners of the square.

Distance R to the corners $= 13{\cdot}5 \times \sqrt{2} = 19{\cdot}1''$.

For a circumscribing square,

$$\text{length of side} = 27 + 3 = 30'', \quad k^2 = \tfrac{1}{6} \times 30^2 = 150$$

hence $\Sigma(r^2)$ is $100 \times 150$ approximately $= 15{,}000$

and from (2), $$F = \frac{2000 \times 19{\cdot}1}{15{,}000} = 2{\cdot}55 \text{ tons}$$

more accurately $$\Sigma(r^2) = 2 \times 2 \times 10 \times 9(4{\cdot}5^2 + 3{\cdot}5^2 + 2{\cdot}5^2 + 1{\cdot}5^2 + 0{\cdot}5^2) = 14{,}850$$

and $$F = \frac{2000 \times 19{\cdot}1}{14{,}850} = 2{\cdot}57 \text{ tons}$$

**183. Design for N or Pratt Girder.**—Plate II. shows the elevation of half of a riveted N or Pratt truss, forming one of the main girders of a double-track railway bridge for which the Author is indebted to Sir Wm. Arrol and Co., Ltd. Part of the cross-section is shown in Fig. 265. Actually, as shown on the plan, the bridge forms a skew span, the angle of skew being 42°, but the design corresponds fairly closely to that for a square span. (Of the nine panel points in the girder only two will be appreciably affected by the angle of skew, viz. the two nearest the abutments at the *acute* end of the span. The loads on these points will be diminished because the cross girders there

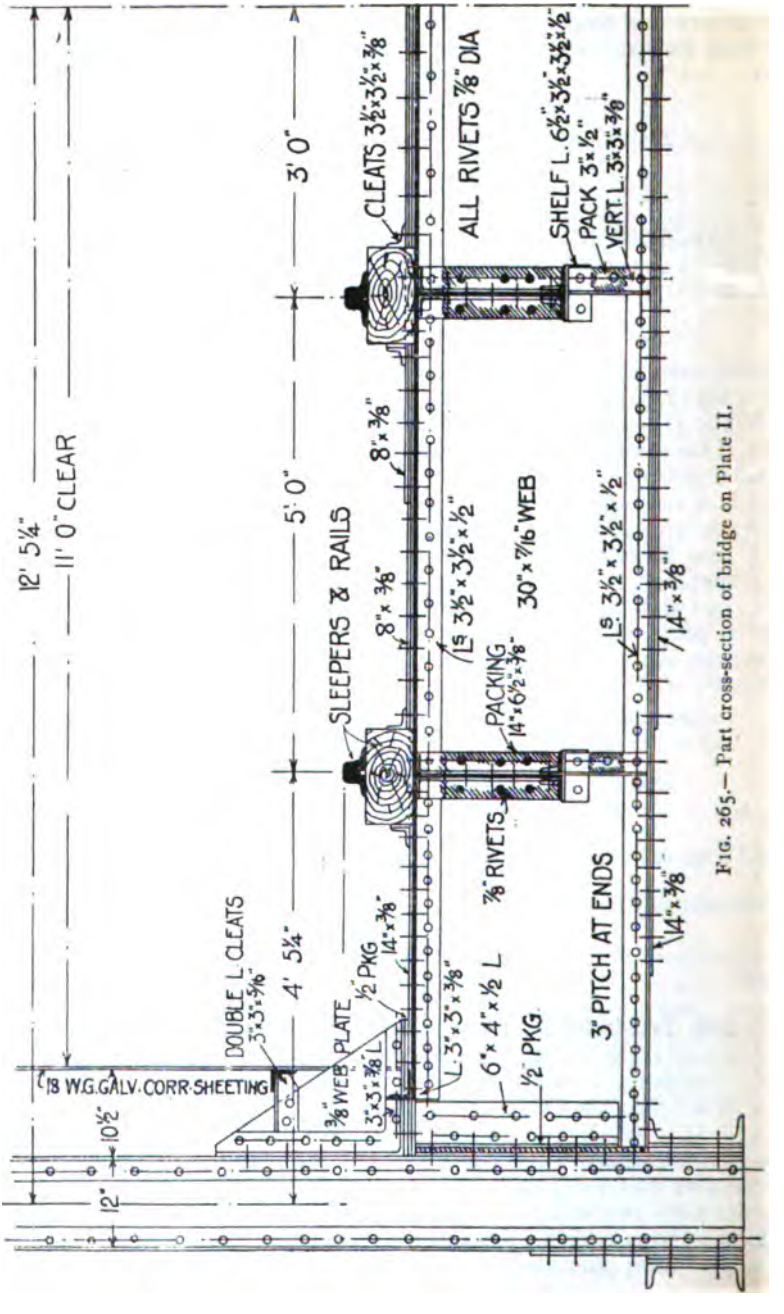



FIG. 265.— Part cross-section of bridge on Plate II.

do not traverse the full breadth and in consequence do not carry the full load.) The cross-section and enlarged elevation of the splice of the top chord are shown in Fig. 262 on Plate IV.

The dimensions on the drawing give the reader many examples in design. As a basis of calculation the following data may be used. Effective span, 92 feet 1 inch. Effective height, 11 feet 6 inches. Panel length, 9 feet $2\frac{1}{2}$ inches. Dead load, 0·6 ton per foot per main girder; equivalent moving load per girder, 1·5 tons per foot plus 23 tons concentrated in the most influential position. For chord members, replacing the concentrated load by twice the amount uniformly distributed, we may take $1{\cdot}5 + \frac{46}{92} = 2$ tons per foot run. Equivalent dead load stress on any member, $= \text{max. stress} + \frac{(\text{range of stress})^2}{\text{max. stress}}$, which is equivalent to using an impact factor (see Art. 41) equal to range of load ÷ max. load. On loads so increased for dynamical contingencies, unit stress of 9 tons per square inch in tension, $9 - 0{\cdot}03\,\frac{l}{k}$ in thrust, and 7 tons per square inch shear in rivets with 14 tons bearing stress may be used. These really correspond to much lower working unit stresses reckoned on the maximum load, *e.g.* for chord members the stress is increased in the ratio $2{\cdot}6 + \frac{2^2}{2{\cdot}6}$ to 2·6, *i.e.* 4·14 to 2·6, hence the tensile allowance reckoned on the maximum loads is $\frac{2{\cdot}6}{4{\cdot}14} \times 9 = 5{\cdot}65$ tons per square inch.

The rivets attaching a web member to a gusset plate have to carry the whole pull or thrust in that member, but the rivets attaching a gusset plate to a continuous chord at a joint have only to carry the increment of chord stress at that joint, *i.e.* the resultant force exerted by the web members meeting at that point.

**184. Pin-joints.**—*Eye-bars.*—The use of knuckle pin joints with forks and eye-bars has already been illustrated in Plate I. Such joints were formerly very common in roofs but are now generally replaced by riveted joints. In America pin-joints have largely been used in bridges, the tie members being made of several eye-bars symmetrically placed with respect to the centre of the length of the pins. The proportions of the forged ends and pin holes in eye-bars vary somewhat, but the form illustrated in Fig. 266, in which the hole diameter $d = \frac{3}{4}h$, where $h$ is the depth of flat bar and the thickness $t = \frac{1}{6}h$ represents about the usual proportions. The head is always made large enough to be stronger than the parallel bar. When the pin is a slack fit in the eye-bar holes the bearing

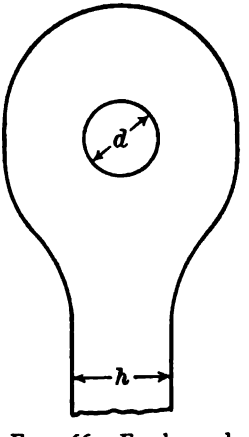


FIG. 266.—Eye bar end.

pressure is very unevenly distributed; in any case the bearing pressure allowed is less than on rivets, being often about 8 tons per square inch. If the ends are not thickened, for equal bearing and tensile resistance,

$$f_t \cdot h \cdot t = d \cdot t \cdot f_b$$

hence

$$\frac{d}{h} = \frac{f_t}{f_b}$$

where $f_t$ and $f_b$ are the working unit stresses for tension and bearing respectively.

*Maximum Bending Stress in Eye-bars with Pin-joints.*—Pin-joints are sometimes said to eliminate any serious degree of secondary stress due to bending from lack of free turning movement at the joints. Let P be the total primary stress in an eye-bar of a pin-connected frame. Then *if the whole pull* P *comes on the pin*, the total frictional resistance at the circumference of the pin before movement takes place is $\mu \cdot P$, where $\mu$ is the coefficient of friction between the pin and the hole.

$$\text{Maximum bending moment at the ends } M = \mu \cdot P \cdot \tfrac{1}{2}d$$

$$\text{Maximum bending stress} = \tfrac{1}{2}\mu P \cdot d \div Z$$

$$= \tfrac{1}{2}\mu P d \div \tfrac{1}{6}th^2 = \frac{3\mu P d}{th^2}$$

Z being the modulus of section for bending stress. Then in order that the bar should move round the pin to prevent a bending stress of, say, $n$ times the primary unit stress (where $n$ is a fraction),

$$\frac{3\mu P d}{th^2} \text{ must be less than } \frac{nP}{th}$$

$$\mu \text{ must be less than } \frac{nh}{3d}$$

*e.g.* if $d = \frac{3}{4}h$, in order to prevent a secondary bending stress greater than 10 per cent. of the primary stress $\mu$ must be less than 0·04, a condition very unlikely to be fulfilled in such a pin-joint. Again, if $\mu$ is so low as 0·25 the secondary stress might reach

$3 \times 0{\cdot}25 \times P \times 0{\cdot}75\ h \div th^2 = \frac{9}{16}\frac{P}{th}$, *i.e.* $\frac{9}{16}$ of the primary unit stress,

before it is relieved by movement about the pin. It may be noted that in a pin-joint in a continuous chord the pin does not bear the whole chord stress but only the increment at the joint, which makes movement possible with lower bending stresses than for discontinuous eye-bars supposed above.

*Stresses in the Pins.*—In addition to the bearing and shearing stresses (the latter being usually low) in the pins, the bending stresses are important. The bending moments on a pin result from forces not all in one plane nor all parallel. The maximum bending moment may be found by resolving the forces exerted by each member hinged at the pin into two perpendicular components, say horizontal and vertical. The component bending moments in two perpendicular planes may then be calculated as in Chap. IV., the component values along the pin axis being proportional to the ordinates of a polygonal diagram such as

Fig. 86. The actual bending moment at any section will in general lie in some intermediate plane, and if at any section $M_h$ = the bending moment in a horizontal axial plane and $M_v$ = the bending moment in the vertical axial plane, the resultant bending moment is $M = \sqrt{M_h^2 + M_v^2}$; the maximum bending moment will occur at the point of application of one of the forces, *i.e.* in the plane of the axis of some member hinged to the pin, and the place of its occurrence may be found by inspection. The varying bending moment along the pin axis may be conceived as represented by the radial ordinates of a winding surface, the generators of which are radial straight lines through the axis of the pin and perpendicular to it. The projections of such a surface on two planes through the pin axis give the diagrams of component bending moments in the respective planes.

**185. Beam and Stanchion Connections.**—The attachments of the ends of an I beam to a stanchion are usually made by "cleats," *i.e.* pieces of angle section riveted to the beam and bolted to the stanchion. Fig. 267 shows the web and flange cleats suitable for such a purpose. The upper flange cleat and web cleats will usually be riveted to the girder, but the lower flange cleat (shown dotted), if used, will usually be riveted to the stanchion. Sometimes connections are made by web cleats only, particularly at the joint of two beams under a floor. The supporting force which such a cleated connection may safely exert is measured by the shearing value of the bolts in single shear, the bearing and the shearing value of the rivets (in double shear), and the smallest of these three values taken. In estimating the shearing and bearing values of the rivets it is to be borne in mind that they are liable to some moment about the centroid of the group in addition to the direct force of the end reaction, in other words to an eccentric force. At an extreme estimate such a moment might be taken as the product of the end reaction into the distance of the centroid from the back of the web cleats; the effect of such eccentricity was estimated in Ex. 1, Art. 182, where it was pointed out that the action of the moment will to some unknown extent be neutralized by the moment exerted by the bolted attachment of the cleats to the stanchion.

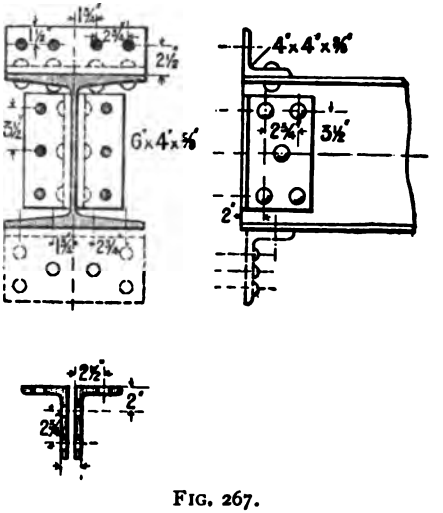



FIG. 267.

The question may suggest itself as to whether such beam connections do to any considerable extent correspond to the assumed end conditions of a "built in" beam. A rough numerical estimate will show

that the moment value of the rivets would be quite inadequate to take the end moments involved, and that the conditions must approximate to those of a beam simply supported at its ends. Before such conditions can obtain some local yielding must take place. If, for example, the

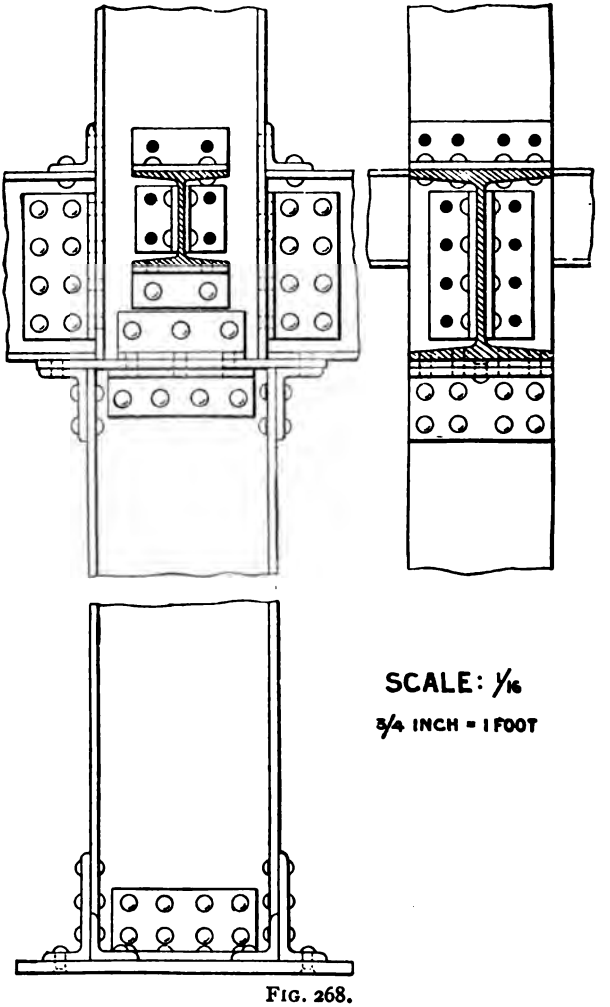



FIG. 268.

central deflection of the beam is say 0·001 of its length the average slope is 0·002, and the end tilt must be of the order of 0·004 of a radian. To accommodate this, *if the rivets are tight fits in the holes* considerable local strain must occur, which may be distributed over the upper bolts in tension, the rivets in shear, and the rivets and rivet holes in bearing

particularly in the web. When once the end slope has been accommodated by slackness or by straining, the rivet stresses may be looked upon as not exceeding the values indicated in Ex. 1, Art. 182. But in any case the conditions are indefinite, and the computations of stress in these connections must be regarded as to a considerable extent conventional.

Fig. 268 represents the joint of an I section stanchion with four horizontal beams attached, such as may occur at the junction of two stories of a steel frame building, and Fig. 269 represents an isometric drawing of such a joint.

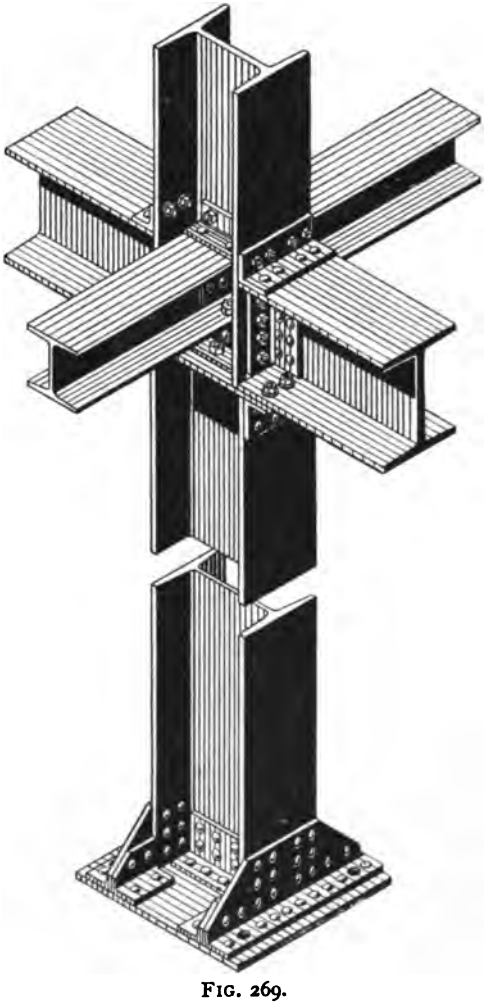


FIG. 269.

Fig. 270 shows the stanchion caps and bases. It is recommended that each angle cleat at the cap should be capable of carrying (calculating on the rivets in single shear) at least half the end reaction of the girder it supports, the other half being taken directly by the stanchion through the cover plate (if used) or through bolts to the upper stanchion in case of a joint, such as Figs. 268 and 269. In the bases, consisting of sole plate and angles to flanges and web, the number of rivets required for a "fixed" end is often taken such that they are capable of transmitting the whole load to the sole or base plate. Thus in the base shown in Fig. 270 there are 27 rivets $\frac{7}{8}''$ diameter attaching the I section stanchion to the angles. Of these, 24 are in single shear and 3 in double shear, and taking the shear stress at 4 tons per sq. inch, and the bearing pressure at 8 tons per sq. inch, the latter will limit the carrying value to about 66 tons. With a machined column end a portion (say about half) of the load may

be assumed to be transferred directly to the base plate instead of through the rivets and angles; thus 66 tons is fully half the maximum working load on the column shown in Fig. 270, *i.e.* its load at its shortest probable length of about 10 feet.

In addition to direct thrust the rivets should in any case be capable of taking any bending moment which *may* arise at the base, whatever the assumed end conditions of the column base. If, however, the base is of an ordinary character and the loads are nominally axial any relatively small bending moment is usually taken as being amply provided for and is therefore disregarded.

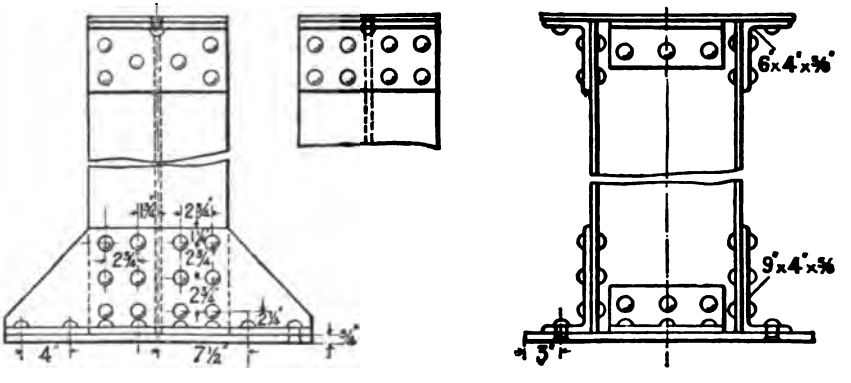



FIG. 270.

Figs. 267 to 270 relate specially to broad flanged sections, among the advantages claimed for which are, simplicity and convenience in such attachments. The flanges and web are thicker than in narrow flanged joists, allowing the use of larger and fewer rivets; the flange area is sufficient to allow of double rows of rivets being used, and consequently for column bases the gusset plates if required at all do not require to be carried so high up the column to provide room for all the necessary rivets. These illustrations are taken by kind permission from the excellent "Structural Steel Handbook" of Messrs. R. A. Skelton & Co. Students of design will find much useful information in the handbooks of the various structural steel makers and merchants.

*Anchorage of Stanchions.*—Stanchions "fixed" at the base must have holding-down bolts as well as riveted base connections capable of withstanding bending moments such as arise from wind and other horizontal loads calculated in Art. 166 and elsewhere in Chap. XV. Let T be the total tension in the bolts on the "windward" side of a stanchion base, let $a$ be the distance of the bolts on the leeward and

windward sides from the centre line of the column base. Let the straining actions at the base be reduced to an axial thrust V and a bending moment M. Then the axial thrust assists the anchorage bolts to resist the bending moment and taking moments about the leeward bolts,

$$T \cdot 2a + V \cdot a = M, \qquad \text{or } T = \tfrac{1}{2}\left(\frac{M}{a} - V\right)$$

*Pressure between Sole Plate and Foundation.*—The unit pressure may be assumed to vary uniformly and consequently, from Art. 112, if A is the area of sole plate the greatest intensity at the leeward edge will be

$$P = \frac{V}{A} + \frac{My_1}{I}$$

where $y_1$ is half the width of the sole plate from the leeward to the windward edge and I is a principal moment of inertia of the sole plate area.

*Stress in Sole Plate.*—The overhang of the sole plate beyond the attachment angles must be limited so that treated as a cantilever with a load varying as above the extreme stress on the face of the plate is within a working value.

## EXAMPLES XVI.

In Problems No. 1 to No. 5 inclusive, as in Art. 183, take the dead load 0·6 ton per foot run, live load 1·5 ton per foot, together with 23 tons concentrated in any position. Impact load = (range of load)$^2$ ÷ maximum load. Unit stress 9 tons per sq. in. in tension, 9-0·03 $\frac{l}{k}$ tons per sq. in. in compression. Rivet shear stress 7 tons per sq. in. bearing stress 14 tons per sq. in.

1. Find suitable sections for the booms at the centre of the span of the girder in Plate II.

2. Find suitable sections for the booms 20 feet from the centre of the span of the girder in Plate II.

3. Design the diagonals S2, S3, S4, and S5 in Plate II.

4. Design the verticals P3 and P6, Plate II.

5. How many $\frac{7}{8}$-inch rivets are required to attach the gusset plate at the foot of the vertical P3 to the lower boom?

6. A beam is attached to end supports by the web cleats only, shown in Fig. 267. Assuming that a reaction of 14 tons acts at the back of the cleats, find the maximum force on any rivet in the web cleats.

7. Twenty-five rivets arranged in the form of a square have a pitch of 3 inches in each direction. What is the greatest stress on any rivet if the group resists a pull of 50 tons parallel to the side of the square and a moment of 100 ton-inches.

8. Solve the previous problem if the resultant pull is parallel to a diagonal of the square.

# CHAPTER XVII

## *PLATE GIRDERS AND BRIDGES*

**186. Types and Proportions.**—The moment of resistance to bending of plate girder sections has been referred to and illustrated in Art. 67 which the reader may revise with advantage before proceeding with the present chapter. The shear stress has been dealt with in Art. 72 and the principal stresses in Art. 73.

In the main, the flanges of a plate girder resist the bending moment and the web resists the shearing force; for a girder simply supported at its ends the bending moment to be resisted will be greatest about the centre of the span and the shearing force greatest at the ends. The plate girder represents a practical approximation to a beam of uniform strength, for its maximum moment of resistance with a full working unit stress allowance at any section is roughly proportional to the maximum bending moment which the girder has to carry. In some cases also the web section is varied, being greatest towards the ends where the shearing force is greatest. The variation in moment of resistance to bending is

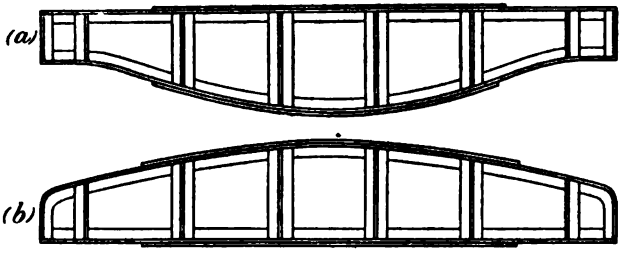



FIG. 271.—Curved flange girders.

accomplished in two distinct ways, viz. (1) in parallel flange girders (Plates III. and IV.) by varying the section of the flanges, (2) in curved flange girders (Fig. 271) by varying the height or distance between the flanges. Of the latter kind (Fig. 271) there are two types, (*a*) the fish-bellied girder with curved bottom flange loaded on the straight top flange such as is used in large travelling cranes and (*b*) the hog-back girder with curved upper flange loaded on the straight bottom flange as used occasionally in bridges. The parallel flange type is the simplest and most economical to construct and is the commonest type. The

curved types may in some cases save material and be lighter, and in some cases are used for the sake of appearance.

Plate girder bridges are used for spans of from 20 to 80 feet and their use tends to extend to larger spans of 100 feet and more.

The proportions vary considerably, but a depth of from 1/12 to 1/10 of the span is about the usual average with single webs, the proportion being somewhat greater in short girders and less in long ones, and in girders having two web plates. The breadth of flanges varies greatly in different classes of work. The flange plates should not overlap the angles by more than say 4″ unless there are stiffeners at short intervals, otherwise there may be local buckling of the compression flange under thrust. For this reason flanges seldom exceed 20″ or 22″ in breadth even in very deep girders and they are frequently not more than 18″. For small spans a width equal to about a third of the depth may be taken as an average proportion.

**187. Curtailment of Flange Plates —Flange Splices.**—In order to reduce the moment of resistance of a parallel flanged girder in proportion to the bending moment, the several plates which with the angles constitute the central and maximum flange section need not all be carried throughout the whole span. It is only necessary to carry any plate so far that the moment of resistance of the remainder of the section is not less than the bending moment to be resisted. A simple way of finding where the plates may be curtailed is shown in Fig. 272; BEFHD represents the bending moment diagram drawn to scale. Then if $A_1$, $A_2$, $A_3$, are the respective areas of section of say the top plate, second plate, and main plate including the angles, etc., the maximum bending moment and moment of resistance HK may be proportionally divided as shown into the moment of resistance of the several parts of the flange by the well-known construction of setting off lengths along BC proportional to $A_3$, $A_2$, $A_1$. If $A_1 + A_2 + A_3$ exceeds the total area A required at the centre to give the moment of resistance HK the point C

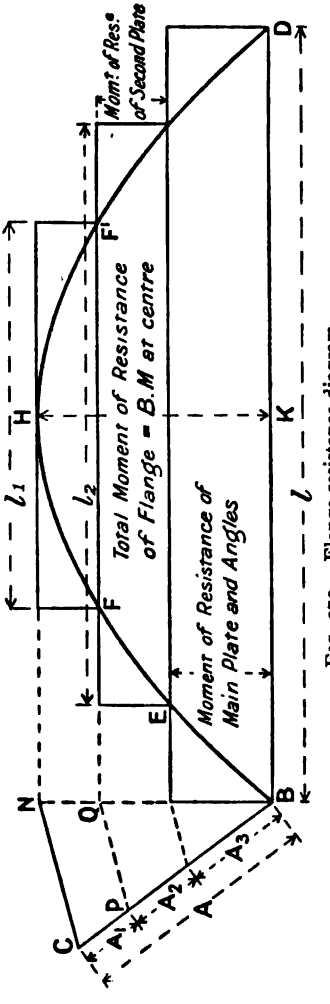



FIG. 272.—Flange resistance diagram.

joined to N must be such that BC represents the smaller quantity A where

$$f \times A \times d = \text{maximum bending moment},$$

or

$$A = \frac{\text{maximum bending moment}}{f \times d} \quad . \quad . \quad . \quad (1)$$

$d$ being the effective depth of girder.

The length of the top plate necessary to keep the moment of resistance up to the amount of the bending moment is found by drawing PQ parallel to CN and a horizontal line through Q meeting the bending moment diagram in F and $F^1$. Then $FF^1$ gives the length of top plate required neglecting the connection to the remainder of the flange. The actual length of plate used often exceeds $FF^1$ (empirically) by a length at each end sufficient to contain such a number of rivets that their resistance shall be equal to the total working stress attributable to the plate, generally some 3 or 4 times the length of the rivet pitch.

The length of second and other plates is similarly determined. Examples are given in the designs in Arts. 191 and 192.

The results may of course be calculated, for the length of any plate is obtained by equating the working moment of resistance of the flange sectional area below it to an expression for bending moment in terms of a variable distance $x$ along the girder, and solving for $x$.

*Uniformly distributed load.*—Let $w$ = load per inch run, $l_1$ = length of top plate, $l$ = span, $f$ = working unit stress due to bending in the flanges, then from the parabolic bending moment diagram,

$$\frac{w}{2}\left\{\left(\frac{l}{2}\right)^2 - \left(\frac{l_1}{2}\right)^2\right\} = f \times d \times (A_2 + A_3) \quad . \quad . \quad . \quad (2)$$

from which $l_1$ may easily be found.

Or more simply

$$\frac{l_1}{l} = \sqrt{\frac{A - (A_2 + A_3)}{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and if $A_1 + A_2 + A_3 = A$ exactly (which is seldom the case in practice)

$$l_1 = l\sqrt{\frac{A_1}{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In these formulæ safety in approximate values of $d$ lies in taking a *high* value; thus for the outer plate the reduction in moment of resistance is $A - (A_2 + A_3)$ multiplied by a quantity rather greater than the effective depth for the whole flange: the plate should not be curtailed before the sections at which the bending moment is similarly reduced.

In the case of girders of variable depth such as those in Fig. 271 equation (2) holds good, $d$ being a variable quantity. For the graphical method the ordinates of the bending moment diagram may be all multiplied by the inverse ratio of the girder depths to that at the central or other section and the depths then taken as constant and equal to that at the chosen section.

*Equivalent Uniformly Distributed Loads.*—When equivalent uniformly distributed loads have been determined applicable to centre

sections only (see Art. 84), a different value and usually a higher one will have to be employed in constructing the flange diagram. The parabola so used would be that which would completely circumscribe the actual maximum bending moment diagram. In order to give the plate lengths directly, without additions to allow for riveting, a still higher table of equivalent loads is sometimes used; an example is given in Art. 192 in which the load (56·5 tons) is about 8 per cent. greater than for the design of the central section.

*Flange Splice.*—The splice of a flange should not generally be at or very near the section of greatest bending moment in the girder. It is made in a similar way to that in Fig. 262 and not more than one member constituting the flange is broken at one place.

**188. Web Stresses and Stiffeners.**—The magnitude and direction of the stresses in the web of a girder of I section have been dealt with in Art. 73. In plate girders of ordinary proportions the deep web alone is quite inadequate to resist buckling under the influence of the principal compressive stress arising mainly from the shear stress. It is therefore reinforced at intervals not usually exceeding the depth of web by stiffeners as shown in Plates III. and IV. The spacing of the stiffeners depends upon the intensity of shear stress allowable in the web. The buckling resistance of the web subject to compressive and tensile stresses at right angles has sometimes been compared to that of a strut, so that if $d$ = unsupported length of web plate between consecutive stiffeners and $t$ = thickness of web, the line of thrust being taken as inclined 45° to the vertical stiffeners, applying Rankine's formula (5) Art. 116 and putting

$$l = \sqrt{2}\,.\,d, \quad k^2 = \tfrac{1}{12}t^2, \quad a = 30{,}000$$

maximum allowable compressive stress = max. allowable web shear stress =

$$\frac{\text{maximum allowable direct stress}}{1 + \frac{1}{1250}\cdot\frac{d^2}{t^2}} \quad . \quad . \quad . \quad . \quad (1)$$

which, if $d$ and $t$ are known gives a rule for checking the web stress or if $t$ and the actual unit shear stress in the web are known gives a rule for finding a suitable value of $d$. Thus if the allowable shear stress is half the tensile unit stress

$$\left(\frac{d}{t}\right)^2 = 1250, \quad d = 35t \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

An American rule with a different constant is,

$$\text{shear unit stress} = \frac{\text{maximum direct unit stress}}{1 + \frac{1}{1500}\cdot\left(\frac{d}{t}\right)^2} \quad . \quad . \quad (3)$$

where $d$ is the pitch of the stiffeners and therefore exceeds the length of unsupported plate by say 4 to 6 inches.

This like Rankine's formula for struts might be reduced to the approximate form of, say,

$$\text{shear stress} = \text{max. direct stress} \times \left(1 - 0{\cdot}0125\frac{d}{t}\right) \quad . \quad . \quad (4)$$

If $d = 40.t$, these rules limit the shear to about half the direct stress.

Common requirements of specifications for plate girders are that stiffeners shall not be further apart than the depth of the web if the distance between the flange angles exceeds say 50 or 60 times the thickness of the web, with a further limitation to about 5 feet whatever the depth of web. The maximum web shear stress is sometimes specified not to exceed half the allowable tensile unit stress; this clause is made with due regard to that for stiffener spacing and *vice versâ*.

Another function of stiffeners is to transmit the concentrated loads at the ends or at the cross girders to the web of a main girder. Hence stiffeners are required at every point of application of a concentrated load. These stiffeners at load points and their rivets are required by

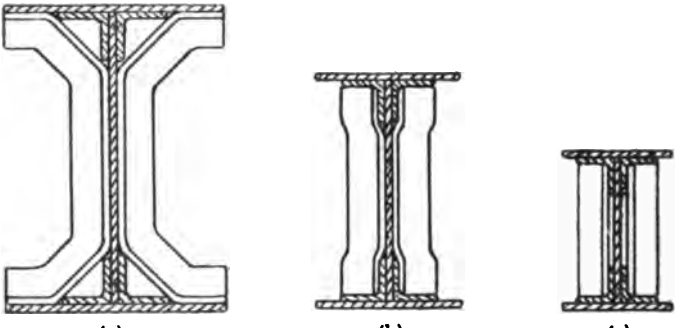



FIG. 273.—Stiffeners.

some specifications to be capable of carrying the whole load applied by them, while by others they are required to be capable of carrying (as a strut) say two-thirds of the vertical shear on the girder at their point of attachment and the whole shear in the case of the end stiffeners, the length of strut being taken as two-thirds or three-quarters of the web length between flange angles.

The stiffeners at points of concentrated loading and at web joints are often made of two angle sections, and a gusset plate, bent over so as to support both flanges and the web, such stiffeners being used in pairs on opposite sides of the web. In other cases either Tee or angle section stiffeners are used either bent over knee shape as shown in Plates III. and IV. and (a), Fig. 273, when the breadth of flange allows, or as in (b) and (c), Fig. 273, bearing tightly against the flange angles, the vertical legs of which they clear either by having packing plates (American, "fillers") behind them (c), or by being "joggled" or "crimped" near their ends (b).

Stiffeners are generally placed on both sides of the web, and where cross girders are used the inside stiffener is turned over and riveted to the top flange, as shown in Plate IV.

In box plate girders stiffening diaphragms are often placed inside, attached by angles to each web plate.

For so complex a structure as a plate girder there can be no very exact theory as to the distribution of web stress, and the above rules relating to spacing must be regarded as largely empirical. Further, the pitch of stiffeners selected will frequently be influenced by the position of cross girders and web splices.

Some idea of the effect of different sectional areas and spacings of stiffeners may be obtained from experiments carried to the point at which the web buckles into wave form; such experiments have been made by Prof. Lilly,[1] who regarding the stiffeners as analogous to the struts in a braced truss (see Fig. 204 say) deduces under certain assumptions a rule of the type (3).

It has been suggested that stiffeners would be more effective if placed at an inclination of 45°, viz. in the direction of the compressive principal stress at the neutral surface of the girder. Constructional advantages, transmission of vertical concentrated loads and more effective support of the compression flange are points which are against such a plan.

**189. Pitch of Rivets Uniting Flanges to Web.**—The rivets attaching the flange angles to the web have to transmit the longitudinal shear between the web and the flanges. Let $p$ be the pitch of the rivets and let R be the working resistance of one rivet. Neglecting any variation in intensity of the shear stress in the web and adopting the approximation mentioned in Art. 72, the intensity of shear stress, horizontally and vertically, is

$$q = \frac{F}{t \cdot h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $t$ = thickness, and $h$ = depth of web, and F = gross shearing force on the section. In a distance $p$ horizontally the total horizontal shearing force to be resisted is $q \cdot p \cdot t$, hence—

$$qpt = R$$

$$p = \frac{R}{q \cdot t} = \frac{Rh}{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which might also be obtained by taking moments about a point P (Fig. 274) of the forces on a section of the web of length $p$, remembering that the only important force on the web is the shearing force F. The expression above for the pitch $p$ shows that in a girder of constant depth $h$ the pitch may be made greater where the variable shearing force F is smaller; for example, towards the middle of the span of a girder carrying a distributed load. Often a pitch suitable for the section of a maximum shearing force is used throughout for convenience instead of a variable pitch. The working resistance R of a single rivet

[1] *Engineering*, Feb. 1, 1907.

may be its resistance to shearing or its resistance to crushing across a diameter.

For attaching the angles to the flange plates twice as many rivets will be necessary if the shearing resistance is the criterion, for the rivets are in single shear; this will require the same pitch $p$ as before on either side of the web, there being then twice as many rivets as are used for attaching the angles to the web. If, however, resistance to crushing is the criterion throughout, a pitch $2p$ might be used to attach the angles to the flange.

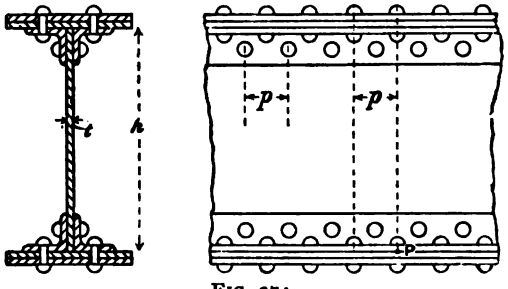


FIG. 274.

If the above rule indicates an inconveniently small pitch, larger angles with double (zigzag or staggered) riveting, a thicker web, or larger rivets must be used.

**190. Web Splices.**—The number of splices in a web depends upon the maximum length of plate of given width obtainable, and also upon conditions of manufacture and erection, great lengths of broad plate being difficult to handle in making and in transport.

The commonest form of web splice is a double covered riveted butt joint, as shown in Plates III. and IV. The number of rivets provided is sufficient to carry the shearing force at the splice, the value of each rivet being measured by its resistance in double shear or in bearing whichever may be the least.

When some one-sixth or one-eighth of the web section is for moment of resistance computation included in the flange area, *i.e.* the web is relied upon to take a share in the moment of resistance, the web splice is calculated to resist the bending moment as well as the shear. But whether the web is assumed to resist any bending moment or not it will almost certainly carry some fraction, the upper limit of which may be taken as one-sixth of the web section divided by the total flange area, inclusive of one-sixth of the web ($f \times$ one-sixth area $\times d = \frac{1}{6} ftd^2$). Or if A = flange area, including one-sixth $td$ where $t$ = thickness and $d$ = depth of web, and M = bending moment at the section, the limit of moment carried by the web may be taken as,

$$\frac{td}{6A} \cdot M$$

The bending moment carried by the web is resisted by the rivets of the web splice, which are thereby stressed in the manner explained and estimated in Art. 182 in addition to the vertical shearing force which they carry. In British practice it is usual to neglect the stresses in the web splice rivets resulting from the bending moment carried by the

web; in consequence the rivets probably sometimes carry a rather higher stress than that computed, unless the joint is relieved by slight movements corresponding to the bending strains in a seamless web, throwing absolutely the whole bending moment on the flanges. But as in Plates III. and IV. the web splices are almost invariably made under web stiffeners, the rivets being ample in number all take their share of the resistance offered by the joint. Such a joint, from experience and from theoretical computation so far as it may reasonably be attempted in such a complex structure, is entirely satisfactory. In America web splices estimated to carry bending moment have sometimes been constructed with six plates—two pairs with their lengths horizontal (one plate being in contact with each of the four flange angles), and the third pair with their lengths vertical and filling the space between the other two pairs: the joint covers forming an I shape on each side of the web. The object of such a joint is to place the rivets advantageously (*i.e.* far from the centroid of the group) to resist bending moment without being thereby highly stressed. Another plan is to use a single pair of vertical plates, but to space the rivets closer together near the flanges and further apart near the centre of the depth of the web.

Such joints may, apart from extra constructional cost, be advantageous if made where the bending moment is great and the shearing force insignificant, but if a web has also to resist considerable vertical shearing force a concentration of rivets near the flanges may cause an unnecessary concentration of stress in that part of the web which is already (see Fig. 113) most heavily stressed. A uniform distribution of rivets corresponds most closely to the distribution of material in the web and will tend least to disturbance or secondary stress in the web. If the joint has to withstand bending moment and shearing force the rivets should of course be sufficient for both purposes. It is well to recall the fact that all calculations on riveted joints are conventional for various reasons, including the neglect of friction, which is always present to a great but unknown extent.

**191. Plate Girder Deck Bridge.**—The various points in the four preceding articles are illustrated in the numerical computations for the design of the girders shown in Plate III. The deck type is one of the most economical, but requires a sufficient available depth for the girders.

*Data.*—To carry a single line of railway. Construction depth, 5′ 8″ (*i.e.* overall from rail level to under side of bridge superstructure.) Effective span, 40 feet. Depth over angles, 4 feet. Equivalent uniformly distributed loads as tabulated in Art. 84, *i.e.* 2·4 tons per foot = 96 tons for bending moment, with 15 per cent. more for shear. Working unit stresses, 7·5 tons per sq. in. tension; 5 tons per sq. in. shearing; 10 tons per sq. in. bearing; all reckoned for dead loads. Variable load unit stress by dynamic formula—(*b*) Art. 41—or the equivalent rule of adding to the maximum stress, impact stress equal to the range of stress (impact coefficient of unity). Allowable dead load stress in the web $\frac{6}{1 + \frac{d^2}{1600 \cdot t^2}}$ tons per sq. in. but not exceeding 3 tons

per sq. in., where $d$ = unsupported distance between stiffeners, which are not to be spaced further apart than the depth of the web. No section to be less than $\frac{3}{8}$" thick.

| | |
|---|---|
| *Dead Load.*—Ballast, permanent way, steel flooring, bracing, etc. (see tabular estimate below) . . . . . | 12·25 tons per girder |
| Assumed weight of girder, checked after design (see below) . . . . . | 7·25 „ |
| Total . . . . | 19·5 tons. |

*Live Load.*—Per girder $96/2 = 48$ tons.

*Girder Flanges.*—Equivalent dead load (taken uniformly distributed)

$$48 + 48 + 19{\cdot}5 = 115{\cdot}5 \text{ tons.}$$

$$\text{Central bending moment} = \frac{115{\cdot}5 \times 40}{8} = 577{\cdot}5 \text{ ton-feet}$$

$$= 6930 \text{ ton-inches.}$$

$$\text{Modulus of section required} = \frac{6930}{7{\cdot}5} = 925 \text{ (inches)}^3.$$

Flange area required with effective depth $48'' = \frac{925}{48} = 19{\cdot}25$ sq. in. The flanges are taken rather broad at 22". The main plate is extended by $2\frac{1}{2}$" overlap on each side to allow attachment of floor plating, giving a total breadth of 27".

*Top Flange.*—Two angle bars $4'' \times 4'' \times \frac{1}{2}''$ (see Table V. Appendix) less four rivet holes, taken $\frac{1}{16}$" over the rivet diameter (see Art. 67) gives 5·625 sq. in. leaving $19{\cdot}25 - 5{\cdot}625 = 13{\cdot}625$ sq. in. Taking a $\frac{7}{16}$" main plate (the minimum suitable thickness for a flange plate which stands alone) $27'' \times \frac{7}{16}''$ allowing for rivet holes, gives 9·461 sq. in., leaving $13{\cdot}625 - 9{\cdot}461 = 4{\cdot}164$ sq. in.

Thickness required for second plate 22" wide, deducting four rivet holes is $\frac{4{\cdot}164}{18{\cdot}25} = 0{\cdot}227''$, hence $\frac{3}{8}$" plate is used. The full area provided is then

| | |
|---|---|
| 2 angles bars less 4 rivet holes $\frac{15}{16}$" diameter . . . . | 5·925 sq. in. |
| 1 main plate less 4 holes $\frac{15}{16}$" and 2 holes $\frac{13}{16}$" diameter . | 9·461 „ |
| 1 outer plate less 4 holes $\frac{15}{16}$" diameter . . . . . | 6·844 „ |
| Total . . . . | 22·230 „ |

*Bottom Flange—*

| | |
|---|---|
| 2 angles as above . . . . . | 5·625 sq. in. |
| $\frac{7}{16}$" main plate 22" less 4 holes $\frac{15}{16}$" . . . | 7·984 „ |
| $\frac{3}{8}$" plate 22" less 4 holes $\frac{15}{16}$" . . . . . | 6·844 „ |
| | 20·453 „ |

Resistance of 22″ × $\frac{3}{8}$″ plate at 7·5 tons per sq. in.

$$22 \times \tfrac{3}{8} \times 7{\cdot}5 = 61{\cdot}875 \text{ tons.}$$

Resistance of $\frac{7}{8}$″ rivet in single shear = 0·61 × 5 = 3·01 tons.
Number of rivets equal to resistance to outer plate

$$= \frac{61{\cdot}875}{3{\cdot}01} = 21 \text{ rivets.}$$

The outer plates are (empirically) prolonged at each end sufficiently to contain 21 rivets beyond its length given by the flange diagrams (Fig. 275).

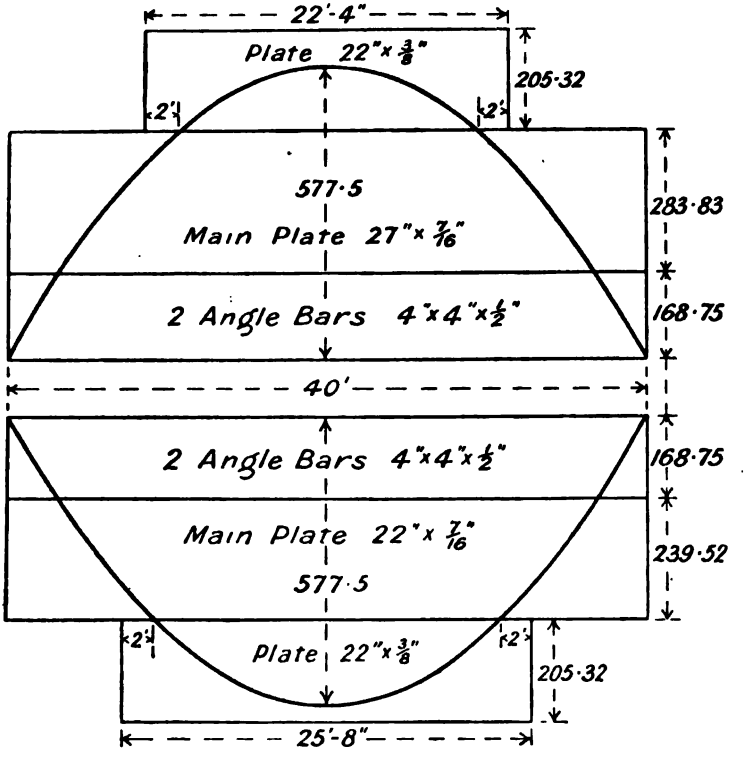



FIG. 275.—Flange resistance diagram (units ton-feet).

*Moments of Resistance for Flange Diagrams.*—
Top flange :

| | | |
|---|---|---|
| 2 angles give 5·625 × 7·5 × 4 = | 168·75 | ton-feet |
| Main plate 9·461 × 7·5 × 4 = | 283·83 | ,, |
| Outer plate 6·844 × 7·5 × 4 = | 205·32 | ,, |
| Total . . . | 657·90 | ,, |

Central ordinate of parabola 577·5 ton-feet.
Bottom flange:

| | | |
|---|---|---|
| 2 angles, as before | = 168·75 | ton-feet |
| $\frac{7}{16}$″ main plate 7·984 × 7·5 × 4 | = 239·52 | „ |
| $\frac{3}{8}$″ outer plate, as before | = 205·32 | „ |
| Total . . . | 613·59 | „ |

The moments of resistance of the various parts and the bending moment are shown in Fig. 275. The lengths of the outer plates prolonged to contain 21 rivets are shown in Fig. 275 and in Plate III.

*Web and Stiffeners.*—

End shear for dead load $\frac{19·5}{2} = 9·75$ tons.

Equivalent uniform live load for shear 1·15 × 48 = 55·2 tons.

End shear for live load = 27·6 tons.

Equivalent dead load shear for any section in the left half of the span (negative shearing force) = dead load shear + maximum live load shear + range of shear. The range of shear without regard to sign is the sum of the extreme opposite shears. Hence the equivalent dead load shear equals the dead load shear + 2(maximum live load shear) + minimum live load shear (*i.e.* the maximum value of opposite sign), all three parts taken with like signs. In Fig. 276 the extreme vertical

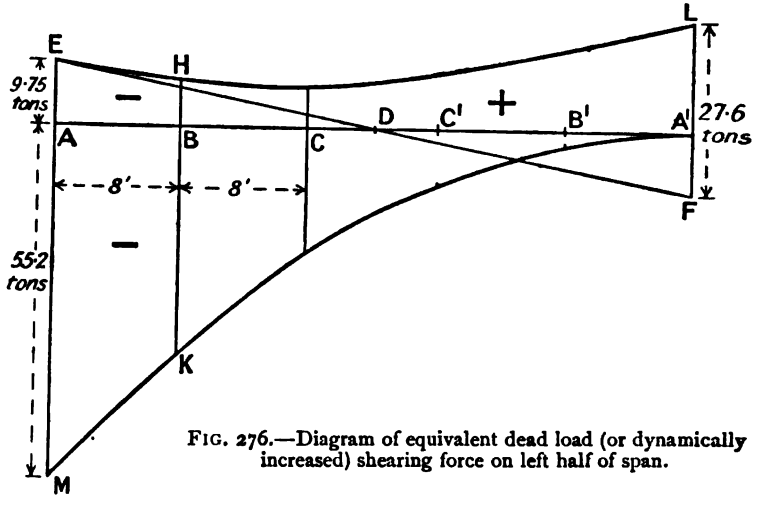



FIG. 276.—Diagram of equivalent dead load (or dynamically increased) shearing force on left half of span.

ordinates across the diagram give for the left-hand half of the span the dynamically increased or equivalent dead load shearing forces: AM = (27·6 × 2); and the parabola MKA′ is drawn on the base AA′ having ordinates proportional to twice the negative shearing force. The ordinates of the curve EHL from the sloping base line EF

represent the positive shearing forces, while ordinates of the line ED represent the dead load shears.

Web lengths of 8 feet are taken as convenient lengths with about 12″ extra on the end sections to cover the bearing plates, the full length of girder being about 42 feet, the effective span for calculation 40 feet, and the clear span between the bearing plates about 38 feet. Placing a stiffener over the inner end of the bearing plate leaves 7′ 4″ to the first joint; one intermediate stiffener would give too wide a spacing, so two may be used with spacings of 28″, 28″, 32″ as shown; these distances being exact multiples of the rivet pitch.

The end shear is $27{\cdot}6 + 9{\cdot}75 + 27{\cdot}6 = 64{\cdot}95$ tons. If we allow for the first pair of intermediate stiffeners to take $\frac{2}{3}$ of this at the full allowance of 7·5 tons per sq. in., the total area required would be $\frac{2}{3} \times 64{\cdot}95 \div 7{\cdot}5 = 5{\cdot}8$ sq. in. Two tees 6″ × 3″ × $\frac{3}{8}$″ (see Appendix) give 6·52 sq. in., and these sections may be used throughout for the intermediate stiffeners. The stiffeners over the end plates and at the web joints are of much more ample strength, consisting of pairs of angles with $\frac{3}{8}$″ gusset plates the full width of the flanges; these gusset plates serve for attachment of the cross bracing. The maximum unsupported distance between stiffeners at the ends = 28 − 6 = 22″.

Assuming $\frac{9}{16}$″ web,

$$\text{shear stress} = \frac{64{\cdot}95 \times 16}{48 \times 9} = 2{\cdot}41 \text{ tons per sq. in.}$$

$$\text{Allowable shear stress} = \frac{6}{1 + \frac{1}{1600}\left(\frac{22}{\frac{9}{16}}\right)^2} = 3{\cdot}07 \text{ tons per sq. in.}$$

which is further limited to 3 tons per sq. in. by the specified conditions.

Assuming 1″ rivets, resistance per rivet in double shear

$$= 0{\cdot}785 \times 1{\cdot}75 \times 5 = 6{\cdot}85 \text{ tons.}$$

Resistance per rivet in bearing

$$= 1 \times \tfrac{9}{16} \times 10 = 5{\cdot}625 \text{ tons.}$$

Hence from (2), Art. 189, pitch of flange rivets

$$= \frac{5{\cdot}625 \times 48}{64{\cdot}95} = 4{\cdot}16, \text{ say } 4''.$$

Bearing resistance of a $\frac{7}{8}$″ rivet

$$= \tfrac{7}{7} \times \tfrac{9}{16} \times 10 = 4{\cdot}92 \text{ tons.}$$

Shear resistance in 48″ depth at 4″ pitch of $\frac{7}{8}$ rivets, from (2), Art. 189

$$= \frac{4{\cdot}92 \times 48}{4} = 59{\cdot}04 \text{ tons,}$$

which is greater than the shear (from Fig. 276) two feet from the end. The rivets are therefore changed to $\frac{7}{8}$″ at the next stiffener. Shear at joint B (Fig. 276) scales 42·3 tons, or, calculating from the parabolic and straight-line equations

$$(\tfrac{4}{5})^2 + 55{\cdot}2 + 0{\cdot}6 \times 9{\cdot}75 + (\tfrac{1}{5})^2 \times 27{\cdot}6 = 42{\cdot}3 \text{ tons.}$$

Assuming $\frac{1}{2}''$ webs,

$$\text{shear stress} = \frac{42{\cdot}3}{48 \times 0{\cdot}5} = 1{\cdot}764 \text{ tons per sq. in.}$$

Allowable stress with stiffeners spaced $\frac{1}{3}$ of 8′, *i.e.* 2′·8″ centres, or 2′ 2″ apart

$$6 \div \left\{1 + \frac{1}{1600} \times \left(\frac{26}{0{\cdot}5}\right)^2\right\} = 6 \div 2{\cdot}69 = 2{\cdot}23 \text{ tons per sq. in.}$$

Assuming $\frac{7}{8}''$ rivets, resistance in bearing

$$= 0{\cdot}875 \times 0{\cdot}5 \times 10 = 4{\cdot}375 \text{ tons.}$$

$$p = \frac{4{\cdot}375 \times 48}{42{\cdot}3} = 4{\cdot}96'', \text{ say } 4'' \text{ pitch.}$$

Number of rivets required (for shear alone) in web joint $\frac{42{\cdot}3}{4{\cdot}375} = 9{\cdot}7$. Nine rivets are provided, but the stiffener has surplus rivets. Shear at joint C (Fig. 276) scales 26·3 tons or

$$(\tfrac{3}{5})^2 \times 55{\cdot}2 + 0{\cdot}2 \times 9{\cdot}75 + (\tfrac{2}{5})^2 \times 27{\cdot}6 = 26{\cdot}3 \text{ tons.}$$

With $\frac{1}{2}''$ web, shear stress $= \frac{26{\cdot}3}{48 \times 0{\cdot}5} = 1{\cdot}10$ tons per sq. in.

Allowable stress with stiffeners spaced 4′ centres = 42″ apart

$$6 \div \left\{1 + \frac{1}{1600}\left(\frac{42}{0{\cdot}5}\right)^2\right\} = 6 \div 5{\cdot}4 = 1{\cdot}11 \text{ tons per sq. in.}$$

The 4″ pitch for $\frac{7}{8}''$ rivets is continued to the centre.

*Other Details.*—The cross bracing of the two main girders, the floor plating, ballast plate and its supports are sufficiently shown in Plate III. The side clearances and lack of parapet girders at the outside of the track take this design outside the usual British railway practice.

On the other hand, the provision of a steel plate floor, low unit stresses (taking account of the high allowance for dynamic increment of live load stress) and ample scantlings and rivets for stiffeners and webs are typical of the practice of first-class British railways which design bridges with a view to long endurance under proper maintenance. It has been remarked that the plate girder bridge ranks next to the brick or masonry arch as a durable structure, and with proper facilities for painting and effective drainage it is probably economical to allow some provision for a long working life. A feature which might cause comment is the attachment of the floor plating *underneath* the main flange plate, thereby putting the rivets in tension; this is sometimes done for more effective drainage and the prevention of leakage at the joint. But in many cases the flooring is placed *over* the main flanges; the matter is one of opinion based on practical experience rather than of theory.

The following table shows an estimate of weights to check the assumed weight of girder. Had there been any serious under-estimate, the design would have required modification accordingly.

single angle drill

single angle

WEIGHTS OF MAIN GIRDERS AND FLOORING.

| Piece. | Number required. | Length. | Total length. | Weight per foot. | Weight in lbs. |
|---|---|---|---|---|---|
| | | | | lbs. | |
| Flange . . . 1′ 10″ × ⅜″ | 1 | 22′ 4″ | 48′ 0″ | 28·05 | 1346 |
| ,, . . . ,, | 1 | 25′ 8″ | | | |
| ,, . . . 2′ 3″ × 7/16″ | 1 | 43′ 0″ | 43′ 0″ | 40·16 | 1727 |
| ,, . . . 4″ × 4″ × ½″ | 4 | 41′ 11″ | 167′ 67″ | 12·75 | 2138 |
| ,, . . . 1′ 10″ × 7/16″ | 1 | 42′ 0″ | 42 | 32·72 | 1374 |
| Bearing pl. . 1′ 10″ × ⅜″ | 2 | 2′ 0″ | 4 | 56·1 | 224 |
| Web . . . 4′ 0″ × 9/16″ | 2 | 8′ 11½″ | 17·92 | 91·8 | 1645 |
| ,, . . . 4′ 0″ × ½″ | 4 | 8′ 0″ | 32 | 81·6 | 2611 |
| Covers . . . 12″ × ⅜″ | 8 | 3′ 4″ | 26·67 | 15·3 | 408 |
| End pls. . . 1′ 10″ × ½″ | 2 | 4′ 0″ | 8 | 37·4 | 299 |
| End angles . 4″ × 4″ × ½″ | 2 | 5′ 4″ | 20·92 | 12·75 | 267 |
| ,, ,, | 2 | 5′ 1½″ | | | |
| Tees . . . 6″ × 3″ × ⅜″ | 9 | 5′ 4″ | 94·12 | 11·08 | 1043 |
| ,, . . . ,, | 9 | 5′ 1½″ | | | |
| Gusset pls. . . 10½″ × ⅜″ | 12 | 3′ 11½″ | 47·5 | 13·39 | 636 |
| Gusset angles 3″ × 3″ × ⅜″ | 12 | 5′ 4″ | 125·5 | 7·18 | 901 |
| ,, ,, | 12 | 5′ 1½″ | | | |
| Packings . . 6½″ × ¼″ | 18 | 6″ | 16·83 | 11·05 | 186 |
| ,, . . ,, | 12 | 6½″ | | | |
| ,, . . ,, | 4 | 4″ | | | |
| | | | | | 14805 |
| | | | Add 5% | rivets | 740 |
| | | | | | 15545 |
| | | | | | =6 t. 18 c. 3 q. 5 lbs. |
| *Flooring*— | | | | | |
| Plate . . . 1′ 6½″ × 7/16″ | 2 | 43′ 0″ | 86 | 27·52 | 2367 |
| ,, . . . 3′ 3″ × 7/16″ | 1 | 43′ 0″ | 43 | 58·01 | 2494 |
| Covers . . 6″ × ⅜″ | 12 | 1′ 1″ | 30 | 7·65 | 230 |
| ,, . . ,, | 6 | 2′ 10″ | | | |
| Tees . . . 6″ × 3″ × ⅜″ | 5 | 2′ 10″ | 25 | 11·08 | 277 |
| ,, . . ,, | 10 | 1′ 1″ | | | |
| Angles . . 3″ × 3″ × ⅜″ | 12 | 2′ 10″ | 70 | 7·18 | 503 |
| ,, . . ,, | 24 | 1′ 6″ | | | |
| Bracing tees 6″ × 3″ × ⅜″ | 12 | 5′ 4″ | 64 | 11·08 | 709 |
| pls. . . 1′ 5″ × ⅜″ | 6 | 1′ 1½″ | 6·75 | 21·68 | 146 |
| Ballast pls. . 12″ × 7/16″ | 2 | 43′ 0″ | 86 | 17·85 | 1535 |
| Ballast angle bars . 3″ × 3″ × ⅜″ | 2 | 43′ 0″ | | | |
| Angles . . ,, | 24 | 5′ 3″ | 232 | 7·18 | 1666 |
| Drips . . ,, | 2 | 10′ 0″ | | | |
| Packings . 3″ × 7/16″ | 4 | 1′ 10″ | 7·33 | 4·46 | 33 |
| | | | | | 9960 |
| | | | Add 5% | rivets | 498 |
| | | | | | 10458 |
| Or 5229 lb. per girder. | | | | | =4 t. 13 c. 1 q. 14 lbs. |

Ballast 9″ deep—200 sq. ft. at 120 lbs. per cubic feet gives 18,000 lbs. per girder. Asphalte 1″ thick, 2600 lbs. and permanent way, say 87·5 × 40 = 3500 lbs. Total dead load = 15,545 + 5229 + 18,000 + 2600 + 3500 = 44,874 lbs. = 20 tons per girder instead of 19·5 tons as used in the calculations.

EXAMPLE.—Estimate the possible maximum stress on any rivet of the web joint nearest the abutments in the above design, assuming the joint to resist bending moment as well as shear.

Bending moment at 0·2 of the span from the end = $6930 \times 0{\cdot}2 \times 0{\cdot}8 \div (0{\cdot}5)^2 = 4440$ ton-inches. $\frac{1}{6}$ of the web area = $\frac{1}{6} \times 48 \times 0{\cdot}5 = 4$ sq. ins. equivalent area at the depth of the flange. Total equivalent flange area approximately = $21 + 4 = 25$ sq. ins. Upper limit to moment of resistance of the web = $\frac{4}{25} \times 4440 = 710{\cdot}4$ ton inches.

For the group of 19 rivets on one side of the joint $\Sigma(r^2)$ approximately = $2(4^2 + 8^2 + 12^2 + 16^2 + 2^2 + 6^2 + 10^2 + 14^2 + 18^2) = 2280$. Hence from Art. 182 (2) approximately (neglecting the distance between the two rows of rivets) the stress on the outermost rivet is

$$\frac{710 \times 18}{2280} = 5{\cdot}60 \text{ tons.}$$

Force per rivet due to vertical shear stress = $\frac{42{\cdot}3}{19} = 2{\cdot}23$ tons.

Total on the outer rivet taking these components as perpendicular,

$$\sqrt{(5{\cdot}6)^2 + (2{\cdot}23)^2} = 6{\cdot}03 \text{ tons.}$$

Probably no such amount of stress would be developed, because either the joint would yield and locally relieve the web of its share of bending stress or the friction of the joint would offer a great resistance to bending moment. Allowing for the resistance of bending moment by the joint rivets, to keep the force per rivet down to the specified 4·375 tons would require an extra row of rivets and correspondingly wider cover plates.

**192. Plate Girder Through Bridge.**—Plate IV. represents a through bridge for a single railway track with a floor consisting of railbearers or stringers supported on cross girders, the whole being covered by plating.

*Data.*—Effective span 40 feet. Construction depth limited to 3 feet. Moving loads uniformly distributed, on 40 feet, 105 tons for central section flange area computation, 113 tons for estimating the curtailment of flange plates (allowing for overlap), 130 tons for shearing force. Cross girder centres 8 feet apart, for which length of railbearer allow uniformly distributed 44 tons for flange areas and 58 tons for shearing force. Maximum pressure per rail on cross girder 14 tons. Working unit stresses for dead loads, 6·5 tons per square inch for tension, 5 tons per square inch for shear, 10 tons per square inch for bearing. For web stress,

$$\frac{6}{1 + \frac{d^2}{1500t^2}} \text{ tons per square inch.}$$

where $t$ = thickness and $d$ = distance between stiffeners which are to be placed at all points of concentrated loading, and elsewhere with centres not further apart than the depth of the web if the ratio of depth to thickness of web exceeds 40. For varying stresses the dynamic formula

to be used or impact stress equal to the range of stress to be added to the maximum stress for use with the dead load unit stresses.

*Rail Bearers or Stringers.*—(See separate elevation, Plate IV.)

Flanges.—Estimated dead load on 8′ length, including ballast 10″ deep, asphalte, rail, chairs, $\frac{7}{16}$″ floor plating, Tee stiffeners and weight of railbearer (1000 lbs.) = 2·8 tons.

Live load per rail, $\frac{1}{2}$ of 44 = 22 tons.

Equivalent dead load 2·8 + 2 × 22 = 46·8 „

Central bending moment $\frac{46{\cdot}8 \times 8 \times 12}{8}$ = 561·6 ton-ins.

Modulus of section required 561·6 ÷ 6·5 = 86·3 (inches)³.

Taking the depth over the angles as 14″, and effective depth 13·5″, say,

Flange area for central section 86·3 ÷ 13·5 = 6·40 ins.

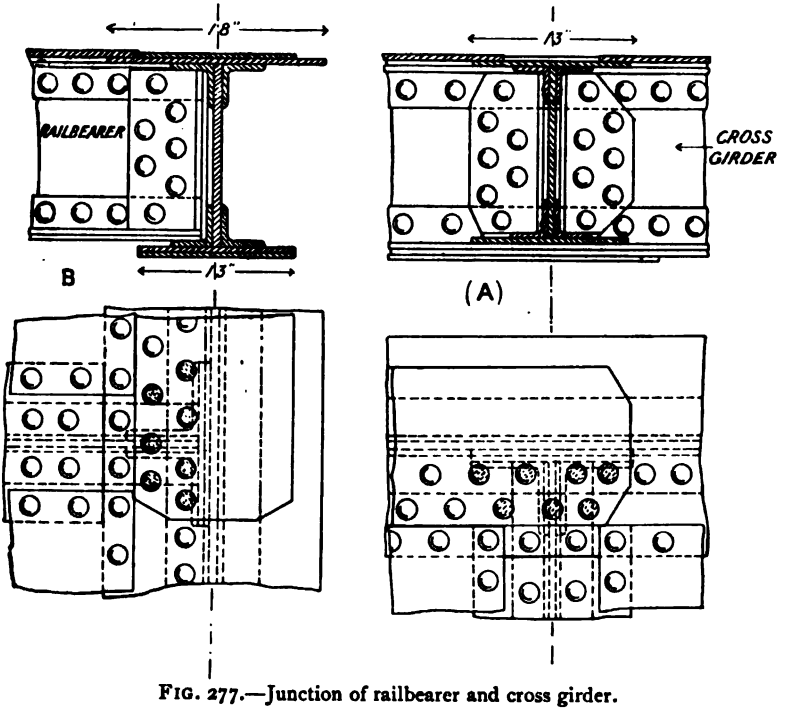



FIG. 277.—Junction of railbearer and cross girder.

Assuming $\frac{7}{8}$″ rivets for flanges and $\frac{3}{4}$″ for flooring attachment, the width required to get in angle and flooring rivets (see (A), Fig. 277) is about 13″; nett width, allowing 4 rivet holes, about 9·5 ins. Two angles

$3'' \times 3'' \times \frac{3}{8}''$ (see Appendix), less 4 rivet holes, give 2·81 sq. ins., leaving $6{\cdot}40 - 2{\cdot}81 = 3{\cdot}59$ sq. ins.

$$\text{Thickness } \frac{3{\cdot}59}{9{\cdot}5} = 0{\cdot}38'', \text{ say } \tfrac{7}{16}'' \text{ plate.}$$

*Railbearer Rivets and Web.*—Assume $\frac{5}{8}''$ web.

Dead load shear at the ends $\frac{1}{2} \times 2{\cdot}8 = 1{\cdot}4$ tons.

Live load shear at the ends $\frac{29}{2} = 14{\cdot}5$ tons.

Equivalent dead load shear at the ends $= 1{\cdot}4 + 29 = 30{\cdot}4$ tons.

Assuming 1" rivets, resistance (in bearing) $= 10 \times 0{\cdot}625 = 6{\cdot}25$ tons.

$$\text{Pitch } p = \frac{6{\cdot}25 \times 14}{30{\cdot}4} = 3'' \text{ pitch.}$$

To begin 4" pitch, try 2 feet from the ends,

Dead load shear $= \frac{1}{2} \times 1{\cdot}4 = 0{\cdot}70$ tons.

[1] Live load shear $= (\frac{3}{4})^2 \times 14{\cdot}5 = 8{\cdot}16$ tons.

[1] Range of shear $= \{(\frac{3}{4})^2 + (\frac{1}{4})^2\} \times 14{\cdot}5 = 9{\cdot}07$ tons.

Equivalent dead load shear $= 17{\cdot}93$ tons.

$$\text{pitch } p = \frac{6{\cdot}25 \times 14}{17{\cdot}93} = 4{\cdot}9''$$

Hence 4" pitch may begin at the stiffeners, which are placed 2 feet from the ends.

Rivets required to transfer the whole end load to the cross girders

$$30{\cdot}4 \div 6{\cdot}25 = 5 \text{ rivets.}$$

$$\text{Shear stress in web } \frac{30{\cdot}4}{14 \times 0{\cdot}625} = 3{\cdot}47 \text{ tons per sq. in.}$$

Unsupported distance $24'' - 7'' = 17''$, allowable stress

$$= 6 \div \left\{1 + \frac{1}{1500}\left(\frac{17}{0{\cdot}625}\right)^2\right\} = 4{\cdot}0 \text{ tons per sq. in.}$$

but even more would be allowable on account of the small depth between the flange angles.

*Cross Girders.*—The effective span is taken as the distance between the main girder centres $= 11'\ 3''$.

*Flange Areas.*—The estimated weight of a cross girder, together with ballast plate and angles, is equivalent to 1 ton, all uniformly distributed.

Dead load at each rail from stringers as above 2·8 tons.

Live load at each rail (given) 14 tons.

Equivalent dead load at each rail $2{\cdot}8 + 2 \times 14 = 30{\cdot}8$ tons.

Central bending moment (see Ex. 3, Art. 57) $= 97{\cdot}7$ tons feet
$= 1172$ ton-inches.

$$\text{Modulus of section required } \frac{1172}{6{\cdot}5} = 180{\cdot}3 \text{ (inches)}^3.$$

[1] As the load on 8' lengths is considerably concentrated, if 14·5 tons is a proper amount for the maximum end shears, the intermediate maximum shears will be greater than those given which are the ordinates of a parabola, but will be less than the ordinates of a straight line (see Arts. 76, 77, 80).

Taking the depth over the angles as 15″ and effective depth 14·75″,

$$\text{Area required} = \frac{180 \cdot 3}{14 \cdot 75} = 12 \cdot 23 \text{ sq. ins.}$$

Two angles $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$ less 4 rivet holes give 4·63 sq. ins., leaving 7·60 sq. ins.

*Bottom Flange.*—Width required for rivets 13″ (see (B), Fig. 277), net width 9·25 ins.

$$\text{Thickness} \frac{7 \cdot 60}{9 \cdot 25} = 0 \cdot 82''$$

say $\frac{7}{8}''$ consisting of $\frac{1}{2}''$ main plate and $\frac{3}{8}''$ outer plate (see cross section in Plate IV.).

*Top Flange.*—Main plate to take the flooring requires 13 + 5 = 18″ width (see (B), Fig. 277). The main plate must be $\frac{7}{16}''$ to lie even with the railbearer without packing. Main plate 18″ × $\frac{7}{16}''$, less 4 holes $\frac{15}{16}''$ and 2 holes $\frac{13}{16}''$, gives 5·52 sq. ins., leaving 7·60 − 5·52 = 2·08 sq. ins. for the outer plate: this being 13″ wide requires a thickness

$$\frac{2 \cdot 08}{9 \cdot 5} = 0 \cdot 22''$$

and $\frac{3}{8}''$ plate is used.

The flange diagram is shown in Fig. 278, in which the weight of cross girders, etc., being a small proportion of the whole load, is taken as acting

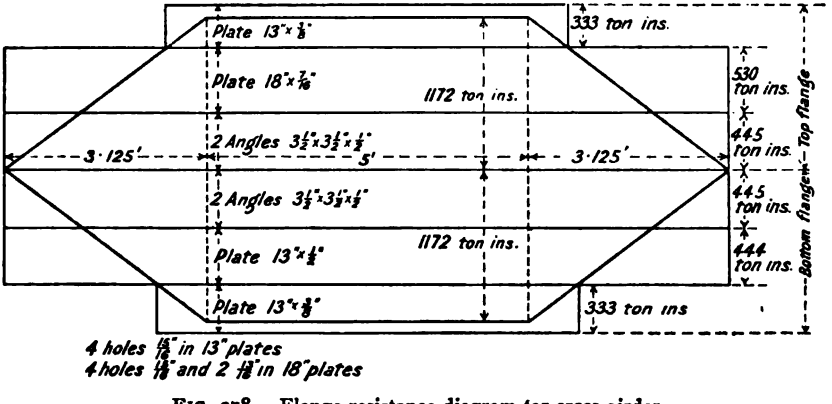



FIG. 278.—Flange resistance diagram for cross girder.

at the railbearers, at each of which the total equivalent dead load is then 2·8 + 0·5 + 14 × 2 = 31·3 tons. Bending moment at and between the stringers, 31·3 × 3·125 × 12 = 1173·8 ton inches. (Compare with 1172 with distributed load.)

Each square inch of metal in the section represents a working

moment of resistance of 6·5 × 14·75 = 96-ton-inches. Hence the total provided is,

Angles, 4·63 sq. ins. equivalent to 4·63 × 96 = 445 ton-inches.
18″ by $\frac{7}{16}$″ plate 5·52 sq. ins. equivalent to 5·52 × 96 = 530 ton-inches.
13″ by $\frac{3}{8}$″ plate 3·47 sq. ins. equivalent to 3·47 × 96 = 333 ton-inches.
13″ by $\frac{1}{2}$″ plate 4·625 sq. ins. equivalent to 4·625 × 96 = 444 ton-inches.

The curtailment of the outer plates might easily be calculated, for if $x$ is its distance from one end, $31{\cdot}3x = 445 + 530$, hence $x = 31$″ for the top flange, and $31{\cdot}3x = 445 + 444$, $x = 28{\cdot}4$″ for the bottom flange.

The live load of 14 tons per rail is a sufficient allowance to give gross lengths, including riveting to the main plate.

*Cross Girder Rivets and Web.*—Assuming $\frac{3}{4}$″ web and 1″ rivets.

Dead load end shear, 2·8 + 0·5 = 3·3 tons.
Equivalent dead load end shear 3·3 + 28 = 31·3 tons.
Resistance of 1″ rivets in double shear 0·785 × 1·75 × 5 = 6·87 tons.

$$\text{Pitch } p = \frac{6{\cdot}87 \times 15}{31{\cdot}3} = 3{\cdot}3, \text{ say } 3'' \text{ pitch.}$$

The rivets are changed to $\frac{7}{8}$″ with 4″ pitch between the stringers where the shearing force is very small.

$$\text{Shear stress in web} = \frac{31{\cdot}3}{15 \times 0{\cdot}75} = 2{\cdot}78 \text{ tons.}$$

*Connection to Main Girder.*—

1″ rivet in single shear, 0·785 × 5 = 3·93 tons.

Number required to transmit all the end shear $\frac{31{\cdot}3}{3{\cdot}93} = 8$.

*Main Girder.*—Take the depth over the angles 4 feet and flanges 21″ wide. Length over all 42′. Estimated weight of girder 7 tons, $\frac{7}{42} = 0{\cdot}167$ tons per foot run, or 1·333 ton for 8′ lengths. This weight may be taken as concentrated like the other loads at the cross girders without any material error.

*Dead Load.*—Pressure at cross girder ends 2·8 + 0·5 = 3·3 tons. Total dead load at each cross girder 3·3 + 1·333 = 4·633 tons.

*Live Load.*—For central section flange area, $\frac{105}{2} = 52{\cdot}5$ tons per girder. The central section is worked out in Example 5, Art. 68.

*Flange Diagram.*—Fig. 279.

Live load per cross girder $56{\cdot}5 \times \frac{8}{40} = 11{\cdot}3$ tons
Equivalent dead load per cross girder $= (11{\cdot}3 \times 2) + 4{\cdot}633 = 27{\cdot}233$ tons
Equivalent bending moment at $a$ and $a'$,

$$8 \times 2 \times 27{\cdot}233 = 435{\cdot}7 \text{ ton-ft.} = 5228 \text{ ton-ins.}$$

Equivalent bending moment at $b$ and $b'$,

$$24 \times 27{\cdot}233 = 654 \text{ ton-ft.} = 7848 \text{ ton-ins.}$$

One square inch of flange section represents a moment of resistance

of $48 \times 6{\cdot}5 = 312$ ton-inches. Taking the section from Example 5, Art. 68—

| | | |
|---|---|---|
| Angle bars give $5{\cdot}62 \times 312$ . . . . . | = 1754 | ton-ins. |
| Mainplate $\frac{9}{16}$″ × 17·25″ gives $9{\cdot}71 \times 312$ . | = 3030 | ,, |
| Outer plate $\frac{1}{2}$″ × 17·25″ gives $8{\cdot}625 \times 312$ | = 2690 | ,, |
| Total at central section | = 7474 | ,, |

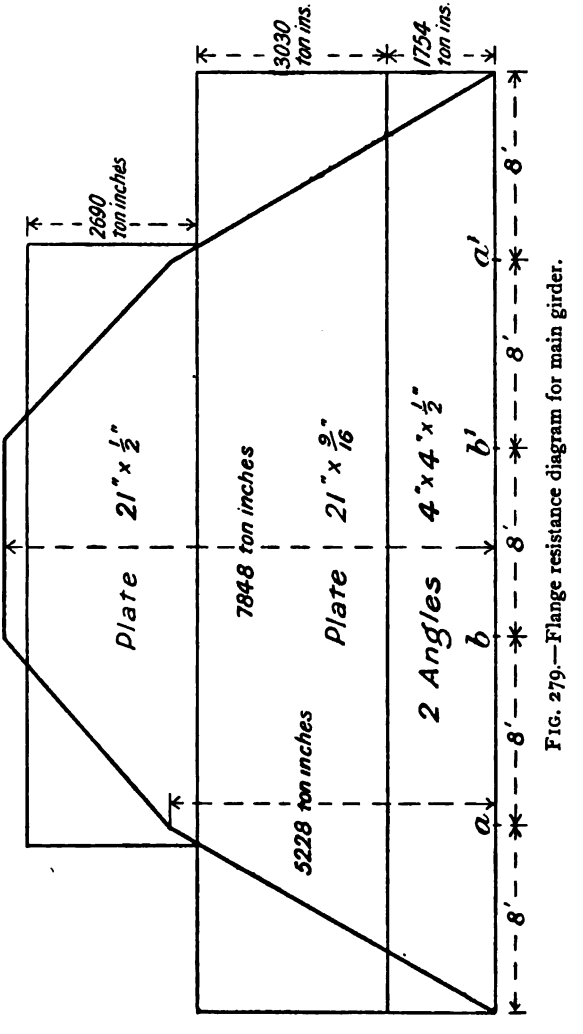



FIG. 279.—Flange resistance diagram for main girder.

As shown in Fig. 279, the total moment of resistance at the centre

is less than 7848 ton-inches, the value between $b$ and $b'$; but this is only the value of the bending moment for the flange diagram, reckoned on the 56·5 tons load. On the 52·5 tons load used for the central section design Example 5, Art. 68, the central bending moment is $615 \times 12 = 7380$ ton-inches.

*Main Girder Web and Rivets.*

Live load per cross girder, $65 \times \frac{8}{40} = 13$ tons

Equivalent dead load per cross girder $= (13 \times 2) + 4{\cdot}633 = 30{\cdot}6$ tons

Equivalent dead load end shear $= 2 \times 30{\cdot}6 = 61{\cdot}2$ tons

Assuming $\frac{1}{2}''$ web and 1″ rivets, the resistance per rivet in bearing being $0{\cdot}5 \times 10 = 5$ tons,

$$\text{rivet pitch for flanges } p = \frac{5 \times 48}{61{\cdot}2} = 3{\cdot}92'',$$

hence 3″ pitch is used at the ends.

The web length for the first panel is $8' + 1'$, hence stiffeners are conveniently placed at $\frac{1}{4}$ of $9' = 2'\ 3''$ centres (exact multiple of pitch). Over the end plate and at cross girders and at the web splice two angles and a gusset plate are used, but for intermediate stiffeners $6'' \times 3'' \times \frac{3}{8}''$ Tee sections are used.

$$\text{Shear stress in web} = \frac{61{\cdot}2}{48 \times 0{\cdot}5} = 2{\cdot}55 \text{ tons per sq. in.}$$

Allowable shear stress at 27″ centres, *i.e.* 21″ unsupported length

$$6 \div \left\{1 + \frac{1}{1500}\left(\frac{21}{0{\cdot}5}\right)^2\right\} = 2{\cdot}75 \text{ tons per sq. in.}$$

Using the conventional method of Art. 143 for the shear in the second panel both for the maximum value and the range, the equivalent dead load shearing force

$$= 4{\cdot}6 + \tfrac{2}{5} \times 3 \times 13 + \tfrac{1}{5} \times 13 + \tfrac{2}{5} \times 3 \times 13 = 38{\cdot}4 \text{ tons}$$

Assuming $\frac{7}{8}''$ rivets and $\frac{1}{2}''$ web, resistance per rivet being (for bearing)

$$10 \times 0{\cdot}875 \times 0{\cdot}5 = 4{\cdot}375 \text{ tons}$$

$$\text{pitch } p = \frac{4{\cdot}375 \times 48}{38{\cdot}4} = 5{\cdot}47''$$

hence 4″ pitch is used.

$$\text{Shear stress in web } \frac{38{\cdot}4}{48 \times 0{\cdot}5} = 1{\cdot}60 \text{ tons per sq. in.}$$

Using two intermediate stiffeners in the second panel gives centres $2'\ 8''$, or 26″ unsupported.

$$\text{Allowable stress, } 6 \div \left\{1 + \frac{1}{1500}\left(\frac{26}{0{\cdot}5}\right)^2\right\} = 2{\cdot}13 \text{ tons per sq. in.}$$

$$\text{Rivets required for shear only in web joint } \frac{38{\cdot}4}{4{\cdot}375} = 9.$$

The shear in the middle panel is small although changing in sign;

$\frac{1}{2}$" web may be used with stiffeners 4' apart, *i.e.* equal to the depth of the web, with $\frac{7}{8}$" rivets at 4" pitch. Dead load shear nil.

$$\text{Live load shear} = (\tfrac{1}{5} + \tfrac{2}{5})\ 13 = 7{\cdot}8 \text{ tons.}$$

$$\text{Range} = 15{\cdot}6 \text{ tons.}$$

$$\text{Equivalent dead load shear} = 7{\cdot}8 + 15{\cdot}6 = 23{\cdot}4 \text{ tons.}$$

$$\text{Shear stress} = \frac{23{\cdot}4}{48 \times 0{\cdot}5} = 0{\cdot}977 \text{ tons per sq. in.}$$

$$\text{Allowable stress} = 6 \div \left\{1 + \frac{1}{1500}\left(\frac{42}{0{\cdot}5}\right)^2\right\} = 1{\cdot}05 \text{ tons per sq. in.}$$

*Other Details.*—The comments under this heading relating to stress allowance at the end of Art. 191 are again applicable to the design in the present article.

**193. Other Types of Bridge Floors. Bridge Bearings.**—Steel troughing placed longitudinally over cross girders, or transversely in lieu of cross girders, is widely used for bridge floors. Particulars of the various sections with their modulii are given in steelmaker's handbooks. Messrs. Dorman, Long & Co.'s *Pocket Companion* contains several illustrations of its use for both rail and road bridges, with examples of the calculations which are very instructive.

Bridge floors are also constructed on short span brick and cement arches called "Jack arches" spanning from one cross girder to the next in lieu of railbearers, or in road bridges sometimes from one main girder to the next, thus replacing cross girders.

Various types of bearings are used for bridges; sometimes a simple bearing or sliding plate attached to the lower flange of the girder rests on a bed plate (see Plate II.), bolted to the bedstone of the abutment; such a bearing has freedom to slide, guided by grooves, to take up expansion if not prevented by friction.

Roller and pin bearings are also used with the same object, but a very general type of bearing is illustrated in Fig. 280, which represents a rocker bearing. A cast-iron rocker rests on a cast-iron bed plate bolted to the bedstone, a projection on the bedplate working in a corresponding groove in the rocker. The function of the rocker is to transmit the pressure centrally to the bedstone, thereby fixing the effective span and preventing pressure concentration at the face edge of the bedstone. When expansion of the girder takes place the rocker may either slide or tilt (with increase of camber of the bridge).

The allowable pressure on bedstone is about 12 tons per square foot for gritstone, and 18 tons for granite, reckoned on the equivalent dead load in both cases; from this and the end reactions the area required may be calculated.

**194. Skew Bridges.**—When the main girders of a bridge are not perpendicular to the abutments, as in Plate II., and the cross girders are placed perpendicular to the main girders, some of the cross girders near the end of the span rest with one end on the abutment, and are shorter than those which span the full distance between the main girders. Consequently such cross girders (which may be lighter than

those of full length) transfer less than the full allowance of load to the panel points at their ends on the main girders. The effect on the main girder bending-moment diagram is to make the ordinates less where the acute angle between the girder and abutment falls inside the bridge than at the other end. Whether such diminution is worth

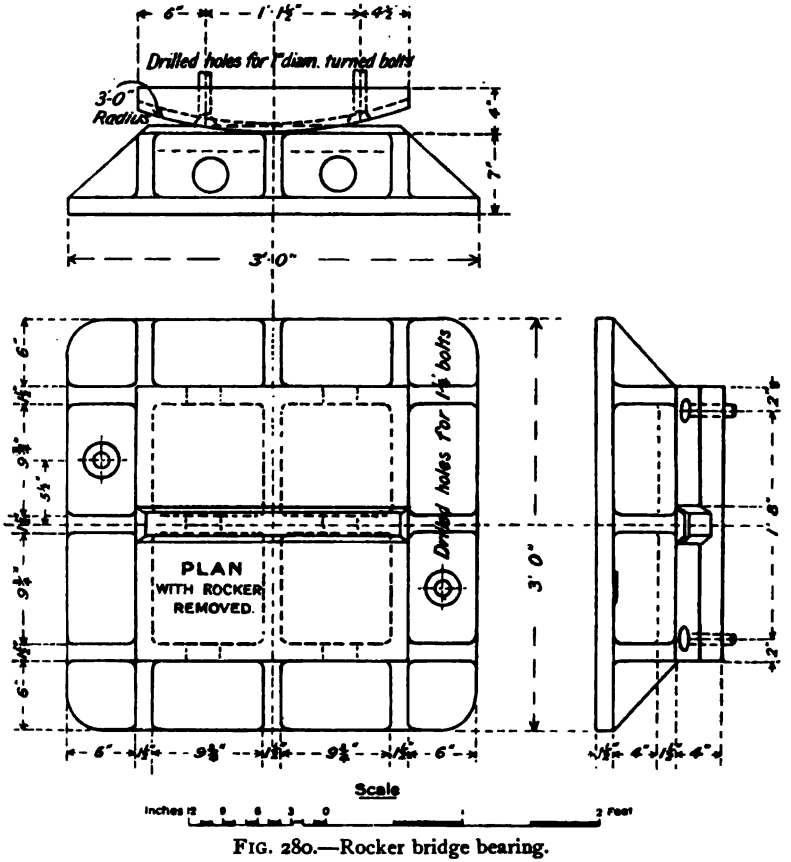



FIG. 280.—Rocker bridge bearing.

taking into account depends upon the angle of skew, and the ratio of length of span to the breadth between the main girders. The bending-moment diagram being not symmetrical with regard to the two abutments, the curtailment of the flange plates is also unsymmetrical. In calculating the sections for such a bridge no new principle is involved.

## EXAMPLES XVII.

1. Find the lengths of the two outer plates in the girder in Problem No. 17, Example V., without any end allowances for riveting to the inner plates.

2. Find the length exclusive of attachment allowance, for the outer $\frac{3}{8}''$ plate of the girder in Problem No. 18, Example V.

3. A plate girder of 50 feet span has a web 48 inches deep and 0·5 inch thick, and carries a uniformly distributed load of 144 tons. Find the necessary thickness of flange plates 16″ wide at the central section if 6″ × 6″ × $\frac{5}{8}''$ angles are used and the working unit stress is 7·5 tons per sq. in. If three plates are used, the two outer ones being each $\frac{3}{8}$ inch thick, find their lengths, allowing 18 inches at each end for attachment.

4. Find the pitch and diameter of rivets for double riveting, attaching the web to the flanges in Problem No. 3, allowing 5 tons per sq. in. in shear and 10 tons per sq. in. in bearing.

5. Find a suitable pitch for the stiffeners near the ends of the girder in Problem No. 3.

6. Calculate the weight from the dimensions in Plate IV. of (*a*) the main girder, (*b*) a cross girder, (*c*) a railbearer.

# CHAPTER XVIII

## *SUSPENSION BRIDGES AND METAL ARCHES*

**195. Hanging Cable and Relation to Linear Arch.**—If we may assume perfect flexibility, *i.e.* no resistance to bending, the form of the centre-line of a hanging chain or cable carrying vertical loads, is that of the funicular or link polygon for the loads and end supporting forces, the horizontal pole distance from the vertical load line being that representing the horizontal tension in the cable. Thus referring to Art. 51, section 3 and Fig. 56, for the cables of negligible weight suspended from P and Q and carrying the four given vertical loads, all possible formations correspond to funicular polygons having their poles on the line $ho_2$. In all cases each vertical load is balanced by the tensions in the two segments of cable meeting on its line of action. The horizontal tension, which evidently cannot vary throughout the cable since no forces having any horizontal component are applied except at the ends, fixes the precise outline of the cable centre line and supplies the remaining condition to fix the pole position in the line $ho_2$. For the horizontal distance of the pole from the line $ae$ represents the horizontal tension to scale or the constant horizontal component of the tensions in the various segments represented by the lines joining the pole to $a$, $b$, $c$, $d$, and $e$.

For a given shape of the hanging cable and given loads the horizontal tension and the position of the pole for the funicular polygon is thus determinate, but if all the loads and the horizontal tensions were altered in the same ratio, the same formation of cable would still hold good. Thus an infinite number of systems of loads having fixed ratios to one another would give a particular formation of cable. Also any given system of loads would give an infinite number of formations, viz. those corresponding to the various poles in the line $ho_2$, or, in other words, those given by different horizontal tensions. Another simple illustration with uniform loading is given in Fig. 281.

An arch supports vertical loads by material exposed to thrust. The funicular polygon represents the direction of resultant thrust at any section, just as for a suspension cable it represents the direction of resultant pull. The funicular polygon in this case is called the line of thrust or linear arch for the given system of loads. Again, an infinite number of funicular polygons corresponding to any given system of loads may be drawn, and to fix the true line of thrust requires some condition additional to the positions of the end supports. There is an important

difference between an arch and a flexible cable in that the line of thrust may pass outside an arch capable of resisting bending, while in the flexible cable the centre line and the line of resistance must coincide. Hence the cable (1), if free to change shape, will accommodate itself to various loadings. (2) If constrained to a particular shape will take up a corresponding system of stress; if this does not correspond to the loading determinable stresses will be exerted on the constraints.

**196. Uniformly Distributed Loads.**—When the load is uniformly distributed over the span, a case approximately realised in some suspension-bridge cables and in telegraph and trolley wires which are tightly stretched and loaded by their own weight, the form of the curve in which the wire hangs is parabolic.

If the uniform loads are applied at short intervals the funicular polygon would be circumscribed by the parabola corresponding to continuous loading, *i.e.* the points of application of the load would lie on a parabola which the cable would follow if the loading were continuous. When the loading is continuous and easily expressed as a function of some convenient variable, algebraic investigation of the curve is most convenient; the uniformly distributed continuous load is the simplest case of this kind.

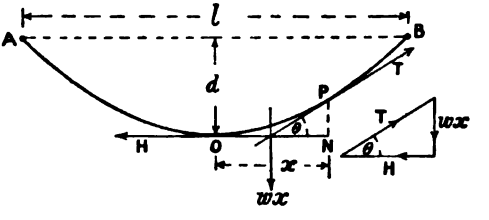


FIG. 281.

Let $w$ be the load per unit length of horizontal span, T the tension at any point P (Fig. 281), and H the constant horizontal component tension. Take the origin at the lowest point O, and the axes of $x$ and $y$ horizontal and vertical respectively. Then the length of wire or chain OP is kept in equilibrium by three forces, viz. T, H, and its weight $wx$, where $x = ON$, the horizontal projection of OP. Then from the triangle of forces, or moments about P

$$\frac{H}{wx} = \frac{x}{2y} \quad \text{or} \quad y = \frac{wx^2}{2H} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the equation to a parabola with its vertex at the origin O. Also

$$H = \frac{wx^2}{2y} = \frac{wl^2}{8d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $l$ is the span AB and $d$ is the total dip. The tension anywhere is

$$T = \sqrt{H^2 + w^2x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which at the points of support A or B reaches the value

$$T = \sqrt{H^2 + \frac{w^2l^2}{4}} = \frac{wl^2}{8d}\sqrt{1 + \frac{16d^2}{l^2}} \quad . \quad . \quad . \quad (4)$$

which does not greatly differ from H if $\frac{d}{l}$ is a small fraction. If the

points of suspension are at levels differing by $h$ (Fig. 282), and $x_1$ is the horizontal distance of the vertex of the parabola from the lower support B, and $d$ is the dip below that support, from (1)

$$H = \frac{wx^2}{2y} = \frac{wx_1^2}{2d} = \frac{w(l - x_1)^2}{2(d + h)} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

from which $x_1$ may be found in terms of $d$, $l$, and $h$. The intensity of tensile stress in the wire, Fig. 281, is

$$p = \frac{T}{A}$$

where A is the area of cross-section, and neglecting the small variation in T

$$p = \frac{H}{A} = \frac{wl^2}{8Ad} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Note that for a hanging wire loaded only by its own weight, $p$ is independent of the area of section A, since $w$ is proportional to A. Also that if $w$ is in pounds per *foot* length, $l$ and $d$ in *feet*, $p$ is in pounds per square *inch* if A is in square *inches*.

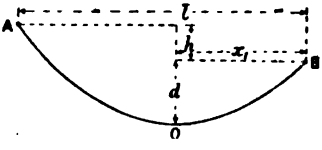


FIG. 282.

The length of such a very flat parabolic arc measured from the origin is approximately[1]

$$x + \tfrac{2}{3}\frac{y^2}{x}$$

hence the total length of cable $s$ is

$$s = l + \tfrac{8}{3}\frac{d^2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

A change of temperature affects the length of such a hanging wire in two ways: the linear contraction or expansion alters the dip; a change in dip corresponds to a change in tension, but owing to elastic stretch or contraction a change in tension corresponds to a change in length independent of temperature changes. The change in dip and in tension resulting from a change in temperature is thus jointly dependent on the change of temperature, coefficient of linear expansion, and the elastic properties of the material.

When the dip is *very* small the elastic stretching greatly modifies the

[1] If $y = cx^2$,

$$\frac{dy}{dx} = 2cx$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 1 + \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2$$

approximately if $\frac{dy}{dx}$ is small;

$$ds = (1 + 2c^2x^2)dx$$

$$s = x + \tfrac{2}{3}c^2x^3 = x + \tfrac{2}{3}\frac{y^2}{x}$$

influence of the temperature changes.[1] For such dips as are used in suspension bridges this effect is negligible.

Let $s_0$ be the initial length of the wire, $d_0$ the initial dip, $p_0$ the initial intensity of tensile stress, $t$ the rise in temperature, $\alpha$ the coefficient of linear expansion, $w$ the weight per unit length, A the area of cross-section, $\frac{w}{A}$ is then the weight per unit volume

$$s_0 = l + \tfrac{8}{3}\frac{d_0^2}{l} \qquad p_0 = \frac{wl^2}{8Ad_0} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

After a change of temperature, neglecting the elastic change in length,

$$s = s_0(1 + \alpha t) \quad \text{or } l + \tfrac{8}{3}\frac{d^2}{l} = \left(l + \tfrac{8}{3}\frac{d_0^2}{l}\right)(1 + \alpha t) \quad . \quad . \quad (9)$$

or to a first approximation, reducing

$$d^2 = d_0^2(1 + \alpha t) + \tfrac{3}{8}\alpha t l^2 = d_0^2 + \tfrac{3}{8}\alpha t l^2 \text{ (when } \alpha t \text{ is small)} \quad (10)$$

or expanding,

$$d - d_0 = \tfrac{3}{16}\alpha t \frac{l^2}{d_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The proportional decrease in stress is

$$\frac{p_0 - p}{p_0} = \frac{\left(\frac{1}{d_0} - \frac{1}{d}\right)}{\frac{1}{d_0}} = \frac{d - d_0}{d} = \tfrac{3}{16}\alpha t\left(\frac{l}{d_0}\right)^2 \text{ approximately} \quad (12)$$

Similarly if the cable without a change of temperature is stretched by an additional distribution load $w$ the fractional stretch is

$$\frac{p}{E} = \frac{T}{AE} = \frac{H}{AE} \text{ (constant) nearly, when the dip is small,}$$

hence we may write $\frac{H}{AE}$ in place of $\alpha t$, and the change of dip is approximately

$$d - d_0 = \frac{3}{16} \cdot \frac{H}{AE} \cdot \frac{l^2}{d_0} \quad \text{or} \quad \frac{3}{128} \cdot \frac{wl^4}{AEd^2} \quad . \quad . \quad . \quad (13)$$

EXAMPLE.—A steel cable has a span of 100 feet and a dip of 10 feet. Find the tension due to a load of 20 tons uniformly distributed horizontally over the span, and also find the length of the cable and the increase of tension due to a fall of temperature of 50° F. if the coefficient of expansion is 0·0000062.

Taking moments about a terminal of the cable

$$H = \tfrac{1}{10} \times 25 \times 10 = 25 \text{ tons}$$

$$T = \sqrt{25^2 + 10^2} = \sqrt{725} = 26{\cdot}9 \text{ tons}$$

$$\text{total length from (7)} = 100 + \tfrac{8}{3} \times \tfrac{100}{100} = 102{\cdot}\dot{6} \text{ feet}$$

$$= 102 \text{ feet } 8 \text{ inches}$$

$$\text{fractional decrease in length} = 0{\cdot}0000062 \times 50 = 0{\cdot}00031$$

[1] Numerical examples are given in the author's "Strength of Materials."

Hence
$$\frac{8}{3}\cdot\frac{d^2}{100} = 102{\cdot}6(1 - 0{\cdot}00031) - 100$$
$$= \tfrac{8}{3} \times \tfrac{100}{100} - 0{\cdot}0318$$
$$d^2 = 100 - 1{\cdot}19 = 100(1 - 0{\cdot}0119)$$
fractional increase in stress $= \frac{1}{2} \times 0{\cdot}0119 = 0{\cdot}00595$
total increase in stress $= 25 \times 0{\cdot}00595 = 0{\cdot}1487$ ton

which may be checked by (12).

**197. Simple Suspension Bridge.**—In the case of a chain carrying a horizontally uniformly distributed load by uniformly spaced hangers as shown in Fig. 283 by the funicular and force polygons, the shape of the chain is a polygon inscribed in a parabola, *i.e.* having vertices on a parabolic curve. It may be noted that concentration of loads at the

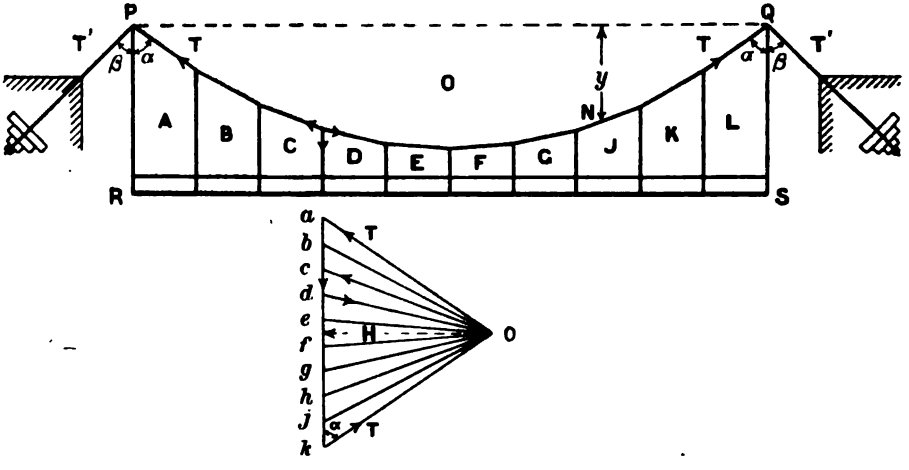



FIG. 283.—Simple suspension bridge.

end of a panel (giving nine concentrated loads for ten panels) with two half loads carried directly at the supports, gives funicular ordinates within the parabola except at panel points, whereas concentration at the centre of segments giving as many concentrated loads as panels, as described in Art. 58, gives ordinates outside the parabola except at junctions of segments.

In Fig. 283 the tensions DO and OC balance the vertical load CD, the triangle of forces for the point CDO being *cdo*.

In a suspension bridge the cable is made either of strands of wire which has high tensile strength, or of eye bars pinned at the panel points forming links in a chain.

In a simple unstiffened suspension bridge the load is carried by a relatively flexible platform or roadway, RS (Fig. 283). If the dead loads being fairly uniformly distributed give initially a parabolic form to the cable, a relatively heavy moving load for different positions would, by giving very unequal pulls on different hangers, cause the cable to take

up shapes varying greatly during the passage of the load. Such variations would cause variations in the shape and gradient of the platform and would be obviously an impossible condition for heavy traffic such as a railway. Such simple unstiffened suspension bridges are, in fact, only used for footbridges and similar light loads for which the dead load is sufficient to prevent great variation in the shape of the bridge. Under uniformly distributed load, in addition to the dead load, the cable would be in the form of the arcs of two or three parabolas.

*Relation of a Suspension Cable to Girder carrying the Same Load.*—If we resolve the end tension T at Q into vertical and horizontal components V and H, then taking any section of the cable as at N, and considering the portion to the right of N, the moment of the external forces is the same as the bending moment on the corresponding section of a rigid girder simply supported at P and Q. In the girder the bending moment (here contra-clockwise) is balanced by the clockwise moment of resistance. In the cable the same clockwise moment is supplied by the horizontal tension at Q, viz. $H \cdot y$, where $y$ is the depth of N below Q.

*Stresses in Anchorage Cables and on Piers.*—Occasionally suspension bridges have side spans between the piers and the shore in which the anchorage cables will form approximately arcs of parabolas similar to that in the centre span. Frequently, however, the cables pass in a straight line (neglecting the sag due to their own weight) from the tops of the piers to anchorages in masonry. If the cable passes over a fixed pulley or fixed rollers at the top of the piers, the tension in the cable is unaltered at those points except for friction of the pulleys. If T is the tension (Fig. 283), then $T = H \operatorname{cosec} \alpha$, and the horizontal (inward) pressure at the top of the piers is

$$H - T \cdot \sin \beta = H(1 - \sin \beta \operatorname{cosec} \alpha) \quad . \quad . \quad . \quad . \quad (1)$$

This horizontal force at the top of a pier will produce bending moments on the pier which will have to be allowed for in the design. The vertical pressure on the pier is

$$T(\cos \alpha + \cos \beta) = H(\cot \alpha + \cos \beta \operatorname{cosec} \alpha) \quad . \quad (2)$$

where half the load may be substituted for $T \cos \alpha$ if the loading is symmetrical and P on the same level as Q.

Frequently to avoid horizontal pressure on the piers the cable passes over saddles free to run on rollers on the tops of the piers. In this case the horizontal pressure is limited to the frictional resistance to the movement of the saddle, and neglecting this the horizontal components of the tensions are the same for the anchorage cables as at the ends of the central span. If $T'$ = tension of anchorage cables,

$$H = T' \sin \beta = T \cdot \sin \alpha \quad \text{or} \quad T' = H \cdot \operatorname{cosec} \beta \quad . \quad (3)$$

The vertical pressure on the pier

$$= T \cdot \cos \alpha + T' \cdot \cos \beta$$
$$= T(\cos \alpha + \sin \alpha \cot \beta) \text{ or } H(\cot \alpha + \cot \beta) \quad . \quad (4)$$

*Bridges with Stiffening Rods.*—Sloping stiffening rods from the top of the piers to the feet of the hangers have been used; these rods do not carry much of the load on the platform but reduce oscillations set up by change in shape of the cable due to alteration in position or amount of the load.

EXAMPLE.—If the cable in the example of Art. 196 passes over a saddle on rollers at a tower and then to an anchorage at an angle of 45° to the horizontal, neglecting friction, find the tension in these backstays and the pressure on the pier.

The slope of the cable at the pier is easily found from the fact that for a parabolic cable it is twice the average slope between the vertex and the point considered (verify by differentiation), viz. the slope is $2 \times \frac{10}{50} = 0{\cdot}4 = \cot a$. Hence, using the previous result,

$$\text{tension in the backstay} = 25 \times \sqrt{2} = 35{\cdot}35 \text{ tons}$$

$$\text{pressure on pier} = 25(0{\cdot}4 + 1) = 35 \text{ tons}$$

**198. Stiffened Suspension Bridges.**—To make a suspension bridge suitable for heavy traffic, it requires stiffening to resist changes of shape in the roadway. This is accomplished mainly in three ways: (*a*) by carrying the roadway on a girder hinged at the two ends of the span, see Fig. 287; (*b*) by two girders each taking half the span, hinged at the piers and hinged together midway between the piers, see Fig. 284; (*c*) by replacing the cable by two stiff suspension girders hinged together midway between the piers; these virtually form an inverted three-hinged arch, see Figs. 290–293. The first two produce statically indeterminate structures, but the third is statically determinate.

In suspension bridges carrying the roadway on stiffening girders the moment of the external forces to either side of a vertical section is balanced in part by the moment of resistance of the girder, and in part by the moment of the tension of the cable at the section. The distribution of resistance between the two depends upon their stiffnesses or elasticities, viz. of the cable and hangers in tension, and of the girder in flexure, and is in accordance with the principles dealt with in Chap. XIV. A treatment on such lines is necessarily lengthy and is outside the scope of this volume. The bending moments and cable stresses are usually estimated on certain simple assumptions as to distribution, but in any given case the results should be used with caution, as their validity will depend upon the relative proportions of cable and girder.

**199. Three-hinged Stiffening Girder.**—It is assumed that whatever the live load on the girders the chain retains its parabolic form which it assumes under the uniformly distributed dead load; such form and the carrying of all the dead load by the cable can be secured by adjustment of the length and tension of the hangers during erection. If the cable remains parabolic, the pull of the hangers, downwards on the cable and upwards on the girders, must be uniformly distributed along the span for all loadings. The function of the stiffening girders is to so distribute the load. The assumed conditions would be approached by very stiff girders and hangers which are equally elastic, *i.e.* the cross-sections proportional to the lengths. Temperature stresses will be to a

considerable extent reduced by the central hinge, but the structure is not really statically determinate on account of the hangers.

Let Fig. 284 represent such a bridge, the cable suspended from A and B, and the girders hinged to F and E and together at C. Then taking A as origin, and measuring $y$ downwards, ACB being a parabola with vertex at C,

$$y = \frac{4d}{l^2}x(l - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then for any live load let M be the bending moment on the girder at the vertical section through any point P on the cable. Let $\mu$ be the bending moment for a girder simply supported at its ends on the span FE with the same loading. Let $w'$ be the load per foot run transferred

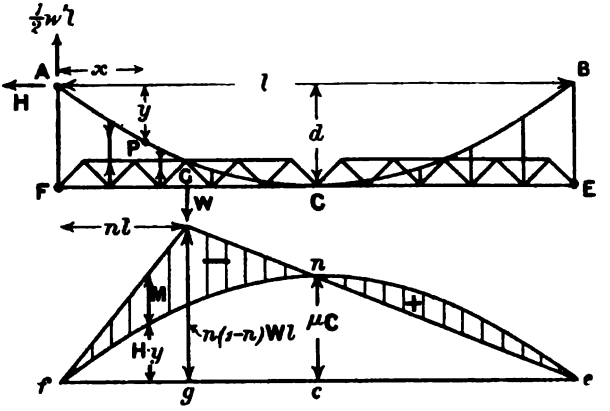



FIG. 284.—Suspension bridge with three-hinged stiffening girder.

to the cable by the hangers, giving an upward vertical reaction $\frac{1}{2}w'l$ at A and an equal amount downward at F acting against the upward reaction calculated as for a simply supported beam FE. Then taking moments, say, to the left of a vertical section through P, and ignoring the equal and opposite forces $w'$ and their reactions,

$$M = \mu + H \, . \, y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\mu$ will be a negative quantity according to the convention of Art. 59 for downward loads, and M may be positive or negative. The value of H, and hence of $w'$, is determined from the fact that M = o at the hinge C where $y = d$, for

$$o = \mu_o + Hd \quad \text{or} \quad H = -\frac{\mu_o}{d} \quad . \quad . \quad . \quad . \quad (3)$$

and as in Art. 196 by moments of forces on the cable,

$$H = \frac{w'l^2}{8d} \qquad \text{hence } w' = -\frac{8\mu_o}{l^2} \quad . \quad . \quad . \quad . \quad (4)$$

hence from (2),

$$M = \mu - \frac{y}{d}\mu_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence to draw the bending-moment diagram for the girder it is only necessary to draw the diagram of bending moments ($\mu$) as for a beam simply supported at F and E, and subtract from each ordinate a quantity equal to the central value $cn$ of $\mu$ reduced in the ratio of the cable depth to the central dip, which is done by drawing a parabola with vertex at $n$, passing through the ends $f$ and $e$. The case illustrated in Fig. 284 is that of a single concentrated weight at W distant $nl$ horizontally from F and A.

The graphical aspect of the matter is that the funicular polygon, whether parabolic or otherwise, which shows the form of cable also represents to scale the diminution of bending moment, H.$y$, *i.e.* the upward bending moment on the girder due to the hanger tensions. The bending-moment diagram ($\mu$) for a beam FE with any loading may be drawn by a funicular polygon (see Art. 58) with any pole distance, but to use this polygon for superposition on the cable polygon it must be to the same scale, *i.e.* have the same pole distance representing H. This may be found by calculation or a trial polygon may be drawn and the central ordinate reduced to the depth $d$ to pass through C (see Art. 51 (*c*)), the pole distance being altered in the inverse ratio of the central ordinates; the ordinates measured from the cable polygon to the polygon for the bending moments on a simply supported beam, then give the bending moments on the stiffening girders. Or since from (2) and (3),

$$M = H\left(\frac{\mu}{H} + y\right) = -H\left(\frac{d}{\mu_c}.\mu - y\right) \quad . \quad . \quad . \quad (6)$$

the negative bending moment at any section is equal to the horizontal thrust multiplied by the length represented to scale by the excess of the load polygon ordinate over the cable polygon ordinate.

*Bending Moments for Simple Loads.*—Consider the bending moment on the girder at any section G distant $nl$, say, less than $\frac{1}{2}l$ from the end F (Fig. 284), due to a load W in all positions. Let $x$ = distance of the load W from A; we may find the bending moment M on the girder from (2), remembering that from (3) $H = \frac{Wx}{2d}$, and using the value (1),

$$Hy = 2Wn(1 - n)x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then for values of $x$ less than $nl$,

$$M = -W(1 - n)x + 2Wn(1 - n)x = -W(1 - n)(1 - 2n)x \quad (8)$$

For values of $x$ greater than $nl$,

$$M = -Wn(l - x) + 2Wn(1 - n)x = Wn\{(3 - 2n)x - l\} \quad (9)$$

If $x$ is greater than $\frac{1}{2}l$,

$$H = W\frac{l - x}{2d} \quad \text{and} \quad Hy = 2Wn(1 - n)(l - x) \quad . \quad (10)$$

hence from (2),

$$M = -Wn(l - x) + 2Wn(1 - n)(l - x) = +Wn(1 - 2n)(l - x) \quad (11)$$

Since (8) is proportional to $x$, and (11) to $(l - x)$, and (9) is linear

in $x$, we may easily draw the influence line *fpqse* (Fig. 285). Writing $x = nl$ in (8) or (9) gives the ordinate

$$-\mathrm{W}ln(1-n)(1-2n) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

at $g$, and writing $x = \frac{1}{2}l$ in (9) or (11) gives the ordinate

$$+\tfrac{1}{2}\mathrm{W}ln(1-2n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

at $c$. Thus $gp$ and $cs$ differ numerically only in the factors $1-n$ and $\frac{1}{2}$, and as $n$ is less than $\frac{1}{2}$, $1-n$ is greater than $\frac{1}{2}$, and the ordinate $gp$ is numerically the greatest, *i.e.* the maximum bending moment at any section occurs when the load is over the section.

*Maximum Moments for Concentrated Loads.*—The maximum negative bending moment is found by differentiating (12) with respect to $n$ and equating to zero, giving

$$6n^2-6n+1=0 \quad \text{or} \quad n=0{\cdot}5\pm0{\cdot}289, \textit{i.e.}\ nl=0{\cdot}211l \ \text{or}\ 0{\cdot}789l \quad (14)$$

distant $0{\cdot}211l$ from F and E. And substituting in (12) the maximum negative bending moment anywhere is found to be

$$0{\cdot}096\mathrm{W}l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The positive bending moment for all sections, from (11), reaches a maximum for $x = \frac{1}{2}l$, viz. $\frac{1}{2}\mathrm{W}ln(1-2n)$, which is the greatest for $n = \frac{1}{4}$ and has the value

$$\tfrac{1}{16}.\mathrm{W}l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This is apparent also from Fig. 284, for the maximum positive ordinate is midway between $c$ and $e$ for all values of $n$, and on the cable diagram is $\frac{3}{4}d-\frac{1}{2}d=\frac{1}{4}d$, which multiplied by the maximum value of H for W at C is $\frac{1}{4}d\times\frac{\mathrm{W}l}{4d}=\frac{1}{16}\mathrm{W}l$. The maximum bending moment curves from (12) and (13) are shown in Fig. 285.

*Maximum Moments for Uniformly Distributed Load.*—As in Art. 88, we may apply the influence line (Fig. 285) to a uniformly distributed load $w$ per foot by writing $\mathrm{W}=1$ and taking the area between the line and the base line. From (9), for $\mathrm{M}=0$, $x=\dfrac{l}{3-2n}$, the length to be loaded for maximum negative bending moment at G. Then maximum negative bending moment at G

$$= w\times\text{area}\,fpq = -\frac{w}{2}\cdot\frac{l}{3-2n}\times\frac{ln(1-n)(1-2n)}{1}$$

$$=-\frac{wl^2}{2}\cdot\frac{n(1-n)(1-2n)}{3-2n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Differentiating this with respect to $n$ and equating to zero gives

$$8n^3-24n^2+18n-3=0 \quad \text{hence} \quad n=0{\cdot}234 \quad . \quad (18)$$

And substituting in (17) the maximum negative bending moment anywhere is

$$-0{\cdot}01883wl^2 \ \text{or}\ -\tfrac{1}{53}wl^2 \ \text{(approx.)}\ \text{at}\ 0{\cdot}234l \ \text{from the ends} \quad (19)$$

the loaded length being $0{\cdot}395l$.

For maximum positive moment the loaded length $qe$ is $l - fq = \frac{2(1-n)}{3-2n} \cdot l$, and the maximum positive bending moment at G is

$$w \times \text{area } qse = \frac{w}{2} \cdot \frac{2(1-n)l}{3-2n} \cdot \frac{ln(1-2n)}{2} = \frac{wl^2n(1-n)(1-2n)}{2(3-2n)} \quad (20)$$

which is the same as (17) except in sign, hence as before the maximum positive bending moment anywhere is

$$+ 0{\cdot}01833wl^2 \text{ or } + \tfrac{1}{55}wl^2 \text{ approx. } 0{\cdot}234l \text{ from the ends} \quad (21)$$

the loaded length in this case being $qe = 0{\cdot}605l$. The maximum bending-moment curves from (17) and (20) are shown in Fig. 285.

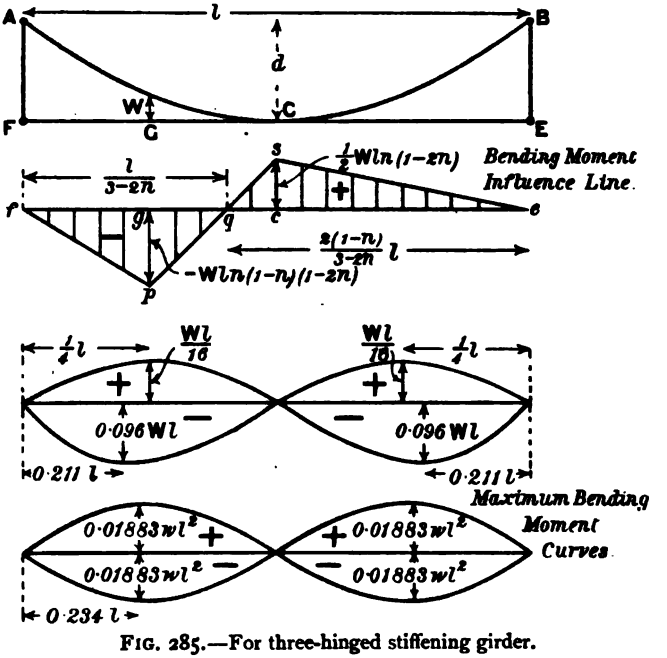



FIG. 285.—For three-hinged stiffening girder.

*Shear Influence Line and Maximum Shears.*—For a single rolling load the positive shearing force (as defined in Art. 59), on the girder is increased by the vertical component of the cable tension, which, if a vertical section be supposed, is additional vertical force on the girder section, hence since the tangent of cable slope is $\frac{dy}{dx}$, the shearing force

$$F = f + H\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where $f$ is the shearing force for a simply supported beam on a span $l$.

And at $nl$ from the end F, differentiating (1) for $\frac{dy}{dx}$ and putting $x = nl$,

$$F = f + H\frac{4d}{l}(1 - 2n) \quad . \quad . \quad . \quad . \quad . \quad (23)$$

For a rolling load W, since for the two halves of the span $H = \frac{Wx}{2d}$ and $\frac{W(l-x)}{2d}$

$$F = f + 2W(1 - 2n)\frac{x}{l} \quad \text{and} \quad f + 2W(1 - 2n)\frac{l-x}{l} \quad (24)$$

These consist of two terms, the first, $f$, is the influence line shown in Fig. 132, and also by *fpse*, Fig. 286, and the second is the line *fte* discontinuous at $t$ shown in Fig. 286. The two parts being superposed,

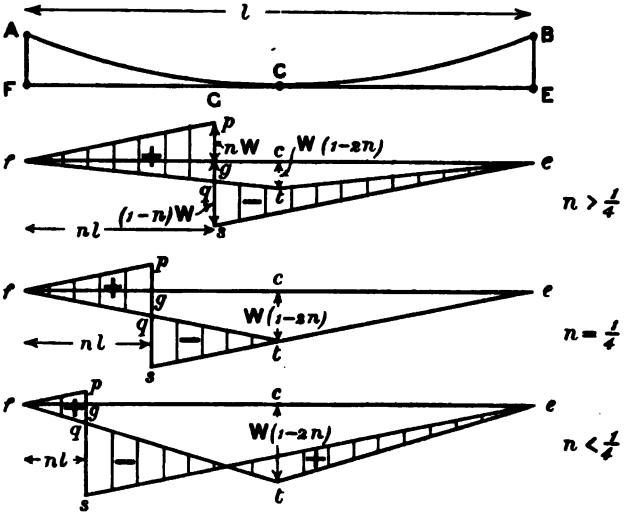



FIG. 286.—Influence lines for shearing force in three-hinged stiffening girder.

the influence line ordinates are measured from the line *fte* across the shaded area to the line *fpse*. The diagram takes the different shapes according to the value of $n$.

The maximum shearing force curves for a single rolling load may be deduced from Fig. 286.

The maximum shearing force curves for a uniformly distributed load may be easily found, also from the areas in the influence diagram, the loaded lengths for the different maximum values being the projections of the areas of like sign in Fig. 286. The positive and negative areas are equal for any value of $n$, and, consequently, the positive and negative maximum curves are similar. The most important value is the end shear ($n = 0$), a particular case of the diagram showing $n$ less than $\frac{1}{4}$

in Fig. 286; then $qs = W = 1$, the loaded lengths are $\frac{1}{3}l$ from $f$ for maximum negative shear at $f$, and $\frac{2}{3}l$ from $e$ for maximum positive shear at $f$, both values being $\frac{1}{2} \times \frac{1}{3}l$. $w = \frac{1}{6}wl$.

At the centre of the span $\frac{dy}{dx}$ being zero it follows from (22) that the maximum shearing force is as in Fig. 116, viz. $\frac{1}{8}wl$. The complete curves are left as an exercise to the reader.

**200. Two-hinged Stiffening Girder.**—(Fig. 287.) The girder is hinged to the piers at each end. It is usually assumed that the cable

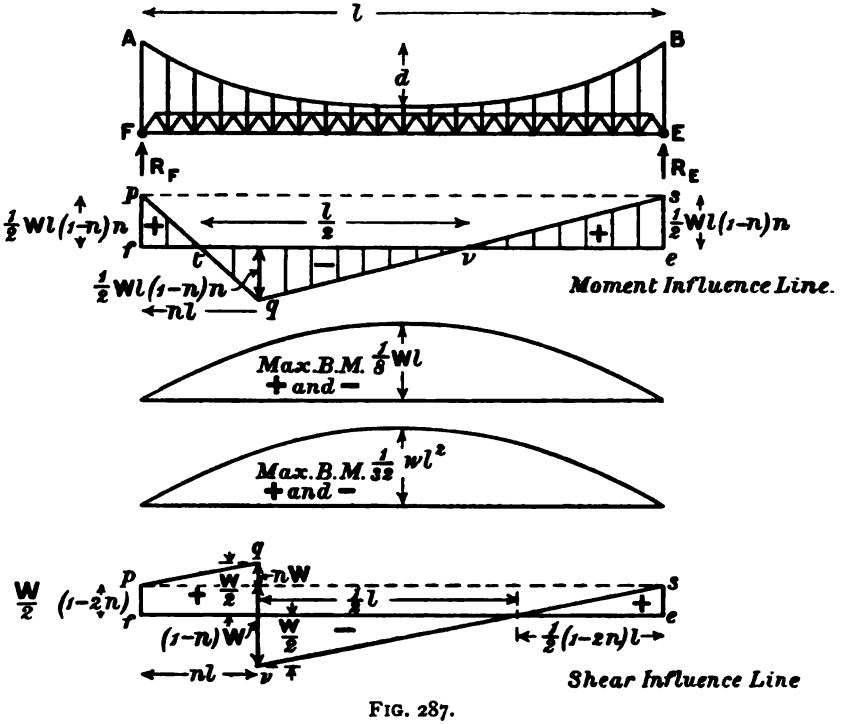



FIG. 287.

remains in the form of a parabola to which it is adjusted by the hangers so as to carry the whole dead load. Such an assumption may be approximate for light loads on a stiff girder, but is not a very reliable assumption without investigation of the proportions of the cable and girder. If the girder were very flexible the parabolic form of the cable would not be retained, and the load would not be uniformly distributed; on the other hand, if the girder were infinitely stiff it would transfer the whole load to the end supports. The assumption is that the hangers carry the whole load, and that the end reactions for unsymmetrical loading are equal and opposite.

Adopting the notation of the previous article (199) for a single load W distant $x$ from F, taking moments about the hinges,

$$R_E = \frac{W}{l}(x - \tfrac{1}{2}l) = -R_F \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also $$w' . l = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then as before, $$M = \mu + Hy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

but $H = \dfrac{w'l^2}{8d} = \dfrac{Wl}{8d}$ (a constant, independent of the load position) (4)

Hence the bending-moment diagram (not shown) is found by the difference of the ordinates of the bending-moment diagram (triangular in this case) for a load W on a simple beam span FE and those of a parabola $Hy$ the central ordinate of which is $\frac{1}{8}Wl$ (when $y = d$). The triangle will intersect the parabola and give some positive ordinates for all positions except the central one, for which it is tangential at the ends.

For other types of loading the same principles hold, the $\mu$ diagram for a simple span is reduced by parabolic ordinates $\dfrac{w'l^2}{8d}y$ or $\dfrac{Wl}{8d} \cdot y$ where W = total load on the span.

*Influence Line for Bending Moment.*—At a distance $nl$ from F the bending moment due to a load distant $x$ from F is from (3),

$$\mu + Hy = \mu + \frac{Wl}{8d}y = \mu + \frac{Wl}{8d}4dn(1 - n) = \mu + \tfrac{1}{2}Wln(1 - n) \quad (5)$$

the first term $\mu$ represents the (negative) ordinate of the influence line for the simple span (Fig. 130), while the second term, $\frac{1}{2}Wln\,(1 - n)$, is constant for all values of $x$, hence the influence line is as shown in Fig. 287; it may be looked upon as the triangle $pqs$ with the rectangle $pfes$ superposed.

*Maximum Bending Moment.*—Due to a single load W, it follows from the influence line diagram that the maximum bending moment, both positive and negative at $nl$ from $f$, is $\frac{1}{2}Wln\,(1 - n)$. For different values of $n$ this gives ordinates of a parabola, the central maximum ordinate being (for $n = \frac{1}{2}$) $\frac{1}{8}Wl$. The diagram is shown in Fig. 287.

For a uniformly distributed load $w$ per foot it is evident from the influence line that the load must extend over a length $tv$ for maximum negative bending moment, and over lengths $ft$ and $ve$ for maximum positive bending moment; in either case writing W = 1 the magnitude is $w \times$ area $= \frac{1}{2}w.\frac{1}{2}l.\frac{1}{2}l(1 - n)n = \dfrac{wl^2}{8}(1 - n)n$, the ordinate of a parabola reaching a maximum[1] value $\frac{1}{32}wl^2$ at the middle of the span where $n = \frac{1}{2}$.

*Influence Line for Shear; and Maximum Shear.*—As in the previous article adding the vertical pull of the cable to the shearing force for a simple beam

$$F = f + H\frac{dy}{dx} = f + \frac{Wl}{8d} \cdot \frac{4d}{l}(1 - 2n) = f + \tfrac{1}{2}W(1 - 2n) \quad (6)$$

[1] The coefficient $\frac{1}{32}$ often quoted is incorrect, and a glance at the influence line will show why.

The influence line (see Fig. 287) is as in Fig. 132 with the ordinates $\frac{1}{2}W(1 - 2n)$ added, or the base line lowered from *ps* to *fe*.

The maximum positive shearing force for a single load is, for a distance *nl* from F, $nW + \frac{1}{2}W(1 - 2n) = \frac{1}{2}W$, and the maximum negative shearing force has the same magnitude. The maximum shearing forces are therefore the same for all sections. For a uniform load *w* per foot the maximum positive shearing force is

$$w\{\tfrac{1}{2}n^2l + \tfrac{1}{2}nl(1 - 2n) + \tfrac{1}{2} \times \tfrac{1}{4}(1 - 2n)^2l\} = \frac{wl^2}{8}$$

which is independent of *n* and therefore the same for all sections. It has been pointed out in Art. 198 that the actual stresses depend upon the relative sections of the cable and the girder; the above theory is only a rough approximation. Obviously the bending moment and shears *pf* and *se* in the influence lines of Fig. 287 should be zero. The superposed rectangles *pfes* should in fact be curves on the bases *ps*, the ordinates depending upon the relative sections of the cable and girder. The assumptions and results of the previous article for the centrally hinged girder will be more reliable than those for the girder without the intermediate hinge.

**201. Temperature Stresses in Stiffening Girder.**—If the resistance of the girder is small compared to the cable resistance so that the cable remains parabolic, the girder must sag or rise with the cable due to temperature variations in such a way as to take a uniformly distributed *change of load*. Hence we can calculate the change in chord stress in the girder due to changes in dip of the cable. This change is estimated to a first approximation in Art. 196 (11) as $\frac{3}{16}\alpha t\frac{l^2}{d_0}$, which will be increase of dip for an increase $t°$ and decrease dip for a fall of temperature $t°$. But from (16) Art. 94, writing D for the depth of the girder, the change of bending stress is $f = \frac{24}{5}\cdot\frac{ED}{l^2} \times$ central deflection. Hence,

$$f = \frac{24}{5}\cdot\frac{ED}{l^2}\cdot\frac{3}{16}\alpha t\frac{l^2}{d_0} = \frac{9}{10}\cdot\frac{D}{d_0}\cdot E\alpha t \quad . \quad . \quad . \quad . \quad (1)$$

where *f* is in the same units as Young's modulus E., the sag $d_0$ being in the same units as the girder depth D. It is interesting to note that this is independent of the length of span and the section of the girder chords, and is proportional to the depth of the stiffening girder.

The temperature stresses in the centrally hinged girder are of about equal magnitude. For from (1) Art. 199, putting $x = \frac{1}{4}l$, the depth of cable is $\frac{3}{4}d$. Hence the change of sag is about three-quarters of the change at the centre; but due to turning about the central hinge the change in level is half the change of sag at the centre, hence the central change of level producing stress is *one-quarter* of that in the case of the girder not hinged at the centre: hence for the half-span length $\frac{1}{2}l$, (1) becomes

$$f = \frac{1}{4} \times \frac{24}{5}\cdot\frac{ED}{(\frac{1}{2}l)^2} \times \frac{3}{16}\alpha t\frac{l^2}{d_0} = \frac{9}{10}\cdot\frac{D}{d_0}\cdot E\alpha t \quad . \quad . \quad . \quad (2)$$

In either type of girder a fall of temperature reduces the sag and causes positive bending moment, *i.e.* tension in the top chord and thrust in the lower chord, while a rise in temperature causes moments and stresses of opposite signs.

EXAMPLE.—A steel suspension bridge has a span of 100 ft., dip 10 ft., and the stiffening girder is 4 ft. deep. Find the change in chord stress due to a change of temperature of 50° F.; take E = 13,000 tons per sq. in., coefficient of expansion 0·0000062.

From (1) or (2) $f = \frac{9}{10} \cdot \frac{4}{10} \times \frac{13000 \times 62 \times 50}{10^7} = 1{\cdot}45$ tons per sq. inch.

**202. Stiffened Cables.**—The suspension bridge in which the cable is replaced by two braced girders hinged together at the centre of the span, forms a statically determinate structure. It has great possibilities of economy for long spans. The determination of reactions and stresses is exactly analogous to those in the three-hinged arch treated in Art. 204.

**203. The Metal Arch and Arched Rib.**—An arch may be looked upon as a curved girder, either a solid rib or braced, supported at its ends and carrying transverse loads which are frequently all vertical; the arch as a whole is subjected to thrust. The *line of resultant thrust* or *linear arch* for an arch carrying vertical loads can easily be drawn when in addition to the vertical loads we know the horizontal component of the thrust of the abutments. The vertical components of the reactions at the abutments are determined algebraically or graphically as for a straight beam and are not affected by the horizontal thrust if the abutments are on the same level, as is evident if we consider moments about an abutment.

Thus in Fig. 288, representing an abutment of an arch with vertical loads AB, BC, CD, if the horizontal thrust H is known, and the vertical reaction V has been determined algebraically or graphically, and the vertical loads *ab*, *bc*, *cd*, etc., and the reaction *oa* = *oh* + *ha* are set off as shown, the line of thrust AO, BO, CO, etc., can be drawn by starting from the centre of the abutment and drawing lines parallel to *oa*, *ob*, *oc*, etc., terminated by the force lines AB, BC, CD, respectively. At the section shown the resultant thrust of the remainder of the arch on the portion shown is DO represented by *do*. The first step to the solution of the stresses in an arch is to determine the horizontal thrust. In one type—the three-hinged arch—the horizontal thrust is statically

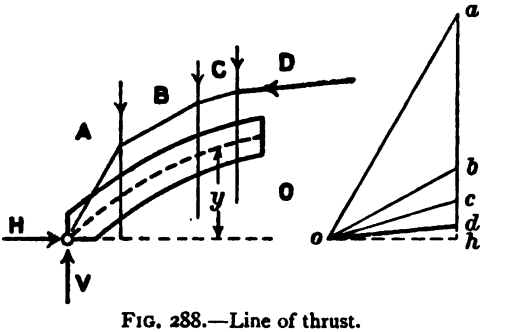



FIG. 288.—Line of thrust.

determinate, but with the two-hinged and not-hinged types the horizontal thrust is statically indeterminate and the structure is subject to indeterminate initial stresses and stresses due to changes of temperature.

The arch may also be compared with a girder with a curved loaded top chord; when the bending moment curve ordinates are proportional to the heights of the girder (see Art. 138) there is no stress in the diagonals, and the vertical shear is carried by the curved top chord in thrust. In the arch the external thrust of the abutments replaces the tension of the lower chord of the girder. If the loading alters, the curved arch has to withstand not the whole bending moments which would arise in a straight girder, but a relatively small difference between them and bending moments of generally about the same average amount over the span. The possible economy of material in the superstructure is obvious; on the other hand, the cost of abutments to withstand the thrust of the arch may more than neutralise this. Steel arch construction is very frequently adopted to span steep gorges, the sides of which provide natural abutments of ample resistance.

The straining actions at any normal cross-section are conveniently resolved into a bending moment and a shearing force, as in the case of a straight beam carrying transverse loads, with the addition in the arched rib of a thrust perpendicular to the section; for, unlike the case of the straight beam, the loads not being all perpendicular to the axis of the rib, the resultant force perpendicular to a radial cross-section is not zero. Thus, at a section AB (Fig. 289) of an arched rib the external forces give rise to (1) a thrust normal P through the centroid C, (2) a radial shearing force F on the transverse section AB, and (3) a bending moment M. These three actions are statically equivalent to a single thrust T through a point D, in the section AB produced, where T is the resultant of all the external forces to the right of a section through C, *i.e.* the resultant of the rectangular components F and P of the force exerted by the right-hand on the left-hand portion of the structure. The distance $CD = \frac{M}{P}$. For continuous loading the linear arch will be a curve having the directions of resultant thrust as tangents. The straining action may thus be specified by the normal thrust, the radial shearing force, and the bending moment, or simply by the linear arch, and when the straining actions are known, the stress intensities in the rib can be calculated. As in straight beams, the shearing force may often be neglected as producing little effect on the stresses. The curvature of the rib not being great, it is usually sufficient to calculate the bending stresses as for a straight beam, as in Art. 63. The uniform compression arising from the thrust P is added algebraically to the bending stress, as in Arts. 111 and 112, and the radial and circumferential shearing stress arising from the radial shearing force may be calculated as in Art. 72,

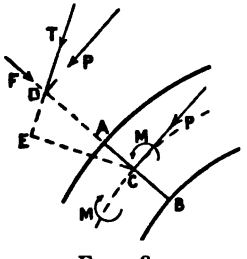


FIG. 289.

and, if necessary, combined with the bending and other direct stress to find the principal stresses, as in Arts. 18, 19, and 73.

In all types, if $y$ is the height of the axis of the arch at any section, and $\mu$ is the bending moment calculated as for a straight horizontal beam under the vertical forces only, the actual bending moment M at this section of the arch (conforming to the convention of Art. 59 as to sign) is the algebraic sum of $\mu$ and the effect $H.y$ of the horizontal thrust, or

$$M = \mu + H.y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\mu$ will always have a negative value for downward loads.

**204. Three-hinged Arch.**—In this statically determinate structure, having a hinge at each abutment or springing, and also at the crown, the horizontal thrust H, and hence the line of thrust or linear arch, are found from the fact that the bending moment at the crown hinge as well as at the springings is zero, *i.e.* the line of thrust passes through this hinge.

*Graphically.*—In Fig. 290 let ACB represent the axis of the arch and W or EF a single load; then since there is no load on the portion

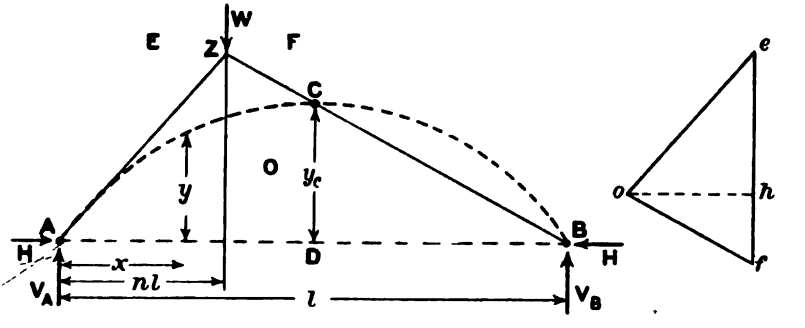



FIG. 290—Three-hinged arch: single load.

CB, the thrust at B must pass through C and be in the direction BC. Hence if BC meets the vertical line EF in Z and the line *ef* is set off to represent W, then completing the triangle *efo* by drawing *fo* parallel to BZ and *eo* parallel to AZ (since Z is the point of concurrency of the three forces), the reactions *oe* and *fo* are completely determined, and the horizontal thrust H is their common horizontal component *oh*.

The graphical problem for the case of several loads is to draw a funicular polygon through the three given points A, B, and C. This has been dealt with in Art. 51 (*c*). In Fig. 291 a trial funicular polygon APSXZA is drawn for any pole $o_1$, and then taking a pole distance $o_2h = \frac{SZ}{CD} \times$ horizontal distance of $o_1$ from *efgkl*, a line of thrust which is the funicular polygon for the pole $o_2$ if started from A will pass through C and B.

*Algebraically.*—In Figs. 290 and 291, if $\mu_0 =$ bending moment for

vertical forces, at the centre D of a span AB, since the bending moment at C is zero

$$\mu_C + H \cdot y_C = 0 \quad \text{or} \quad H = -\frac{\mu_C}{y_C}$$

Hence for any other section

$$M = \mu + Hy = \mu - \mu_C \times \frac{y}{y_C}$$

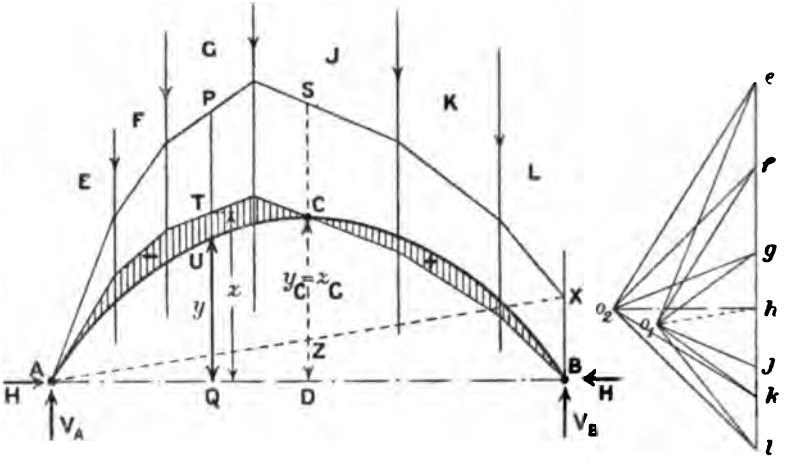



FIG. 291.—Three-hinged arch : several loads.

*Eddy's Theorem.*—All ordinates of the linear arch from the base AB are proportional to the bending moment of the vertical forces alone, and the ordinate at C is equal to $H \cdot y_C$ or H multiplied by the ordinate of the linear arch ; hence the value of $\mu$ everywhere is $-$ H $\times$ ordinate ($z$) of the linear arch (QT representing $z$ to scale) ; and the actual bending moment for a section through any point U on the axis of the arch is $-Hz + Hy = -H(z - y)$, *i.e.* $-$H multiplied by the height $(z - y)$ of the linear arch above the axis of the arch. If the linear arch lies below the axis of the arch the bending moment is positive, the signs being as in Art. 59. (Positive moments tend to produce increased convexity of the axis upwards.) The intercepts between the arch axis and the linear arch represent the bending moment to the same scale on which CD represents $H \cdot y_C$, viz. $p \cdot q \cdot o_2h$ lb.-ft. to one inch, where the other scales are $p$ lbs. to one inch, $q$ ft. to one inch, and $o_2h$ is measured in inches (see Art. 50).

The normal thrust at U, say, may be obtained by multiplying the resultant thrust (represented by $go_2$) by the cosine of the angle between the tangent to the arch axis at U, and the direction of thrust GO (or $go_2$) ; the transverse or radial shearing force may be obtained by multiplying the resultant thrust ($go_2$) by the sine of its inclination to the tangent of the arch axis at U.

Algebraically the resultant thrust may be obtained by compounding

the constant horizontal thrust H with the *vertical* shearing force determined as for a straight horizontal beam.

It is evident that if the centre line of the arched rib is of the same form as the curve of $\mu$, the bending moment M is everywhere zero, *e.g.* in the case of an arched rib carrying a load uniformly spread over the length of span the bending-moment diagram of $\mu$ is a parabola (Art. 57, Fig. 81) symmetrically placed with its axis perpendicular to and bisecting the span; if the rib is also such a parabola the bending moment is everywhere zero.

EXAMPLE 1.—A symmetrical parabolic arched rib has a span of 40 feet and a rise of 8 feet, and is hinged at the springings and crown. If it carries a uniformly spread load of $\frac{1}{2}$ ton per foot run over the left-hand half of the span, find the bending moment, normal thrust, and radial shearing force at the hinges and at $\frac{1}{4}$ span from each end.

Taking the origin at D, Fig. 291, say, the equation to the curved axis or parabolic curve of the centroids is—

$$x^2 = c(8 - y) \quad \text{and at A, } x = 20 \quad y = 0 \qquad \text{hence } c = 50$$

and $$x^2 = 50(8 - y) \quad \text{or} \quad y = 8 - \frac{x^2}{50} \qquad \frac{dy}{dx} = -\frac{x}{25}$$

which gives the tangent of slope anywhere on the rib.

The vertical components of the reactions are evidently—

$$V_A = \tfrac{3}{4} \times 20 \times \tfrac{1}{2} = 7{\cdot}5 \text{ tons} \qquad V_B = 2{\cdot}5 \text{ tons}$$

Taking moments about C—

$$7{\cdot}5 \times 20 - 10 \times 20 \times \tfrac{1}{2} - H \times 8 = 0 \qquad H = 6{\cdot}25 \text{ tons}$$

*Normal Thrust at A.*—

Resultant thrust $R_A = \sqrt{(7{\cdot}5)^2 + (6{\cdot}25)^2} = 9{\cdot}763$ tons

Tangent of inclination to horizontal $= \frac{V_A}{H} = \frac{7{\cdot}5}{6{\cdot}25} = 1{\cdot}2 = \tan 50{\cdot}20°$

Tangent of slope of rib from $\frac{dy}{dx}$ is

$$\frac{20}{25} = 0{\cdot}8 = \tan 38{\cdot}67°$$

Inclination of $R_A$ to centre line of rib $= 50{\cdot}20 - 38{\cdot}67 = 11{\cdot}53°$.

Normal thrust at A $= 9{\cdot}763 \times \cos 11{\cdot}53° = 9{\cdot}56$ tons
Shearing force at A $= 9{\cdot}763 \times \sin 11{\cdot}53° = 1{\cdot}95$ tons

Between A and C at $x$ feet horizontally from D

$$M = -7{\cdot}5(20 - x) + \tfrac{1}{4}(20 - x)^2 + 6{\cdot}25y = -2{\cdot}5x + \tfrac{1}{8}x^2$$

This reaches a (negative) maximum for $x = 10$ when $M = -12{\cdot}5$ ton-feet. The vertical shearing force is then $7{\cdot}5 - 10 \times \frac{1}{2} = 2{\cdot}5$ tons (upward external force to the left), the slopes of the rib and the thrust are the same, viz. $\tan^{-1} 0{\cdot}4$, and the normal thrust is equal to the resultant thrust, viz.

$$\sqrt{(6{\cdot}25)^2 + (2{\cdot}5)^2} = 6{\cdot}73 \text{ tons}$$

At the crown, *vertical* shearing force $= -7{\cdot}5 + 10 = 2{\cdot}5$ tons (downward to the left).

$$\text{Thrust } T_C = \sqrt{(6{\cdot}25)^2 + (2{\cdot}5)^2} = 6{\cdot}73 \text{ tons}$$

The direction and magnitude of the thrust on all the right-hand side of the rib is constant, being in the line BC (as in Fig. 290).

At 10 feet from B the bending moment, which is evidently the maximum value on BC, is

$$-2{\cdot}5 \times 10 + 6{\cdot}25 \times 6 = +12{\cdot}5 \text{ ton-feet}$$

*i.e.* 12·5 ton-feet tending to produce greater curvature of the rib.

$$\text{At B, tangent of inclination of thrust} = \frac{2{\cdot}5}{6{\cdot}25} = 0{\cdot}4 = \tan 21{\cdot}8°$$

tangent of inclination of rib (as at A) is

$$0{\cdot}8 = \tan 38{\cdot}67°$$

Inclination of reaction at B to centre line of rib $= 38{\cdot}67 - 21{\cdot}8 = 16{\cdot}87°$.

$$\text{Normal thrust at B} = 6{\cdot}73 \cos 16{\cdot}87° = 6{\cdot}44 \text{ tons}$$
$$\text{Shearing force at B} = 6{\cdot}73 \sin 16{\cdot}87° = 1{\cdot}95 \text{ tons}$$

**205. Three-hinged Spandrel-braced Arch.**—When the reactions have been obtained algebraically or graphically, as described in the previous article, the determination of the dead load stresses in the members of this structure, illustrated in Fig. 292, gives rise to no special point. The stresses may be found by the method of sections or by a stress diagram, half of which for uniform panel loads is shown at (*a*) in Fig. 292.

The use of influence lines will make the determination of the moving load stresses clear. Taking a vertical section through the panel GF, the stress in GF is found from moments about E. Now from (1), Art. 203, the bending moment at $E = M^E = \mu + Hy$. Hence taking unit load, say, the influence line for E is found by superposing the influence line A′QB′ (*b*) Fig. 292 for a beam of span $l$ (see Fig. 130), and that for the terms $Hy$ in which $y$ is a constant (height of E above AB) and $H = -\frac{\mu_C}{y_C}$, so that $Hy = -\frac{y}{y_C} \cdot \mu_C$. The influence line for $\mu_C$ is a particular case of that in Fig. 130, and has a central ordinate when the load is at C of $-\frac{l}{4}$; hence the influence line for $Hy$ has a central ordinate $ND' = +\frac{y}{y_C} \cdot \frac{l}{4}$. The complete influence line is shown at (*b*), Fig. 292, the base line being A′NB′.

The projections of the shaded triangular areas (see Art. 88) show the portions to be loaded with uniform moving load for maximum negative and positive bending moments at E, corresponding to maximum thrust and tension respectively in GF. And if expressions be written for these areas they give the extreme bending moments at E for unit load per foot, and hence the stress in GF for any uniform load $w$ per foot by multiplying by $w$ and dividing by FE. The projection of the intersection Z′ gives the section at which a concentrated load would give

zero bending moment at E. This is also shown at Z, the intersection of AE and BC, for a load over Z has a reaction in the line AE which is the only external force to the left of E, and has zero moment about E. If the load moves to the left or right of Z, the reaction line moves above or below E, giving a negative or positive moment at E.

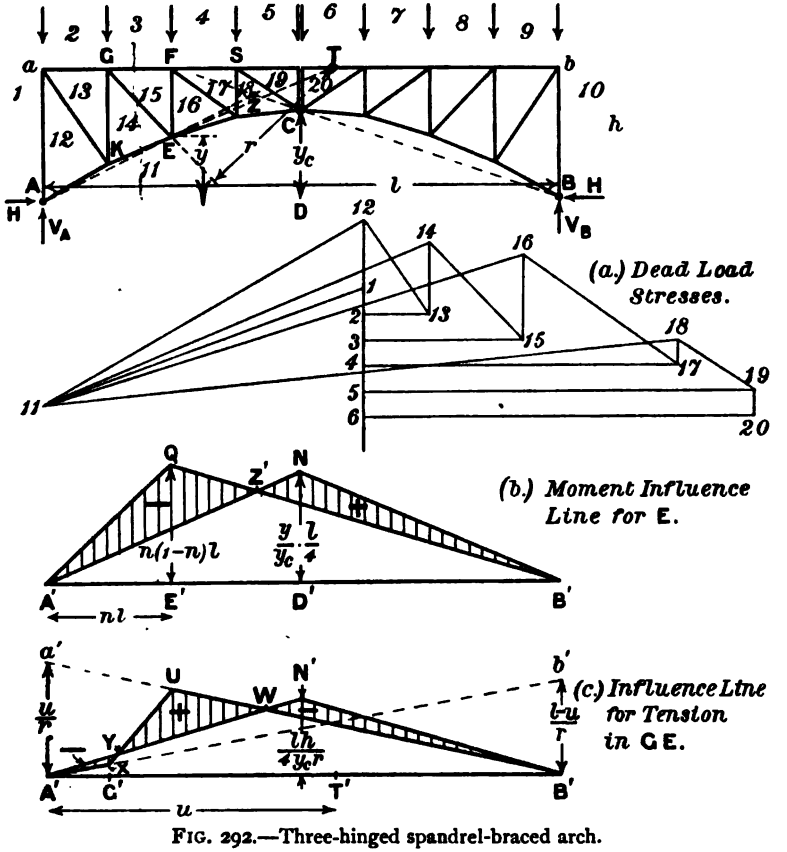



FIG. 292.—Three-hinged spandrel-braced arch.

The influence line for the point F, say, will be the same as regards the line A'QB', but will differ in having the point N raised above A'B' in the ratio that F is higher than E above the line AB.

If all the panel points on the curved rib AECB lie on a parabola, it follows that with uniformly distributed load the maximum opposite (positive and negative) bending moments at any of these points are of equal magnitude, for they arise from loadings on complementary portions of the span, and if both these portions are loaded simultaneously, the linear arch is a parabola passing through these panel points and causing zero bending moment at them.

For the moving load stress in the diagonal GE, take moments about T the intersection of GF and KE. Let $r$ = the perpendicular distance of T from GE, and let $u = a\mathrm{T}$, $l - u = b\mathrm{T}$. Then for unit load moving from $a$ to G for the structure to the right of a vertical section we find

$$\text{Tension on GE} = \frac{1}{r}\{\mathrm{V_B}(l-u) - \mathrm{H}h\} = \mathrm{V_B}\frac{l-u}{r} - \mathrm{H}.\frac{h}{r}$$

where $h$ = height $\mathrm{A}a = \mathrm{B}b$.

The first term is represented by the line A'X, a part of A'$b'$, in Fig. 292 (c), reaching a value $\frac{l-u}{r}$ (for unit load) at $b$. The second term is represented by the line A'N'B'. For loads from F to $b$ the first term becomes $\mathrm{V_A}.\frac{u}{r}$, and is represented by the line B'U, a part of B'$a'$, reaching $\frac{u}{r}$ for unit load at $a$. The variation in tension arising from vertical loads passing over GF is evidently linear, hence joining UX completes the influence line for the first term. Superposing the negative ordinates of A'N'B' on the positive ordinates of A'XUB' gives the resultant influence line for tension in GE measured from A'N'B' as a base. The areas give the magnitudes for the tensions for unit load per foot with loads over the portions of the span projected vertically from the shaded areas. The distance along the span to the change point W might also be found by the intersection of AT with BC produced.

The influence line for stress in a vertical member may be deduced in a similar manner.

The uniformly distributed load equivalent to any given train load will not be that for a simply supported girder; the effect of load concentration will be very strongly marked in its effect on the extreme stresses.

*Approximate Method.*—The foregoing stress calculations for uniform loads from influence line areas may be described as exact, but as in Chapter XII., a conventional calculation may be made by assuming full panel loads. Thus for the maximum negative moment at E, instead of taking a load from $a$ to a point over Z and Z', full panel loads at $a$, G, F, and S may be assumed. And for maximum positive bending moment at E, full panel loads from the centre to $b$. Or again, for maximum live load tension in GE, instead of load over the horizontal length between the change points Y and W, full panel loads at F and S only, may be assumed, and for maximum live load thrust, full panel loads at $a$ and G and from the centre to $b$. When the stresses in all the members are required, it is convenient to tabulate stress coefficients, *i.e.* stresses for unit loads at each panel point in succession. The dead load and maximum and minimum moving load stresses are then easily selected by adding the appropriate coefficients and multiplying the results by the actual panel loads.

*Example.*—Fig. 293. Find the dead load and extreme moving load coefficients for stress in members FG and FQ.

The coefficients are given in tabular form, and the reader is left to work out the results for other members. The position of the moment centre and their distances from the members may be scaled from a drawing or calculated.

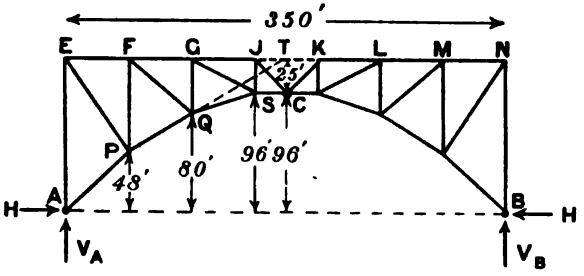



FIG. 293.

TABLE OF COEFFICIENTS OF TENSILE STRESS FOR UNIT PANEL LOADS IN FIG. 293.

For member FQ, T is in PQ produced, FT $= \frac{7}{8} \times 50 = 114$ feet. FQ $= \sqrt{50^2 + 41^2} = 64{\cdot}65$ feet. Distance of FQ from T $= 114 \times \frac{41}{64{\cdot}65} = 72{\cdot}3$ feet, which may also be obtained by drawing to scale.

| Unit load at | $V_A$ | H | Member FG, moment centre Q, arm GQ = 41 ft. | Member FQ, moment centre T, arm 72·3 ft. |
|---|---|---|---|---|
| F | $\frac{6}{7}$ | $\frac{24}{80}$ | $-\frac{1}{41}(\frac{1}{7}\times 250 - \frac{24}{80}\times 80) = -0{\cdot}363$ | $\frac{1}{72{\cdot}3}(\frac{1}{7}\times 186 - \frac{24}{80}\times 121) = -0{\cdot}0683$ |
| G | $\frac{5}{7}$ | $\frac{24}{48}$ | $-0{\cdot}363\times 2 = -0{\cdot}726$ | $\frac{1}{72{\cdot}3}(\frac{2}{7}\times 164 - \frac{24}{48}\times 121) = +0{\cdot}7487$ |
| J | $\frac{4}{7}$ | $\frac{24}{32}$ | $-\frac{1}{41}(\frac{4}{7}\times 100 - \frac{24}{32}\times 80) = +0{\cdot}131$ | $\frac{1}{72{\cdot}3}(\frac{4}{7}\times 164 - \frac{24}{32}\times 121) = -0{\cdot}0113$ |
| K | $\frac{3}{7}$ | $\frac{24}{32}$ | $\frac{1}{41}(\frac{24}{32}\times 80 - \frac{3}{7}\times 100) = +0{\cdot}480$ | $\frac{1}{72{\cdot}3}(\frac{3}{7}\times 164 - \frac{24}{32}\times 121) = -0{\cdot}3350$ |
| L | $\frac{2}{7}$ | $\frac{24}{48}$ | $\frac{2}{3}\times 0{\cdot}480 = +0{\cdot}320$ | $-\frac{2}{3}\times 0{\cdot}3350 = -0{\cdot}2233$ |
| M | $\frac{1}{7}$ | $\frac{24}{80}$ | $\frac{1}{3}\times 0{\cdot}480 = +0{\cdot}160$ | $-\frac{1}{3}\times 0{\cdot}3350 = -0{\cdot}1117$ |
| F, G, J, K, L, M | 3 | $\frac{24}{8}$ | 0 | 0 (approximately) |
| F, G | | | $-1{\cdot}089$ | — |
| J, K, L, M | | | $+1{\cdot}091$ | — |
| G | | | — | $+0{\cdot}7487$ |
| F, J, K, L, M | | | — | $-0{\cdot}7492$ |

The points A, P, Q, S, etc., lie on a parabola, hence the full live load and the dead load bending moment and top chord stresses are zero, and the (complementary) extreme live-load stresses in the top chord are equal and opposite. And considering the top joints, the diagonals evidently carry the horizontal components of the top chord stresses; hence they also have zero full-load stresses and equal and opposite

maximum and minimum stresses. Also at full loads the verticals must just carry the panel loads, and the arched lower chord must have a constant horizontal component throughout, and vertical components just equal to the vertical shearing force. These tests form satisfactory checks for the tabulated stresses in any member of the structure. It is instructive to check the calculations by the more exact method, using the influence line areas.

*Deflection.*—The deflection of the central hinge C and all other points may be found as described in Arts. 155–157. Adopting the graphical method of Art. 157, the two halves of Fig. 292 may be treated separately as if A*a* and B*b* remained vertical, say, and their lower ends fixed. The changes in length in AC and BC are thus found, and hence, applying the graphical method again to the triangle ABC, A and B remaining fixed, the deflection of C is found, and, if desired, the deflection of all other points may be drawn in.

**206. Flexural Deformation of a Curved Rib.**—The bending of a curved rib results in an alteration in the shape, and in particular,

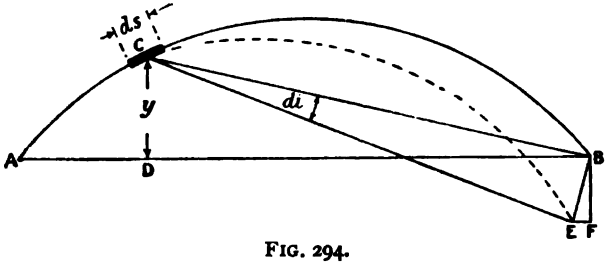



FIG. 294.

chords joining points on the original centre line may be considerably altered in length. Let ACB (Fig. 294) represent the centre line of a curved rib which is subjected to a variable bending moment. To find the alteration in the length AB, consider the effect of the bending of an element of length *ds*; if the remaining part of the bar were unchanged while the element *ds* turned through an angle *di*, the rib at A being supposed fixed in position and direction, B would move to E, the horizontal projection of this displacement being

$$EF = EB\cos B\hat{E}F = CB.di.\cos B\hat{E}F = di.CB\cos B\hat{C}D = DC.di \text{ or } y.di$$

And from Art. 93 the change of curvature

$$\frac{di}{ds} = \frac{M}{EI} \text{ or } di = \frac{M}{EI}ds$$

where I is the moment of inertia of cross-section, and E is the modulus of direct elasticity.

Hence the alteration EF in the chord AB resulting from the bending of the element *ds* is $\frac{M}{EI}.y.ds$; and the total alteration due to bending is

$$\int \frac{My}{EI}ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the integral or sum being taken between limits corresponding to the ends A and B. Bending moments producing greater curvature evidently cause decrease of length of the chord, and those producing decrease of curvature cause increase in length.

Similarly the displacement of B perpendicular to AB is

$$\int \frac{Mx}{EI} ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $x$ is measured along AB from A.

If A represents a hinge fixed in position about which the rib can freely turn, and if B instead of being free is constrained to move in any given locus, the position of B after strain may be found by finding its displaced position, say, B′, by components (1) and (2) as if the rib were fixed at A, and then striking an arc with centre A and radius AB′ to intersect the given locus. For small strains, as in Art. 157, the arc will be a straight line perpendicular to the original chord AB. Hence the actual strained position of B is found by projecting perpendicularly to AB. Hence taking A fixed in position and B constrained to remain in the line AB the shortening of the chord AB due to flexure is as given by (1).

If the strain due to a variable thrust T along a rib of cross-section A is taken into account, an arc $ds$ is shortened by an amount $\frac{T}{AE} ds$, and if $\theta$ is the inclination of the rib to the chord, the corresponding shortening of the chord element $dx$ is $\frac{T}{AE} ds \cos\theta$ or $\frac{T}{AE} dx$. Hence the additional shortening of the chord is

$$\int \frac{T \cos\theta}{AE} ds \quad \text{or} \quad \int_0^l \frac{T}{AE} dx \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $l$ = total length of chord.

In a vertically loaded arch rib the constant horizontal thrust $H = T \cos\theta$, hence the decrease in the chord is

$$H \int \frac{ds}{AE} \quad \text{or} \quad \frac{HS}{AE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where S = total length of arch rib along the axis.

The correction (4) is small and is only important for very flat arches or deep ribs; it omits a correction due to the change in curvature which is itself only important for arches of great curvature. Thus for very approximate results correction (4) may be added to (1) for flat arches and omitted in other cases.

*Deflection of the Crown Hinge of a Three-hinged Arch Rib.*—The deflection of C in Fig. 291 may be found by calculating the changes in AC and BC by (1) corrected by (4) if necessary, and then proceeding as in Art. 157 (Figs. 229 and 233). The integrals in (1) may not be easily calculable algebraically; in such a case they may be found by approximate methods, dividing the arcs into a number of short lengths $\delta s$ and taking for M the values at the centres of the short lengths.

**207. Arched Rib hinged at the Ends.**—A rib hinged at the ends differs from one having three hinges, in that bending stress may result from expansion or contraction of the rib if the hinges at the ends are rigidly fixed in position. The stresses in such a rib are statically indeterminate unless some condition beyond the zero bending moment at the two hinges is assumed. It is usual to suppose that before loading the rib is free from stress, and that after the load is applied the hinged ends remain at the same distance apart as previously, *i.e.* the span remains unchanged. This condition allows of the horizontal thrust being calculated from the principle of displacement. With the notation of Arts. 203 and 204, let M be the bending moment at any cross-section of which G, Fig. 295, is the centroid; then

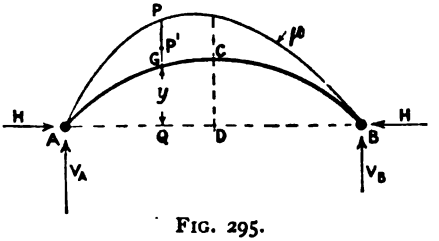


FIG. 295.

$$M = \mu + H.y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and from Art. 206 (1), the total decrease of span, neglecting the effect of the normal thrust, is

$$\int_A^B \frac{M.y}{EI} ds = \int_A^B \frac{(\mu + Hy)yds}{EI}$$

where I is the moment of inertia of cross-section and $ds$ represents an element of the arc AGCB; and by the assumption that the hinges remain in the same position

$$\int \frac{(\mu + Hy)y}{EI} . ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{or,} \quad -\int \frac{\mu}{EI} . yds = H \int \frac{y^2}{EI} ds \quad \text{and} \quad H = \frac{-\int \frac{\mu y}{EI} ds}{\int \frac{y^2}{EI} ds} \quad . \quad (3)$$

the summations being taken over the whole length of the rib. In a large built-up arched rib I will generally be variable, but if not, and E is constant, (3) reduces to

$$H = -\frac{\int \mu . yds}{\int y^2 ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $y$, $\mu$, and $ds$ can be expressed as functions of a common variable this value of H may be found by ordinary integration, and in any case it may be found approximately when the curve of $\mu$ has been drawn by dividing the arc AGCB into short lengths $\delta s$ and taking the sums of the products $\mu . y . \delta s$ and $y^2 . \delta s$, using values of $\mu$ and $y$ corresponding to the middle of the length $\delta s$. If I varies, products $\frac{\mu}{I} y . \delta s$ and $\frac{y^2}{I} . \delta s$ must be used in the summations.

In a circular arch $y$, $ds$ and horizontal distances can easily be expressed as functions of the angle at the centre of curvature, and if the moment $\mu$ can be expressed as in Chapter IV. as a function of horizontal distances along the span, the integrals in (4) can easily be found. In the case of concentrated loads the integral containing $\mu$ can be split into ranges over which $\mu$ varies continuously. When H has been found, M and the normal thrust P may be found from (1) as in the previous article, or graphically from the linear arch drawn by a funicular polygon with a pole distance proportional to H. For a very flat arch the correction (4), Art. 206, may be added to the left side of (2), which adds a term $\frac{S}{AE}$ to the denominator in (3) and a term $\frac{IS}{A}$ or $k^2 s$ to the denominator of (4).

*Alternative Method.*—As an alternative, to find H we may adopt the principle of mimimum resilience (Arts. 158 and 160). Again, neglecting the deformation due to normal thrust from ((5) Art. 108), the total resilience U is

$$U = \tfrac{1}{2}\int M di = \tfrac{1}{2}\int \frac{M^2}{EI} ds = \tfrac{1}{2}\int \frac{(\mu + Hy)^2}{EI} ds = \tfrac{1}{2}\int (\mu^2 + 2\mu Hy + H^2 y^2)\frac{1}{EI} ds$$

and since $\frac{dU}{dH} = 0$, $\int \frac{\mu y}{EI} ds = H\int \frac{y^2}{EI} ds$ or $H = -\int \frac{\mu y}{EI} ds \div \int \frac{y^2}{EI} ds$ (3*a*)

*Movement of Supports.*—If the two hinges instead of remaining a constant distance apart are forced a distance $\delta x$ apart by the thrust, $\delta x$ must be added to the right-hand side of equation (2) and to the numerator of (3), or $EI \cdot \delta x$ to the numerator of (4).

*Graphical Method.*—If the force scale is $p$ pounds to 1 inch, the correct pole distance for drawing the linear arch is $h = \frac{H}{p}$, and if the linear scale is $q$ inches to 1 inch, P′ (Fig. 295) being a point on the linear arch or line of thrust

$$-\mu = P'Q \times p \cdot q \cdot h \text{ (Art. 58) and } y = q \cdot GQ$$

hence from (3), $$H = ph = \frac{\int \frac{P'Q \times GQ}{EI} ds \times p \cdot hq^2}{\int \frac{GQ^2}{EI} ds \times q^2}$$

therefore $$\frac{\int \frac{P'Q \times GQ}{EI} \cdot ds}{\int \frac{GQ^2}{EI} ds} = 1$$

If the diagram of bending moments $\mu$ be drawn to *any* scale, the ordinates PQ being $n$ times the true ordinates P′Q

$$\frac{\int \frac{PQ \cdot GQ}{EI} \cdot ds}{\int \frac{GQ^2}{EI} \cdot ds} = n$$

To get the true ordinates P'Q of the linear arch, each ordinate such as PQ must be altered in the ratio 1 to $n$ or multiplied by $\frac{1}{n}$, *i.e.* by

$$\frac{\int \frac{GQ^2}{EI} ds}{\int \frac{PQ \cdot GQ}{EI} ds}$$

a ratio which can be found for any case graphically, by approximate summation after subdivision of the curve into a number of equal lengths.

*Reaction Locus; Single Load* W.—In dealing graphically with the effect of concentrated loads, such as live panel loads on a two-hinged arch, it is sometimes convenient to construct a locus of the intersections of the reactions. Let Fig. 290 represent a two-hinged arch, in which BZ does *not* necessarily pass through C, and let $z$ be the height of Z above AB, then if the horizontal distance of W from A is $nl$, by similar triangles

$$\frac{z}{nl} = \frac{eh}{oh} = \frac{V_A}{H} = \frac{(1-n)W}{H}$$

$$\text{and} \quad z = n(1-n)l \cdot \frac{W}{H} \quad \text{the locus required} \quad . \quad . \quad (5)$$

*Parabolic Rib; Single Load* W.—The case of a parabolic rib is much simplified if we make the reasonable supposition that the value of I varies proportionally to the secant of the angle of slope of the rib, which is unity at C (Fig. 295), $I = I_0$ say. Then elsewhere $I = I_0 \frac{ds}{dx}$, and substituting this value in (3), E being constant gives

$$H = -\int \mu y dx \div \int y^2 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Then in Fig. 290, for a *two*-hinged arch, with single load W, $nl$ from A, $y = \frac{4y_c}{l^2} x(l-x)$, splitting the integration into two ranges

$$-\int_0^l \mu y dx = \frac{y_c 4W(1-n)}{l^2} \int_0^{nl} x^2(l-x)dx + \frac{4Wny_c}{l^2} \int_{nl}^{l} x(l-x)^2 dx$$

hence from (6)

$$H = \tfrac{5}{8} \frac{Wl}{y_c} n(1-n)(1+n-n^2) \quad . \quad . \quad . \quad . \quad (7)$$

and substituting this in (5), we get the locus

$$z = y_c \cdot \frac{8}{5(1+n-n^2)} \qquad \text{and } z_c = 1{\cdot}28 y_c \quad . \quad . \quad . \quad (8)$$

*Circular Rib; Single Load* W.—Using the notation of Fig. 290, 296, or 319, but taking the rib as *hinged* at both ends, but the load being in the angular position $\beta$, *i.e.* R sin $\beta$ to the left of the centre of the span, the value of H is easily found by taking half the value for two loads, W, symmetrically placed at angular positions, $\beta$ and $-\beta$. From

$x = 0$ or $\theta = \alpha$ to $x = a$ or $\theta = \beta$, $\mu = Wx = WR(\sin\alpha - \sin\theta)$, and from $x = a$ or $\theta = \beta$ to $x = \frac{1}{2}l$ or $\theta = 0$, $\mu = Wa = W(\sin\alpha - \sin\beta)$, and writing $y = R(\cos\theta - \cos\alpha)$ and $ds = -Rd\theta$, equation (4) gives

$$H = \tfrac{1}{2}\frac{-\int_0^{2\alpha}\mu y R d\theta}{-\int_0^{2\alpha} y^2 R d\theta} = \tfrac{1}{2}\frac{\int_0^{\alpha}\mu y d\theta}{\int_0^{\alpha} y^2 d\theta}$$

$$= \frac{W}{2}\cdot\frac{R^2\int_\beta^\alpha(\sin\alpha-\sin\theta)(\cos\theta-\cos\alpha)d\theta + R^2(\sin\alpha-\sin\beta)\int_0^\beta(\cos\theta-\cos\alpha)d\theta}{R^2\int_0^\alpha(\cos\theta-\cos\alpha)^2 d\theta}$$

$$= W\cdot\frac{\frac{1}{2}(\sin^2\alpha-\sin^2\beta)+\cos\alpha(\cos\beta-\cos\alpha-\alpha\sin\alpha+\beta\sin\beta)}{\alpha-3\sin\alpha\cos\alpha+2\alpha\cos^2\alpha}. \quad (9)$$

which takes the simple form $\frac{W}{\pi}\cos^2\beta$ for a semicircular arch when $\alpha = 90°$.

**208. Temperature Stresses in Two-hinged Rib.**—If an arched rib were free to take up any position it would expand, due to increase of temperature, and remain of the same shape. But if the ends are hinged to fixed abutments the span cannot increase, and in consequence the rib exerts an outward thrust on the hinges, and the hinges exert an equal and opposite thrust on the rib; a fall in temperature would cause forces opposite to those called into play by an increase. In either case the horizontal reactions arising from temperature change produce a bending moment as well as a direct thrust or pull in the rib. The change in span arising from these bending moments and that arising from temperature change neutralise one another or have a sum zero.

Let $\alpha$ be the coefficient of linear expansion (see Art. 31), and $t$ be the *increase* of temperature of the rib; then the horizontal expansion, being prevented by the hinges, is—

$$\alpha t \,.\, l$$

where $l$ is the length of span. Hence if M is the bending moment produced at any section of the rib, the centroid of which is at a height $y$ above the horizontal line joining the hinges, and $ds$ is an element of length of the curved centre line of the rib, from Art. 206 (1)

$$\alpha t l - \int\frac{M}{EI}\cdot y\,.\,ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and since M arises from the horizontal thrust H

$$M = Hy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

hence

$$\alpha t l - H\int\frac{y^2}{EI}ds = 0 \quad \text{or} \quad H = \frac{\alpha t l}{\int\frac{y^2}{EI}ds} \quad . \quad . \quad . \quad (3)$$

and if E and I are constant, this becomes

$$H = \frac{EI\alpha tl}{\int y^2 ds} \quad \ldots\ldots\ldots \quad (4)$$

the integrals being taken in either case over the whole span.

The bending moment anywhere, $H.y$, being proportional to $y$, the ordinates of the centre line of the rib measured from the horizontal line joining the hinge centres are proportional to the bending moment, thus giving a bending-moment diagram; the straight line joining the hinges is the line of thrust or "linear arch" for the temperature effects. The stresses at any section due to bending, and due to direct thrust or pull, may be calculated separately and added, the former being the more important. If $h$ is the rise of the rib above the hinges at the highest point or crown, and $d$ is the depth of the section, taken as constant and symmetrical about a central axis, the maximum bending moment due to temperature change is

$$H.h = \frac{EI\alpha tlh}{\int y^2 ds}$$

and the resulting change of bending stress at outside edges of this section is

$$f = \frac{Hh}{I} \times \frac{d}{2} = \frac{E\alpha tlhd}{2\int y^2 ds} \quad \ldots\ldots\ldots \quad (5)$$

In the case of a circular rib the term $\int_{\theta=\alpha}^{\theta=-\alpha} y^2 ds$ in the notation of Fig. 296 may be replaced by $R^3(\alpha - 3 \sin \alpha \cos \alpha + 2\alpha \cos^2 \alpha)$ as in (9), Art. 207. In the parabolic rib, if $\frac{ds}{I} = \frac{dx}{I_0}$, using $\int_0^l y^2 dx$, the value is $\frac{16}{15} h^2 l$.

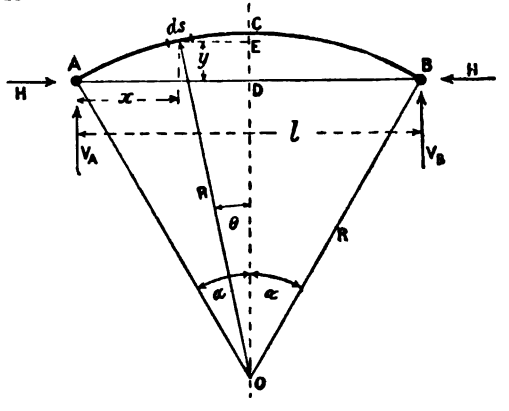



FIG. 296.

EXAMPLE.—A circular arched rib of radius equal to the span is hinged at each end. Find the horizontal thrust resulting from a rise of temperature of 50° F., the coefficient of expansion being 0·0000062 per degree Fahrenheit. If the depth of the rib is $\frac{1}{40}$ of the span, and E = 13,000 tons per square inch, find the extreme change in the bending stresses.

From Fig. 296

$$l = R \qquad \sin \alpha = \tfrac{1}{2} \qquad \alpha = \frac{\pi}{6} \qquad \cos \alpha = \frac{\sqrt{3}}{2} \qquad ds = -R d\theta$$

$$y = R\left(\cos\theta - \frac{\sqrt{3}}{2}\right)$$

$$\int_{\theta=-\alpha}^{\theta=\alpha} y^2 ds = 2\int_0^{\alpha} R^3(\cos^2\theta - \sqrt{3}\cos\theta + \tfrac{3}{4})d\theta = R^3 \cdot \frac{5\pi - 9\sqrt{3}}{12}$$

$$= 0{\cdot}00996R^3$$

hence, from (4), the horizontal thrust

$$H = \frac{EI\alpha/R}{0{\cdot}00996R^3} = \frac{50 \times 0{\cdot}0000062EI}{0{\cdot}00996R^2} = 0{\cdot}03112\frac{EI}{R^2}$$

The bending moment at the crown is

$$HR\left(1 - \frac{\sqrt{3}}{2}\right) = 0{\cdot}03112 \times 0{\cdot}134\frac{EI}{R} = 0{\cdot}00417\frac{EI}{R}$$

hence the extreme change of bending stress is

$$0{\cdot}00417\frac{EI}{R} \times \frac{R}{80I} = 0{\cdot}0000521 \times 13{,}000 = 0{\cdot}677 \text{ ton per sq. inch}$$

**209. Two-hinged Spandrel-braced Arch.**—This is a statically indeterminate frame of the kind dealt with in Art. 159. A preliminary design may be based on a known structure or on calculations from reactions deduced from (3) or (4), Art. 207, taking the braced arch as a rib having a constant value of I. The stresses in the members of such a design may then be calculated after the horizontal thrusts H have been determined for each position of the load by (6), Art. 159, assuming an infinitely stiff member between the hinges to take all the horizontal force. This will necessarily be tedious for all the live loads, but the work is much facilitated by determining the locus of the reaction lines with the panel load lines. This may be done by determining the intersections for say three points on the half span (including the centre) and drawing a smooth curve through them. Tabulation from conventional whole-panel loads as in Art. 205, or influence lines, may be used.

**210. Arched Rib fixed at the Ends.**—The arched rib fixed or clamped in direction at both ends is statically indeterminate, and bears to the rib virtually hinged at each end much the same relation as that of the straight built-in beam to the beam simply supported at each end. The principles of Chap. VIII. hold good for the built-in arched rib. In order to draw the linear arch or otherwise find the bending moment at any section X of such a rib (Fig. 297), it is necessary to know the fixing couples applied at the built-in ends and the horizontal thrust, or three other quantities which make the problem determinate from the simple principles of statics.

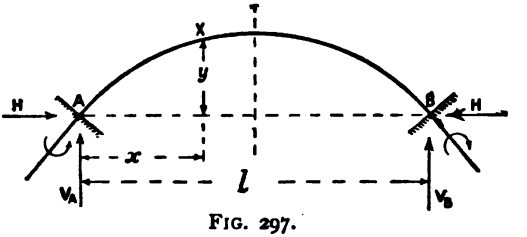



FIG. 297.

*First Method.*—We may write, as in Arts. 102 to 104, allowing for the effect of horizontal thrust

$$M = \mu + M_A + (M_B - M_A)\frac{x}{l} + Hy \quad . \quad . \quad . \quad . \quad (1)$$

where $\mu$ is the bending moment on a straight horizontal freely supported beam carrying the same vertical loads, $M_A$ and $M_B$ are the fixing couples at the ends A and B respectively, H is the constant horizontal thrust, and $y$ is the height of the rib at X above the supports A and B. Bending moments being reckoned positive if tending to increase *convexity upwards* as in Art. 204, the fixing couples $M_A$ and $M_B$ will generally be positive quantities, as in Chap. VIII.

The three unknown quantities $M_A$, $M_B$, and H may be found from the following three conditions :—

(1) The assumption that A and B remain fixed leads, as in Art. 207, from (1), Art. 206, to the equation

$$\int \frac{My}{EI} \cdot ds = \int \frac{\mu y}{EI} ds + M_A \int \frac{y ds}{EI} + \frac{M_B - M_A}{l} \int \frac{xy ds}{EI} + H \int \frac{y^2}{EI} ds = 0 \quad (2)$$

the integrals being taken over the complete length of the curved centre line of the rib; if E and I are constant they may be omitted from each term.

(2) The assumption, as in Art. 103, that the total bending or change from original direction over the whole length of an arch is zero when the ends are firmly fixed gives

$$\int \frac{M}{EI} ds = \int \frac{\mu}{EI} ds + M_A \int \frac{ds}{EI} + \frac{M_B - M_A}{l} \int \frac{x ds}{EI} + H \int \frac{y ds}{EI} = 0 \quad (3)$$

the integrals being over the whole length of the curve, and EI being omitted when constant.

(3) If the ends A and B remain at the same level, as in Arts. 103 and 206 (2)

$$\int \frac{Mx}{EI} ds = \int \frac{\mu x}{EI} ds + M_A \int \frac{x ds}{EI} + \frac{M_B - M_A}{l} \int \frac{x^2 ds}{EI} + H \int \frac{xy}{EI} ds = 0 \quad (4)$$

the integrals being over the whole length of curve between A and B, and EI being omitted when constant.

The three equations (2), (3), and (4) are sufficient to determine the three unknown quantities $M_A$, $M_B$, and H. If all the variables entering into the integrals can easily be expressed in terms of a single variable, ordinary methods of integration may be used. If not, some approximate form of summation by division of the arch AB into short lengths $\delta s$, or graphical methods such as are explained in Art. 103, may be used.

In the case of symmetrical loading, $M_A = M_B$ and equation (4) becomes unnecessary; in that case equations (2) and (3) reduce to

$$\int \frac{\mu y}{EI} \cdot ds + M_A \int \frac{y ds}{EI} + H \int \frac{y^2 ds}{EI} = 0 \quad . \quad . \quad . \quad . \quad (5)$$

$$\int \frac{\mu}{EI} ds + M_A \int \frac{ds}{EI} + H \int \frac{y ds}{EI} = 0 \quad . \quad . \quad . \quad . \quad (6)$$

which are still further simplified if E and I are constants.

*Second Method.*—Just as (1) represents a modification in the bending moment $\mu$ for a simply supported beam, we may, as in Art. 103 and Fig. 158, look upon the rib fixed at both ends as a curved cantilever fixed at A (Fig. 297) and carrying certain loads, and in addition the otherwise free end B subjected to (1) a vertical supporting force $V_B$, (2) a horizontal thrust H, and (3) a fixing couple $M_B$. Let $m$ be the bending moment produced at any section if the rib at B were free. Then

$$M = m - V_B(l-x) + M_B + H.y \quad . \quad . \quad . \quad (7)$$

The conditions stated for equations (2), (3), and (4), taking EI as constant, give—

$$\int My\,ds = \int my\,ds - V_B\int(l-x)y\,ds + M_B\int y\,ds + H\int y^2\,ds = 0 \quad . \quad (8)$$

$$\int M\,ds = \int m\,.\,ds - V_B\int(l-x)\,ds + M_B\int ds + H\int y\,ds = 0 \quad . \quad (9)$$

$$\int Mx\,ds = \int mx\,ds - V_B\int(l-x)x\,ds + M_B\int x\,ds + H\int xy\,ds = 0 \quad (10)$$

In the case of symmetrical loading $V_B$ = half the load, and (10) may be omitted.

As in Art. 207, the equations (2) to (10) inclusive may easily be deduced from the principle of minimum resilience by writing the resilience—

$$U = \tfrac{1}{2}\int\frac{M^2}{EI}ds$$

and substituting for M from (1) and then putting—

$$\frac{dU}{dH} = 0, \quad \frac{dU}{dM_A} = 0, \quad \frac{dU}{dM_B} = 0$$

*Symmetrical Arches.*—In the (usual) case of symmetry of the curved centre line about the vertical centre line, we may simplify the equations. For writing for the whole length $\int ds = S$, $\int y\,ds = \bar{y}\,.\,S$, where $\bar{y}$ is the mean height

$$\int(l-x)y\,ds = \int xy\,ds = \tfrac{1}{2}l\int y\,ds = \tfrac{1}{2}l\,.\,\bar{y}\int ds = \tfrac{1}{2}l\,.\,\bar{y}\,.\,S$$

$$\int(l-x)\,ds = \int x\,ds = \tfrac{1}{2}l\int ds = \tfrac{1}{2}l\,.\,S. \quad \text{Also} \int_0^S y^2\,ds = 2\int_0^{\frac{1}{2}S} y^2\,ds$$

Inserting these, equations (8), (9), and (10) become—

$$\int my\,ds - \tfrac{1}{2}l\,.\,\bar{y}\,.\,S\,.\,V_B + M_B\,.\,\bar{y}\,.\,S + H\int y^2\,ds = 0 \quad . \quad . \quad (8a)$$

$$\int m\,ds - \tfrac{1}{2}l\,.\,S\,.\,V_B + M_B\,.\,S + \bar{y}\,.\,S\,.\,H = 0 \quad . \quad . \quad . \quad . \quad (9a)$$

$$\int mx\,ds - (\tfrac{1}{2}l^2\,.\,S - \int x^2\,ds)V_B + M_B\,.\,\tfrac{1}{2}lS + \tfrac{1}{2}l\bar{y}\,.\,SH = 0 \quad (10a)$$

H is independent of the third condition, for from (8*a*) and (9*a*)—

$$H = \frac{\bar{y}\int_0^S m\,ds - \int_0^S my\,ds}{2\int_0^{\frac{1}{2}S} y^2\,ds - (\bar{y})^2 S} \quad \text{or} \quad \frac{\int_0^S m(\bar{y}-y)\,ds}{2\int_0^{\frac{1}{2}S} y^2\,ds - (\bar{y})^2\,.\,S} \quad . \quad (11)$$

$V_B$ is independent of the first condition, for from (9*a*) and (10*a*)

$$V_B = \frac{\frac{1}{2}l\int_0^S ds - \int_0^S mxds}{\int_0^S x^2ds - \frac{1}{4}l^2S} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and from the symmetry

$$\int_0^S x^2ds = 2\int_0^{\frac{1}{2}S}(\tfrac{1}{2}l - x)^2ds + \left(\frac{l}{2}\right)^2 S \text{ (see Theorem 1, Art. 52)}$$

hence (12) becomes

$$V_B = \frac{\int_0^S m(\frac{1}{2}l - x)ds}{2\int_0^{\frac{1}{2}S}(\frac{1}{2}l - x)^2ds} \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

And from (9*a*)

$$M_B = \tfrac{1}{2}lV_B - H\,.\,\bar{y} - \frac{\int_0^S mds}{S} \quad . \quad . \quad . \quad . \quad (13)$$

Also from moments about the crown the bending moment there is

$$M_C = H(h - \bar{y}) - \frac{I}{S}\int_0^S mds + m_C \quad . \quad . \quad . \quad . \quad (13a)$$

*Approximate Summations.*—If the above integrals cannot be easily calculated algebraically, approximate summations may be made. If the whole length S be divided into *2n equal* parts, it is only necessary to write $\Sigma$ for $f$, $\delta s$ for $ds$, $2n\,.\,\delta s$ for S, $\frac{1}{n}\Sigma y\delta s$ for $\bar{y}$, and divide out the factor $\delta s$ common to the numerators of (11), (12), (12*a*), and (13).

*Varying Moment of Inertia.*—If the moment of inertia (I) of cross-sections of the rib is variable, the factor $\frac{I}{\bar{I}}$ will be retained in each of the equations (8), (9), and (10), and the subsequent summations.

It may happen that $I = I_0\frac{ds}{dx}$ approximately, where $\frac{ds}{dx}$ = secant of inclination of the rib, and $I = I_0$ at the crown. In this case the common factor $I_0$ disappears, and $ds$ is replaced by $dx$ and the limit S by $l$, and $\bar{y}$ becomes the mean height of the enclosed area instead of that of the curved boundary. In approximate solutions the lengths $\delta s$ must not be equal but inversely proportional to the value of I at the centre of each length $\delta s$, *i.e.* so that $\delta s \div I$ = constant for each length chosen. The factor $\delta s \div I$ then may be divided out from the expressions for H, $M_B$, and $V_B$.

*Single Concentrated Load.*—For a single concentrated load W distant $a$ horizontally from A (Fig. 297 or Fig. 319), from A to the load $m = W(a - x)$, and beyond the load $m = 0$; hence, writing equations (11), (12*a*), and (13) for ordinary integration or for approximate summation

when the centre line is divided into $2n$ *equal* parts, remembering that $\bar{y} = \frac{1}{2n}\Sigma(y)$ for the whole length or $\frac{1}{n}\Sigma(y)$ for the half-span

$$H = W.\frac{\bar{y}\int_0^{x=a}(a-x)ds - \int_0^{x=a}(a-x)yds}{2\int_0^{\frac{1}{2}S} y^2 ds - (\bar{y})^2 S}$$

or

$$\frac{W}{2}.\frac{\sum_0^{\frac{1}{2}l}(y) \times \sum_0^{a}(a-x) - n\sum_0^{a}\{(a-x)y\}}{n\sum_0^{\frac{1}{2}l}(y^2) - \left\{\sum_0^{\frac{1}{2}l}(y)\right\}^2} \quad . \quad . \quad (14)$$

$$V_B = \tfrac{1}{2}W.\frac{\frac{1}{2}l\int_0^{x=a}(a-x)ds - \int_0^{x=a}x(a-x)ds}{\int_0^{\frac{1}{2}S}(\frac{1}{2}l-x)^2 ds}$$

or

$$\frac{W}{2}.\frac{\sum_0^{a}\{(a-x)(\frac{1}{2}l-x)\}}{\sum_0^{\frac{1}{2}l}(\frac{1}{2}l-x)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which is equal to $V_0$, the vertical shearing force at the crown, if there is no change between B and the crown, *i.e.* if $a$ is less than $\frac{1}{2}l$; and for a load W similarly placed on the right-hand half of the arch $V_0$ is merely changed in sign, V being equal to the vertical upward external force to the right (or downward to the left) of any section in accordance with the convention for F in straight beams (Art. 59).

$$M_B = \tfrac{1}{2}l.V_B - H\bar{y} - \frac{W}{S}\int_{x=0}^{x=a}(a-x)ds$$

or

$$\tfrac{1}{2}l.V_B - \frac{H}{n}\sum_0^{\frac{1}{2}l}(y) - \frac{W}{2n}\sum_0^{a}(a-x) \quad . \quad . \quad . \quad (16)$$

Also

$$M_C = H(h-\bar{y}) - \frac{W}{S}\int_{x=0}^{x=a}(a-x)ds$$

or

$$H\left\{h - \frac{1}{n}\sum_0^{\frac{1}{2}l}(y)\right\} - \frac{W}{2n}\sum_0^{a}(a-x) \quad . \quad . \quad . \quad . \quad (17)$$

The advantage of these forms of H, $V_B$, and $M_B$ is that the limits over which the summations are to be taken are short; consequently in tabulating numerical values there are few terms. The values derived from (2), (3), and (4) involve more terms in the summations since $\mu$ is not anywhere zero, although some reduction is obtained by taking the value of H for two loads $\frac{1}{2}W$ symmetrically placed apart from the crown.

*Movement of Supports.*—If the support B moves relatively to A, the movement can be taken into account by adding suitable terms to the

fundamental equations. If B moves $\delta x$ horizontally from A, $\delta y$ downwards below A, and rotates $\delta i$ clockwise, the terms $\delta x$, $\delta i$, and $\delta y$ will have to be added to the right-hand sides of equations (2), (3), and (4) respectively, or EI times these terms to the right-hand sides of equations (8), (9), and (10) respectively.

*Correction due to Shortening of Rib by Normal Thrust.*—As for the rib hinged at both ends, approximately for very flat ribs the correction (4), Art. 206, may be added to the left side of equation (2) or EI times this amount to the left side of equation (8).

*Stresses, etc.*—When H, $V_B$, and $M_B$ are determined, the bending moment anywhere is obtained from (7), and the normal thrust as explained in Arts. 203 and 204. If it is desired to draw the linear arch, the vertical line is drawn and the pole set off from $V_B$ and H. A starting point is found either at a distance $M_B \div H$ vertically below B or $M_B \div V_B$ horizontally to the left of B, or $M_C \div H$ below the crown.

EXAMPLE 1.—Find the unknown quantities for a parabolic arch, rise of centre line $h$, and distance between centres of fixed ends $l$, carrying a load W distant $a$ horizontally from the left-hand support centre, taking $I = I_0 \frac{ds}{dx}$.

It is only necessary to write $dx$ instead of $ds$ in (14), (15), and (16). The equation of the centre line is

$$y = \frac{4h}{l^2}x(l-x) \quad \text{and} \quad \bar{y} = \tfrac{2}{3}h.$$

$$\int_0^a (a-x)dx = \tfrac{1}{2}a^2, \qquad \int_0^a (a-x)ydx = \frac{4h}{l^2}\int_0^a x(a-x)(l-x)dx = \frac{a^3h(2l-a)}{3l^2}$$

$$\int_0^{\frac{1}{2}l} y^2dx = \frac{16h^2}{l^4}\int_0^{\frac{1}{2}l}(l^2x^2 - 2lx^3 + x^4)dx = \tfrac{4}{15}h^2l, \qquad l(\bar{y})^2 = \tfrac{4}{9}h^2l$$

$$\int_0^a x(a-x)dx = \tfrac{1}{6}a^3, \qquad \int_0^{\frac{1}{2}l}(\tfrac{1}{2}l - x)^2dx = \tfrac{1}{24}l^3$$

Substituting these in (14), (15), (16), and (17) with $dx$ for $ds$, we find

$$H = \frac{15W}{4}\cdot\frac{a^2(l-a)^2}{l^3h} \qquad V_B = V_C = W\frac{a^2(3l-2a)}{l^3}$$

$$M_B = -\frac{Wa^2}{2l^3}(3l-5a)(l-a) \qquad M_C = \frac{Wa^2}{4l^3}\{5(l-a)^2 - 2l^2\}$$

EXAMPLE 2.—Find the horizontal thrust in a symmetrical circular arch, radius R, fixed at the ends, and carrying a single load W. The arch subtends an angle $2\alpha$ at the centre of curvature, and the arc between the crown and the load subtends an angle $\beta$. Notation as in Fig. 296 or Fig. 319, $s = R(\alpha - \theta)$, $ds = -Rd\theta$.

$$S = 2R\alpha \qquad y = R(\cos\theta - \cos\alpha) \qquad (a-x) = R(\sin\theta - \sin\beta)$$

$$\tfrac{1}{2}l - x = R\sin\theta \qquad \bar{y} = R\int_0^\alpha yd\theta \div R\alpha = \frac{R}{\alpha}(\sin\alpha - \alpha\cos\alpha)$$

$$\int_{x=0}^{x=a}(a-x)(\bar{y}-y)ds = R^3\int_\beta^\alpha(\sin\theta - \sin\beta)\left(\frac{\sin\alpha}{\alpha} - \cos\theta\right)d\theta$$

The remaining integrals in (11) are simple, and the result is

$$H = \frac{W}{2} \cdot \frac{\sin \alpha(2 \cos \beta + 2\beta \sin \beta - 2 \cos \alpha - \alpha \sin \alpha) - \alpha \sin^2 \beta}{\alpha^2 + \alpha \sin \alpha \cos \alpha - 2 \sin^2 \alpha}$$

The calculations of $V_B$ and $M_B$ offer no difficulty.

$$V_B = \frac{W}{2} \cdot \frac{2(\alpha - \beta) - \sin 2\alpha + 4 \cos \alpha \sin \beta - \sin 2\beta}{2\alpha - \sin 2\alpha}$$

$$M_C = M_B - \tfrac{1}{2} \cdot V_B \cdot l + HR(1 - \cos \alpha)$$

$$= HR\left(1 - \frac{\sin \alpha}{\alpha}\right) + \tfrac{1}{2}WR \cdot \frac{\cos \alpha - \cos \beta + (\alpha - \beta) \sin \beta}{\alpha}$$

For a numerical example of the approximate method, see Art. 217.

**211. Temperature Stresses in Fixed Rib.**—With the same notation as in Art. 208, for the direction AB (Fig. 297), in which expansion is prevented as for the two-hinged rib

$$\alpha t l - \int \frac{My}{EI} \cdot ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also as in (3), Art. 209,

$$\int \frac{M ds}{EI} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and as in (4), Art. 209,

$$\int \frac{Mx}{EI} ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let H and V be the vertical and horizontal thrusts at either end of the span resulting from a temperature change of $t$ degrees (V is equal and opposite at the two ends and is taken positive when upwards at A), and let $M_A$ be the fixing couple at the supports due to the temperature change; then

$$M = M_A - V.x + H.y \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This value of M substituted in the three equations (1), (2), and (3), gives the necessary equations to find $M_A$, V, and H. The bending moment anywhere in the rib then follows from (4).

If the rib is symmetrical about a vertical axis through the middle of the span, V is zero, and the two equations (2) and (3) reduce to one, and equation (4) becomes

$$M = M_A + Hy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which, being substituted in (1) and (2), gives

$$M_A \int \frac{y}{EI} \cdot ds + H \int \frac{y^2}{EI} \cdot ds - \alpha t l = 0 \quad . \quad . \quad . \quad . \quad (6)$$

and

$$M_A \int \frac{ds}{EI} + H \int \frac{y}{EI} ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

from which $M_A$ and H may be found.

The "line of thrust" in this case is a straight horizontal line the distance of which above AB (Fig. 297) is

$$-\frac{M_A}{H} = \frac{\int y ds}{\int ds} = \text{mean height of centre line (if EI = constant)}$$

In the uncommon case of an unsymmetrical rib the line of thrust would be inclined to the line AB, passing at distances $\frac{M_A}{T}$ and $\frac{M_B}{T}$ respectively from A and B, where T is the thrust the components of which are H and V, and $M_B$ is the fixing moment at B, viz. $M_A - V.l.$

The necessary integrals for equations (6) and (7) have been given in the preceding articles for the circular rib and for the parabolic rib in which $\frac{ds}{I} = \frac{dx}{I_0}$.

EXAMPLE.—Solve the problem at the end of Art. 208, in the case of an arched rib rigidly fixed in direction at both ends. Find also the points of zero bending moment.

In this case

$$\int_{\theta=\frac{\pi}{6}}^{\theta=-\frac{\pi}{6}} yds = 2R^2\int_0^{\frac{\pi}{3}}\left(\cos\theta - \frac{\sqrt{3}}{2}\right)d\theta = R^2\left(1 - \frac{\sqrt{3}\pi}{6}\right) = 0{\cdot}0931R^2$$

$$\int_{\theta=\frac{\pi}{6}}^{\theta=-\frac{\pi}{6}} y^2ds = 0{\cdot}00996R^3 \text{ (see Art. 208)}, \qquad \int_{\theta=\frac{\pi}{6}}^{\theta=-\frac{\pi}{6}} ds = \frac{\pi}{3}R = 1{\cdot}0472R$$

Substituting these values in (6) and (7)—

$$0{\cdot}0931M_A \cdot R^2 + 0{\cdot}00996R^3 \cdot H - 0{\cdot}00031EIR = 0$$
$$1{\cdot}0472M_A \cdot H + 0{\cdot}09310R^2H = 0$$

$$-M_A = \frac{0{\cdot}09310}{1{\cdot}0472}HR = 0{\cdot}0889oHR$$

hence $$H = 0{\cdot}1845\frac{EI}{R^2} \qquad M_A = -0{\cdot}0164\frac{EI}{R}$$

At the crown (Fig. 296) $y = \left(1 - \frac{\sqrt{3}}{2}\right)R = 0{\cdot}134R$

and $$M_0 = -0{\cdot}0164\frac{EI}{R} + 0{\cdot}1845 \times 0{\cdot}134\frac{EI}{R} = +0{\cdot}0083\frac{EI}{R}$$

The maximum bending moment is $M_A$ at the supports, and at those sections the extreme change in bending stress is

$$f = \frac{-M_A \times d}{2I} = \frac{0{\cdot}0164EI}{2RI} \times \frac{R}{40} = 0{\cdot}000205 \times 13{,}000$$
$$= 2{\cdot}665 \text{ tons per square inch}$$

which is nearly four times the value for the similar hinged arch in Art 208.

The points of zero bending moment occur when $Hy = -M_A$.

$$y = \frac{-M_A}{H} = 0{\cdot}0889R = R\left(\cos\theta - \frac{\sqrt{3}}{2}\right)$$

$$\cos\theta = \frac{\sqrt{3}}{2} + 0{\cdot}0889 = 0{\cdot}9549 \qquad \theta = 17{\cdot}3°$$

Distance from support $= x = R(\frac{1}{2} - \sin\theta) = 0{\cdot}2026R$ or 0·2026 of the span.

## EXAMPLES XVIII.

1. Limiting the dip to $\frac{1}{10}$ of the span, find the greatest span which a uniform steel wire may have without exceeding a stress of 7·5 tons per square inch due to its own weight—viz. 0·28 lb. per cubic inch.

2. A suspension bridge cable of 80 feet span has to support a total load of $\frac{1}{2}$ ton per foot of span, and its dip is 8 feet. Find the maximum pull in the steel cables, and their cross-sectional area and length if the working stress is to be 5 tons per square inch. If the cable passes over a saddle and the backstay is inclined 30° to the horizontal, find the tension in the backstay and the pressure on the pier. If the cable passes over a pulley, find the horizontal and vertical pressures on the pier, and draw triangles of forces for both cases.

3. A chain consisting of eyebar links has a span of 99 feet, and 10 hangers which divide the span into 11 equal parts, and each hanger carries a load of 2 tons. The right-hand end is 16 feet and the left-hand end is 4 feet above the lowest point in the centre line of the chain. Draw the form of the chain, and write down the tension in the successive links from the left-hand end.

4. A suspension cable of 100 feet span and 10 feet dip is stiffened by a three-hinged girder. The dead load is $\frac{1}{4}$ ton per foot run. Determine the maximum tension in the cable and the maximum bending moment in the girder due to a concentrated load of 5 tons crossing the span, assuming that the whole dead load is carried by the cable without stressing the girder. Find the bending moment in the girder at $\frac{1}{10}$ of the span from either pier when the concentrated load is 25 feet from the left-hand pier.

5. If the girder in Problem No. 4 is traversed by a uniform load of $\frac{1}{10}$ of a ton per foot, find the maximum positive or negative bending moment in the left-hand half of the girder due to live load and the lengths covered by the load when these maxima occur.

6. Find the maximum shearing forces at $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{3}{8}$ of the span with the data in Problem No. 4.

7. Solve Problem No. 6, but using the loads of No. 5.

8. Solve Problem No. 4 if the central hinge is omitted.

9. Solve Problem No. 5 if the central hinge is omitted.

10. Solve Problem No. 6 if the central hinge is omitted.

11. Solve Problem No. 7 if the central hinge is omitted.

12. Find the change in the stress in the chords of a two or three hinged stiffening girder of a suspension bridge due to a change of 60° F. in temperature if the dip is 20 feet and the depth of the girder 7 feet. ($E = 13{,}000$ tons per square inch. Coefficient of expansion $62 \times 10^{-7}$.)

13. A symmetrical three-hinged arch rib is of circular form, has a span of 50 feet and a rise of 10 feet. If the uniformly distributed load is 1 ton per foot of span, find the horizontal thrust and the bending moment at $\frac{1}{4}$ span (horizontally) from one end.

14. A parabolic arched rib, hinged at the springings and crown, has a span of 50 feet and a rise of 10 feet; if the load varies uniformly with the horizontal distance from the crown from $\frac{1}{2}$ ton per foot of span at the crown to 1 ton per foot run at the springings, find the horizontal thrust and the bending moment at $\frac{1}{4}$ span. What is the normal thrust and the shearing force 5 feet from one of the abutments?

15. If the rib in Problem No. 14 has a concentrated load of 5 tons, 12·5 feet from one support, find the horizontal thrust and the bending

moments on the rib at the $\frac{1}{4}$ span points ; also if it has 5 ton loads at 12·5 feet from *each* end support.

16. Find approximately in terms of the panel loads W, say, the extreme live load stresses in EF and EP (Fig. 293).

17. Find by the exact method the extreme moving load stresses in FG (Fig. 293) for a uniform load $w$ per foot.

18. Find the horizontal thrust for the arch in problem No. 14 if it is hinged at the ends only.

19. A parabolic two-hinged arched rib has a span of 40 feet and a rise of 8 feet, and carries a load of 10 tons at the crown. The moment of inertia of the cross-section of the rib is everywhere proportional to the secant of the angle of slope of the rib. Find the horizontal thrust and the bending moment at the crown.

20. Solve Problem No. 19 if the load is at (*a*) $\frac{1}{4}$ span, (*b*) $\frac{1}{8}$ span.

21. A circular arched rib 40 feet radius hinged at both ends and subtending an angle of 90° at the centre carries a load of 1 ton at a horizontal distance of 20 feet from midspan. Find what horizontal thrust is caused by this load.

22. Find the maximum intensity of bending stress in a circular arched rib 50 feet span and 10 feet rise, hinged at each end, due to a rise in temperature of 60° F., the constant depth of the rib being 12 inches. (Coefficient of expansion $\frac{5}{9} \times 10^{-5}$. E = 12,500 tons per square inch.)

23. Solve Problem No. 19 if the rib is fixed at both ends.

24. A semicircular arched rib of span $l$, and fixed at both ends, carries a load W at the crown. Find the bending moment, normal thrust, and shearing force at the ends and crown.

25. Solve Problem No. 21 if the rib is fixed at both ends.

26. A piece of steel 1 inch square is bent into a semicircle of 20 inches mean radius, and both ends are firmly clamped. Find the maximum bending stress resulting from a change in temperature of 100° F. in the steel. What is the angular distance of the points of zero-bending moment from the crown of the semicircle? (Coefficient of expansion $62 \times 10^{-7}$. $E = 30 \times 10^6$ pounds per square inch.)

27. Solve Problem No. 22 if the rib is fixed at both ends.

# CHAPTER XIX

## *EARTH PRESSURE, FOUNDATIONS, MASONRY STRUCTURES*

**212. Earth Pressure.**—In order to compute the forces to which various foundations and masonry structures, etc., such as retaining walls, are subjected, the pressures exerted by and on plane faces of earth are required. There are numerous theories as to the pressure exerted, differing somewhat in the assumptions made and the expressions deduced. Most theories are based upon the supposition that earth is a granular mass entirely lacking in cohesion and having for each kind of earth a definite angle of repose or natural slope which it will assume if left unsupported for a sufficient time. The various theories give results which in most practical cases do not materially differ from one another. There is very little experimental evidence that the calculated pressures form a reliable guide to the actual conditions which vary with many circumstances. The cohesion in moist, well-rammed earth is often very considerable, and in consequence many structures are able to withstand earth pressure which, if the granular earth theories gave correct values, would be quite unsafe.

It is well to recognise that earth pressures cannot be calculated with anything approaching the accuracy usually possible in say stress computations for a simple steel framework or a simply supported steel beam. One theory is here given in some detail, and for others the reader is referred to books specially devoted to such matters.

*Rankine's Theory of Earth Pressure.*—Notation. $\phi$ = angle of repose of earth = maximum angle which any resultant force across any internal face can make to the normal without slipping occurring. $w$ = weight of unit volume of earth (say lbs. per cubic foot).

(*a*) *Vertical Wall Face: Horizontal Earth Surface.*—When slipping is about to take place downwards across the plane where the resultant force is most oblique to the normal, it follows from (6), Art. 17, that the smaller principal stress is,

$$p_y = p_x \cdot \frac{1 - \sin \phi}{1 + \sin \phi} \quad \ldots \ldots \quad (1)$$

where $p_x$ is the intensity of the maximum principal stress.

At a depth $h$ in the earth the maximum principal stress will be vertical, *i.e.* perpendicular to a horizontal face and equal to $wh$, since

one square foot say supports a column of earth $h$ feet high, having contents $h$ cubic feet and weight $wh$ lbs. Hence the horizontal pressure is,

$$p_y = wh \cdot \frac{1 - \sin\phi}{1 + \sin\phi} \quad \ldots\ldots\ldots \quad (2)$$

which is proportional to $h$. Hence the total pressure per foot length of a vertical face is (average intensity × area),

$$P = \tfrac{1}{2}wh \cdot \frac{1-\sin\phi}{1+\sin\phi} \cdot h = \tfrac{1}{2}wh^2 \frac{1-\sin\phi}{1+\sin\phi} \text{ or } \tfrac{1}{2}w \cdot h^2 \tan^2(45° - \tfrac{1}{2}\phi) \quad (3)$$

or $\rho \times \dfrac{1 - \sin\phi}{1 + \sin\phi}$ times that of water pressure on the same wall face, where $\rho$ is the specific gravity of the earth. The force P acts as shown in Fig. 298 at a depth of $\frac{2}{3}$ of $h$ from the horizontal earth surface.[1]

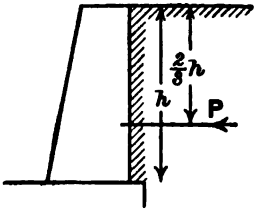



FIG. 298.

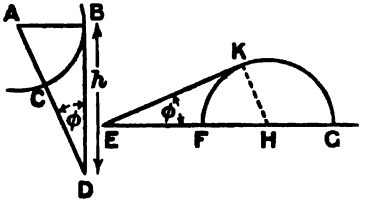



FIG. 299.

It will be noticed from (2) and (3) that the magnitudes are those for the pressure of a liquid of density $w$ per unit volume multiplied by the coefficient $\dfrac{1 - \sin\phi}{1 + \sin\phi}$, which is unity for a liquid, which may be defined by the static property $\phi = 0$.

Simple graphical constructions for P and for $wh\dfrac{1 - \sin\phi}{1 + \sin\phi}$ are shown in Fig. 299; AC = AB, BD = $h$, angle $A\hat{D}B = \phi$, then $P = \frac{1}{2} \cdot w \cdot CD^2$. And if EG = $wh$ to scale and the semicircle FKG is drawn to touch EK inclined $\phi$ to EG, $EF = wh\dfrac{1 - \sin\phi}{1 + \sin\phi}$.

(*b*) *Sloping Wall Face: Horizontal Earth Face.*—Let the slope be $\theta$ to the vertical (see Fig. 300). Then the intensity of pressure across the face at a depth $h$ by (3), Art. 15, is

$$p = \sqrt{p_x^2 \sin^2\theta + p_y^2 \cos^2\theta} = \sqrt{p_n^2 + p_t^2} \quad \ldots \quad (4)$$

where $p_x = wh$ and $p_y$ as before is $wh \cdot \dfrac{1 - \sin\phi}{1 + \sin\phi}$, hence $p$ is propor-

[1] For the depth of the centre of pressure see any elementary book dealing with the mechanics of fluids.

tional to $h$. The total pressure per foot length of wall face is $\frac{1}{2}p \times$ area or substituting

$$P = p \cdot \tfrac{1}{2}h \cdot \sec\theta = \tfrac{1}{2}wh^2\sqrt{\tan^2\theta + \tan^4(45 - \tfrac{1}{2}\phi)} \quad . \quad (5)$$

and this pressure acts in the direction given by (5), Art. 15.

*Graphical Construction.*—The value of $p$ may be found graphically by the method given in Fig. 17, the principal stresses $p$ and $p_y$ being known as above. The graphical construction is shown in Fig. 301, where $p_x$ is across a horizontal plane. The proof, with the notation of Art. 15, is as follows:—

$$\begin{aligned} ED &= \tfrac{1}{2}(p_x - p_y)\sin 2\theta = p_t \\ BD &= \tfrac{1}{2}(p_x + p_y) - \tfrac{1}{2}(p_x - p_y)\cos 2\theta \\ &= \tfrac{1}{2}p_x(1 - \cos 2\theta) + \tfrac{1}{2}p_y(1 + \cos 2\theta) \\ &= p_x\sin^2\theta + p_y\cos^2\theta = p_n \\ BE &= \sqrt{BD^2 + ED^2} = \sqrt{p_n^2 + p_t^2} = p. \end{aligned}$$

$$P = \tfrac{1}{2}ph\sec\theta \text{ or } \tfrac{1}{2}\,p \cdot AB \quad . \quad . \quad . \quad . \quad . \quad (6)$$

acting parallel to BE through a point in AB, $\frac{2}{3}$ of AB from A.

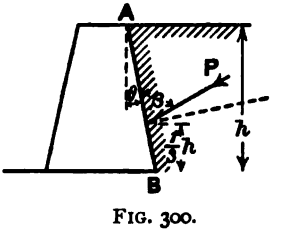



FIG. 300.

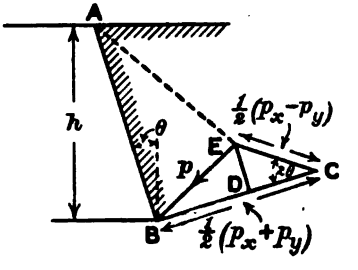



FIG. 301.

(*c*) *Vertical Wall surcharged at Slope* $\alpha$.—Rankine assumed that the pressure on a vertical face was parallel to the earth slope, *i.e.* inclined $\alpha$ to the horizontal. Then at any depth $h$ the vertical pressure $p_1$ on a plane face inclined $\alpha$ to the horizontal, forms with the resultant pressure intensity $p_2$ on a vertical face a pair of conjugate stresses (so called because they may be represented by a pair of conjugate semi-diameters of the ellipse of stress), *i.e.* $p_1$ is parallel to the face across which $p_2$ acts, and $p_2$ is parallel to the face across which $p_1$ acts. Now referring to Fig. 302,

$$p_1 = \frac{wh}{\sec\alpha} = wh\cos\alpha \text{ (not a principal stress)} \quad . \quad . \quad . \quad . \quad (7)$$

The stresses $p_1$ and $p_2$ may be looked upon as resultants of principal stresses $p_x$ and $p_y$ (in directions unknown), and therefore (Art. 17), of a normal stress $\frac{1}{2}(p_x + p_y)$ added geometrically to a stress $\frac{1}{2}(p_x - p_y)$ acting at an unknown angle $2\theta$ to the normals of the faces giving resultants inclined $\alpha$ to the normals. Fig. 303 represents the vector diagram as

in Art. 17, AC being proportional to $\frac{1}{2}(p_x + p_y)$ and BC $(=CD)$ to $\frac{1}{2}(p_x - p_y)$. The two stresses inclined $\alpha$ to the normals are represented by AB and AD, so that

$$\frac{p_2}{p_1} = \frac{AB}{AD} \quad \ldots \ldots \ldots \ldots \quad (8)$$

and substituting in terms of AC and BC and writing from (9), Art. 15 or (5) Art. 17, when slipping is about to take place, $\frac{p_x - p_y}{p_x + p_y} = \sin\phi$, and using (7) we get

$$p_2 = wh \,.\, \cos\alpha \,.\, \frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\phi}} \quad \ldots \quad (9)$$

which reduces to (1) when $\alpha = 0$.

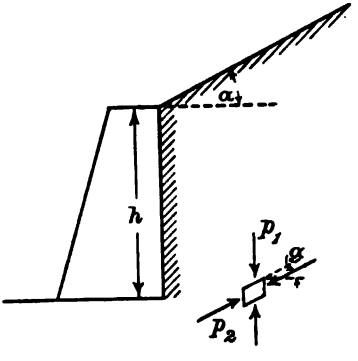



FIG. 302.

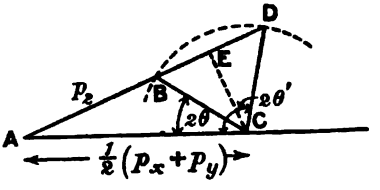



FIG. 303.

The total pressure per foot length of face is

$$P = \tfrac{1}{2}p_2 \,.\, h = \tfrac{1}{2}wh^2 \,.\, \cos\alpha \,.\, \frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\phi}} \quad . \quad (10)$$

If the surcharge reaches the maximum possible angle $\phi$ (10) becomes the maximum possible pressure on a vertical wall, viz.

$$P = \tfrac{1}{2}wh^2 \,.\, \cos\phi \quad \ldots \ldots \ldots \quad (11)$$

*Graphical Constructions.*—The value $p_2$ is easily found graphically as shown in Fig. 304, by drawing a semicircle BEC centred at O, and from E a tangent AE to meet the circumference in E, and from A drawing AC inclined $\alpha$ to AD cutting the semicircle in B and C, and then drawing CD perpendicular to AC. Then AB represents $p_2$ and AC represents $p_1$ on the scale that AD represents $wh$. The point E is the limiting position of both B and C for maximum surcharge.

P may be represented by the weight of a triangular prism of earth

XYZ (Fig. 305), one foot long, perpendicular to the figure if YZ $= h \cdot \frac{AB}{AC}$ (Fig. 304), or $h \cdot \frac{AB}{AD}$ (Fig. 303).

(*d*) *Sloping Surcharged Wall* (Fig. 306).—The resultant pressure on a sloping face AB may be found by finding that on BC as in the previous case, and adding geometrically the weight of a triangular prism

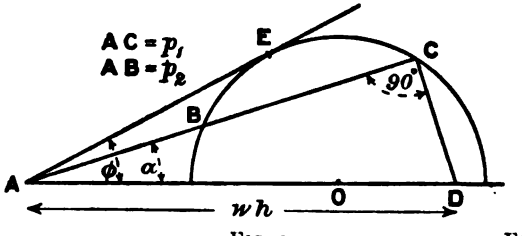



FIG. 304.

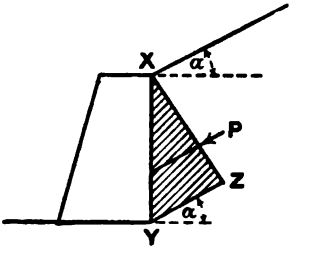



FIG. 305.

of earth ABC. The algebra involved is quite simple but occupies much space, so is not set forth here. It should be noticed that BC is equal to $h(1 + \tan\theta \cdot \tan\alpha)$ or $h\{\cos(\alpha - \theta) \cos\alpha \cdot \cos\theta\}$. The final result, which the reader may verify for himself, is

$$P = \tfrac{1}{2}wh \cos(\alpha - \theta) \sec^2\theta \sec\alpha \times \sqrt{\sin^2\theta + 2K \tan\alpha \sin\theta \cos(\alpha - \theta) + K^2 \cos^2(\alpha - \theta) \sec^2\alpha} \quad (12)$$

where K is the ratio $\cos\alpha \cdot \dfrac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\phi}}$

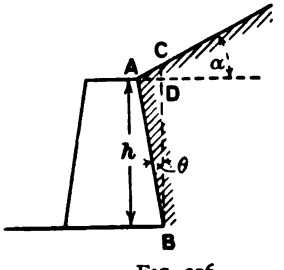



FIG. 306.

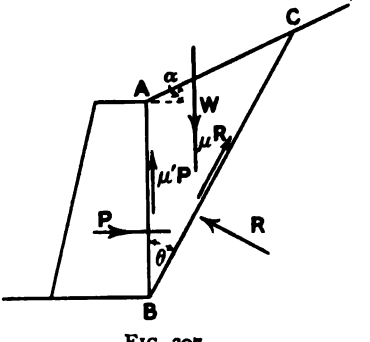



FIG. 307.

The inclination of P to the horizontal is $\beta$ where

$$\tan\beta = \frac{1}{K} \sin\theta \sec(\alpha - \theta) + \tan\alpha \quad . \quad . \quad . \quad (13)$$

*Wedge Theories.*—Another method of estimating the pressure on say a vertical face AB is to consider it as supporting a wedge or triangular prism of earth ABC (Fig. 307), which would slip away if the face were

removed. This involves the *assumption* that the surface of rupture would be a *plane* of rupture such as BC, inclined say $\theta$ to the vertical. From the principles of statics the value of the normal component pressure P exerted by the wall on the wedge may be written in terms of $\theta$ and constants. To find the maximum value of P this expression may be differentiated with respect to $\theta$ and the result equated to zero, and hence $\theta$ obtained. By substituting this value of $\theta$ the maximum value of P may be found. Various assumptions may be made with respect to the angle of friction $\beta$, say between the earth and the wall. The commonest is to make $\tan \beta = \mu' = \tan \phi = \mu$. If we put $\beta = 0$ (*i.e.* $\mu' = 0$) and $\alpha = 0$, $\theta = 45 - \frac{1}{2}\phi$, and we obtain Rankine's value (3), which may also be written $\frac{1}{2}wh^2(\sqrt{1+\mu^2} - \mu)^2$. If we put $\mu' = \mu$, resolving horizontally and vertically and eliminating R,

$$P = \tfrac{1}{2}wh^2 \cdot \frac{1 - \mu \tan \theta}{(2\mu \cot \theta + 1 - \mu^2)(1 - \tan \alpha \tan \theta)} \quad . \quad . \quad (14)$$

and differentiating this with respect to $\theta$ and equating to zero to find the conditions which give a maximum normal thrust P, we find

$$\tan \theta = \frac{2\mu^2 - \sqrt{2\mu(1+\mu^2)(\mu - \tan \alpha)}}{(1+\mu^2)\tan \alpha - \mu(1-\mu^2)} \quad . \quad . \quad . \quad (15)$$

which when substituted in (14) gives the maximum value of P. The actual thrust on the face AB according to this theory is the resultant of $\mu'$P downward and P horizontally.

Taking the particular case of level earth, *i.e.* $\alpha = 0$, and substituting in (15) and (14), we find,

$$\tan \theta = \frac{\sqrt{2(1+\mu^2)} - 2\mu}{1 - \mu^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$P = \tfrac{1}{2}\, wh^2 \frac{1 + 3\mu^2 - 2\mu\sqrt{2(1+\mu^2)}}{(1-\mu^2)^2} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

**213. Resistance and Stability of Masonry, Brickwork, etc.**—Masonry and similar structures are usually employed (without steel reinforcement) mainly to resist compressive forces. This, of course, (Art. 7) causes shear stress on surfaces oblique to that which withstands thrust.

(1) Owing to eccentricity of the resultant thrust (Arts. 111 and 112) bending stresses arise which unless the eccentricity is suitably limited will involve to tensile stress. It is a general practice to disregard any tensile resistance which such structures are capable of exerting in virtue of the adhesion of the mortar or cementing material, and to attempt to limit the possible eccentricity of thrust so as to prevent tensile stress.

(2) The shearing resistance of joints is likewise taken as negligible, and consequently the obliquity of thrust across any joint should be limited to the angle of friction (Fig. 308), *i.e.* the tangential stress on a joint should not exceed the frictional resistance to sliding. The

main function of mortar is not adhesive resistance, but uniform distribution of thrust across joints.

(3) The resistance to thrust (really dependent on the shearing resistance on oblique planes) is limited by the strength of the stone or brick or concrete. To allow for concentration of pressure due to uneven bedding a high factor of safety is usually adopted.

*Middle Third Rule.*—The majority of masonry and brickwork joints are of rectangular section, and from Art. 112 and Fig. 170 it is evident that to avoid tension the eccentricity of thrust must be limited to $\frac{1}{3}$ of the breadth of the joint, *i.e.* the thrust must fall *within the middle third* of the joint. In Fig. 308, to avoid tension at B, the resultant thrust P must not fall outside DE. If it falls to the left of D, tension at B may open the joint. The result is a smaller bearing surface, giving increased intensity of compressive stress at A. In many cases with the ample margins allowed no serious consequences may ensue, but too great an opening of the joint may result in failure by shearing associated with compression in the neighbourhood of A.

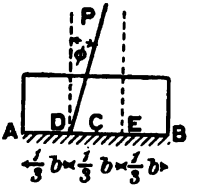


FIG. 308.

*Stresses in Masonry and Brickwork.*—Stone and brick are in general much less homogeneous than say steel, and further they are often not even approximately isotropic; they have different strengths and elasticities in different directions. A masonry or brickwork structure varies in properties even more than does a single piece of the component material, hence the application of the principles previously deduced for ideal homogeneous and perfectly elastic material must be regarded as conventional to a considerable degree. For many such structures strength considerations are not the primary ones, but where considerable loads are to be carried the principles already dealt with, allowing ample margins, form the basis of calculation.

**214. Foundations.**—Provided that the earth is sufficiently firm to support a structure without piles or other form of reinforcement, the area of a foundation is found by dividing the total weight borne by the known allowable unit pressure, which will not cause a serious amount of compression. The allowable unit pressure in loose earths is about from 1 to 2 tons per square foot, rising to say 10 tons on good rock. In loose ground to prevent the earth being squeezed out laterally, the horizontal pressure intensity at the depth of the foundation must be a certain amount. If W = total weight on a foundation in tons, and A = its area in square feet, the unit pressure is $p = \frac{W}{A}$. According to Rankine's theory of earth pressure, Art. 212 (1), the least lateral pressure to prevent movement must be $p \frac{1 - \sin \phi}{1 + \sin \phi}$. Again, to support this horizontal pressure at the outside of the base, there must similarly be a vertical unit pressure $\frac{1 - \sin \phi}{1 + \sin \phi}$ times the horizontal

pressure, or $p\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2$; if this pressure is supplied by sinking the horizontal surface its intensity is $wh$, where $w$ = weight of earth in tons per cubic foot, and $h$ = depth in feet, hence the minimum depth of a foundation for stability is given by

$$wh = \frac{W}{A}\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2 \quad \text{or} \quad h = \frac{W}{Aw}\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^2 \quad . \quad . \quad (1)$$

EXAMPLE.—Find the necessary depth of a concrete foundation 6 feet wide carrying a wall which supports 8 tons per lineal foot, including its own weight. The weight of concrete foundation is $1\frac{1}{4}$ cwt. per cubic foot; weight of earth 1 cwt. per cubic foot; angle of repose 30°.

$$\text{Total load per foot run} = 8 + \frac{1{\cdot}25 \times 6 \times h}{20} = 8 + 0{\cdot}375h$$

$$\left(\frac{1-\sin 30^\circ}{1+\sin 30^\circ}\right)^2 = \frac{1}{9}, \text{ hence from (1)}$$

$$h = \frac{8 + 0{\cdot}375h}{0{\cdot}05 \times 6} \times \frac{1}{9} \quad \text{hence } h = 2{\cdot}6 \text{ feet.}$$

*Footings.*—The steps in which the area of the base of a wall or pier is increased to the full area of the base are called footings (Fig. 309). Due to the upward pressure at the foundation, there is bending action on the overhanging portion EB, which may be treated as a cantilever uniformly loaded, giving a maximum bending moment at ED. It is desirable to keep the tensile bending stress at E within safe limits, and also to see that the shear stress on ED is within safe limits, remembering Art. 72 that the maximum value may be about 1·5 times the mean value. Actually the thickness DE for a given type of masonry or brickwork structure and the projections of successive courses are determined by empirical rules, which allow ample margins in these respects.

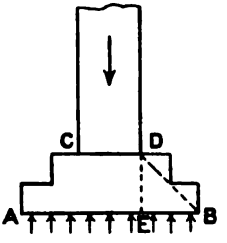


FIG. 309.

*Grillage Foundations.*—The necessity of deep excavations to secure a wide base for a foundation to carry a heavy load, such as that carried by a large stanchion, may be obviated by the use of two or more tiers of steel joists set in concrete. In poor bearing soils a single layer of steel joists may even be used in the concrete of a heavy wall foundation. The practical requirements are that all joists of a tier should be spaced sufficiently far apart to allow of concrete being well rammed between them, say 4 inches or 5 inches between flanges. The joists are kept in proper position by cast-iron separators, or by bolts passing through steel tubes about 1 inch diameter. Fig. 310 shows such a stanchion foundation having two layers of joists. At least 12 inches depth of concrete below the joists is generally allowed. The resistance of the joists to both bending and shearing must be considered in designing

such a foundation, and the joists must not be placed too far apart to provide sufficient shearing resistance in their webs. For a given stanchion base as many (usually 3 or 4) joists are placed in the first tier as can be spaced sufficiently far apart as to allow of proper ramming.

The calculation of bending moment in the joists under a stanchion base is a conventional one, for it depends upon what assumption is

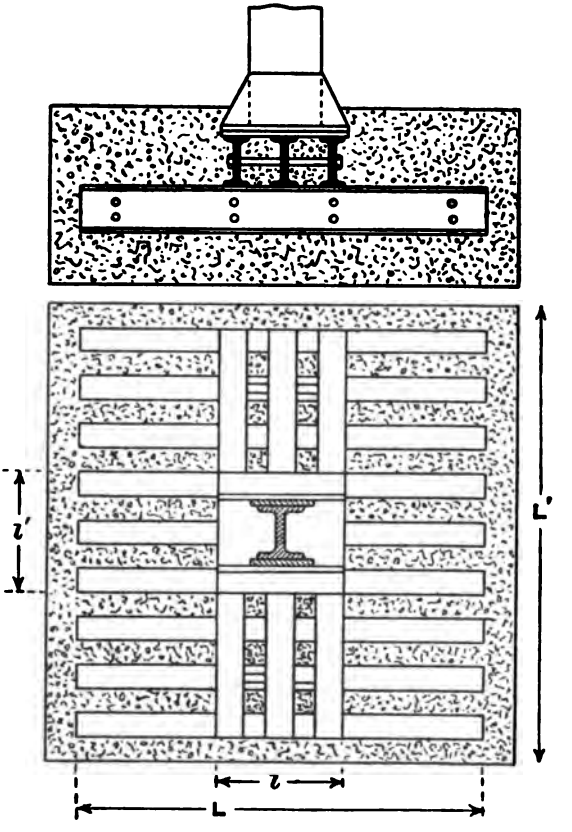



FIG. 310.—Grillage foundation.

made as to the distribution of the pressure exerted by the base on the grill.

Let Fig. 311 represent a stanchion base resting on a single layer of steel beams; if they are embedded in concrete the upward pressure may be taken as uniformly distributed, say $\frac{W}{L}$ per foot, where W = total load and L = length of beam in feet. Then if the beams are so flexible in comparison with the (shorter) stanchion base that they bend so as to

rest on the edges of the base, the diagram of bending moments is of the type shown (for different proportions) in Fig. 184, and as shown by the curve ADFEB (Fig. 311). In this case the maximum bending moment at D is as for a cantilever of length $\frac{1}{2}(L - l)$ equal to the overhang, and loaded $\frac{W}{L}$ per foot, viz.:

$$\tfrac{1}{8}\frac{W}{L}(L - l)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If, on the other hand, the stanchion base is so flexible as to bend with the beam, the downward pressure might be taken as a uniformly distributed load $\frac{W}{l}$ per foot, giving the bending moment diagram ADCEB where DEC is a parabola, and the maximum OC is

$$\tfrac{1}{2}W \cdot \tfrac{1}{4}L - \tfrac{1}{2}W \cdot \tfrac{1}{4}l = \tfrac{1}{8}W(L - l) \quad (2)$$

or $\frac{L}{L-l}$ times as great as (1). The actual distribution of pressure, and therefore the bending moment, depends upon the relative flexural stiffness of the parts, and is statically indeterminate: both values (1) and (2) are conventional. If the base were *very* flexible, the pressure might be more concentrated towards the centre of the base, giving some such ordinates as indicated by DGE (Fig. 311). In the case of the lower tier of joints the downward load is necessarily partially distributed by the upper tier.

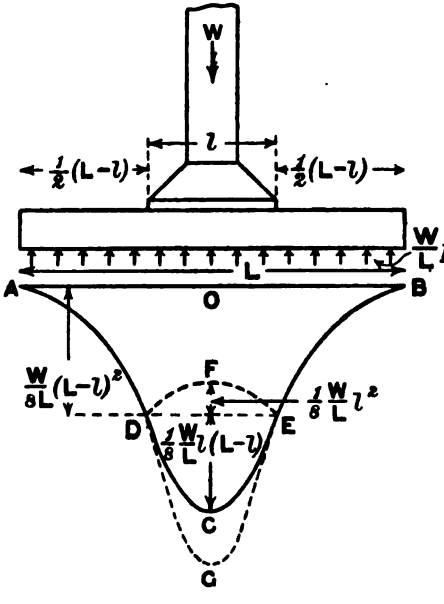


FIG. 311.

Frequently both grillage and stanchion base will be made exactly square; if not, however, appropriate values of L or L′ and $l$ or $l'$ in (1) and (2) must be adopted.

The maximum shearing force, whatever the distribution of pressure, will be practically at the edge of the stanchion base and will be

$$\tfrac{1}{2}\frac{W}{L}(L - l) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

EXAMPLE.—A stanchion designed to stand a direct load of 120 tons has a base 30″ square. Arrange a suitable grillage for the foundation

to give a pressure of not more than 2 tons per square foot. Use British Standard Beams, and limit the bending stress in the flanges to $7\frac{1}{2}$ tons per square inch and the mean shearing stress to 4 tons per square inch, taking the web area as the thickness multiplied by the depth of joist.

Area required $= \frac{120}{2} = 60$ sq. ft.; concrete, say 8′ by 8′; joists, 7′ long and 6″ from edges. For the lower tier use 8 joists spaced 12″ apart, the outside ones being 6″ from the edges.

$$L - l = 84 - 30 = 54''.$$

Then from (2), maximum bending moment $= \frac{1}{8} \times 120 \times 54 = 810$ ton-inches.

$$\text{Modulus of section per beam} = \tfrac{1}{8} \times \frac{810}{7{\cdot}5} = 13{\cdot}6 \text{ (inches)}^3.$$

Referring to Table I. Appendix, the B.S.B. 12, 8″ × 4″, with a modulus of 13·92, will suit.

The shearing force (3), is $\frac{1}{2} \times 120 \times \frac{64}{84} = 38{\cdot}6$ tons
$= 4{\cdot}82$ tons per beam.

The allowable shearing force per beam is $0{\cdot}28 \times 8 \times 4 = 9{\cdot}3$ tons, which is ample.

The clear spaces between flanges is $12'' - 4'' = 8''$.

For the upper tier, taking three joists, the modulus required is

$$\frac{810}{3 \times 7{\cdot}5} = 36 \text{ (inches)}^3.$$

And Table I. gives for B.S.B., 20, 12″ × 5″, modulus 36·66 (inches)³

$$\text{Shearing force per beam} \frac{38{\cdot}7}{3} = 12{\cdot}9 \text{ tons.}$$

Allowable shearing force per beam $0{\cdot}35 \times 12 \times 4 = 16{\cdot}8$ tons, which allows a margin.

The three flanges occupy $5 \times 3 = 15''$, leaving 15″ for two spaces or 7·5″ clear between flanges.

**215. Resistance of Retaining Walls.**—It has already been indicated that neither the earth pressure on a retaining wall nor the stress caused in masonry by given forces can be accurately calculated. The following conditions with which a retaining wall is made to comply must therefore be regarded as more or less empirical. Let P (Fig. 312) be the estimated earth pressure per foot run from one of the theories given in Art. 212; let W be the weight of masonry in the wall per foot run, and G be the position of the centre of gravity of this piece of the wall. Then combining W and P by the triangle of forces EKL drawn to scale (or by trigonometrical calculation), the resultant pressure R on the basis AB is ascertained in magnitude and direction. It is then necessary that—

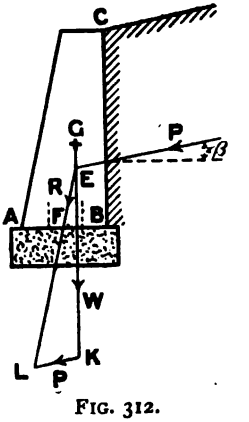


FIG. 312.

(1) R shall cut AB in a point F within the middle third to avoid a vertical component tension at the inner toe B.

(2) That the intensity of vertical compressive stress at A calculated by (2), Art. 112 (taking a normal thrust $W + P \sin \beta$), shall be within the working limits suitable to the masonry used.

(3) The inclination of R to the normal to the base of the wall shall be less than the safe angle of friction between the masonry and the foundation.

The wall should also satisfy the same conditions for all horizontal sections. The position of F may easily be found by equating the opposite moments about, say, A or B, of the forces acting on the wall including the reaction of the foundation which is equal and opposite to R.

*Practical Proportions.*—The above conditions, in conjunction with the commoner rules for earth pressure, often lead to retaining wall proportions which are unnecessarily wasteful. The late Sir Benjamin Baker stated that as a result of his experience he made the width of the bases of retaining walls for average ground equal to $\frac{1}{3}$ of the height from the footings to the top. Also that a thickness of $\frac{1}{4}$ of the height with a batter of 1 or 2 inches per foot on the face was sufficient with favourable backing and foundation, while with a solid foundation the thickness need *never* exceed $\frac{1}{2}$ of the height. He also stated that good filling gives a thrust equivalent to that of a fluid weighing 10 lbs. per cubic foot; hence, allowing a factor of safety of 2, the wall should be capable of sustaining the pressure exerted by a fluid weighing 20 lbs. per cubic foot (see also end of Art. 216).

EXAMPLE.—A trapezoidal retaining wall is 24 feet high, the base is 8 feet wide, and the top 6 feet. If the earth weighs 110 lbs. per cubic foot and its angle of repose is 50°, and if it stands level with the top of the wall, find, according to Rankine's rule, the centre of pressure on the base of the wall, and the extreme intensities of normal stress on the base assuming that the intensity varies uniformly and the masonry weighs 150 lbs. per cubic foot.

The total horizontal force on the wall per foot, by Rankine's rule, is

$$\tfrac{1}{2} \times 110 \times 24^2 \times \frac{1 - 0{\cdot}766}{1 + 0{\cdot}766} = 4200 \text{ lbs.}$$

Weight of masonry per lineal foot is $\frac{1}{2}(8 + 6)24 \times 150 = 25{,}200$ lbs.

Horizontal distance of centre of gravity from the inner side of the wall $= \{(6 \times 24 \times 3) + (\frac{1}{2} \times 24 \times 2 \times 6\frac{2}{3}) \div 168 = 3{\cdot}5236'$.

Taking moments about the inner toe of the wall—

Distance of centre of pressure $\times$ 25,200 $= 25{,}200 \times 3{\cdot}5236 + 4200 \times \frac{24}{3}$.

Distance of centre of pressure $= 3{\cdot}5236 + 1{\cdot}3333 = 4{\cdot}8569$ feet;

*i.e.* 0·8569 foot from the centre of the base, and therefore well within the middle third, which extends $\frac{8}{6}$ ft. $= 1' - 4''$ from the centre.

**216. Masonry Dams.**—The thrust on the face of a masonry dam being, unlike that on a retaining wall, due to water pressure, is calculable with considerable exactness. But there is not, and cannot well be, any exact computation of the state of internal stresses in dams. They form, however, such large, costly, and important structures that

much attention has, during recent years particularly, been paid to the estimation of such stresses.

Most existing dams have been designed so far as strength and stability are concerned with a view to fulfilling the three conditions laid down in Art. 215 for retaining walls for earth. It may be pointed out, however, that the falling of the resultant within the middle third of the horizontal section only ensures that, *assuming a uniformly varying distribution* of normal stress across the section, there is no tensile component across this section.

The dam may be regarded as a vertical cantilever of cross-sectional dimensions (breadth) comparable with its height. To apply the theory of long uniform beams to such a case is at best a rough approximation. The normal stresses across the horizontal sections will not generally be principal stresses, and there will be tangential components or shearing stress on such sections, which may involve tensile stress across some other plane. Most recent theories of stresses in dams have been supported by some experimental approximations (deduced from models) as to the distribution of horizontal shearing stress in the dam. Such data and theories as representing anything like actual conditions in a masonry dam must for the present be regarded as tentative, and are here only given by references at the end of this article.

*Water Pressure.*—The water face of a dam is usually so little curved in its vertical section that the water pressures on either part or the whole may be taken as if the face were plane. The pressures per foot run of the dam for whole or part of the depth are then easily estimated by the rules of hydrostatics applicable to immersed rectangles. Thus if Fig. 313 represents a section of a dam with water up to the sill at A, the pressure per foot run on the curved face AB may be taken as that on a rectangle of length AB and breadth one foot, the mean intensity being that at E midway between the ends A and B, *i.e.* at half the vertical depth of B. Further, this pressure is perpendicular to AB, and acts through a point F in AB distant one-third of AB from B. Strictly, the weight of water within the space between the straight and curved faces AB should be added geometrically to this pressure, but it is usually negligible, and such approximation is on the safe side. A still closer approximation would be obtained by dividing AB into a number of straight faces, and summing geometrically the partial pressures, and finding the position of the resultant by a funicular polygon.

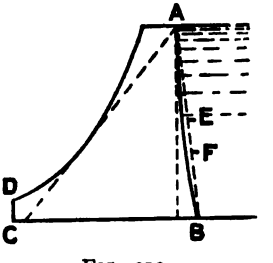


FIG. 313.

*Middle Third Rule and Lines of Thrust or Resistance.*—If the main criterion as to stability is accepted as being that the line of resultant thrust shall pass within the middle third of horizontal sections, it becomes desirable to test a vertical section of a given design by drawing lines called *lines of resistance*, or lines of thrust which give the direction and position of the resultant thrust on all horizontal sections under the

extreme conditions of no water pressure and full water pressure. Fig. 314 shows how to determine graphically a single point in the lines of resistance for the reservoir full and the reservoir empty. The section chosen is at the floor-level of the reservoir, but the construction is the same for any other horizontal section. Taking one foot perpendicular to the figure the centre of gravity G of the masonry ABCD is found by the well-known trapezoidal rule of joining NM, the middle points of AD and BC, and finding the intersection G with SQ, where AQ = CB and CS = AD. The weight of masonry W acts through G, and its line of action cuts the base CB in K, which is a point in the line of resistance for the reservoir empty. With the reservoir full the line of action of the pressure $P = \frac{1}{2}wh^2$ (where $w$ = weight of 1 cubic foot of water say 62·4 lbs.) is $\frac{1}{3}$ of $h$ above B and cuts GK in E. A triangle of forces EFH gives the direction of the resultant thrust EH, which cuts CB in L, which is a point in the line of resistance for the full reservoir. Then L and K and similarly determined points for all other horizontal sections are required to fall within the middle third of the various horizontal sections.

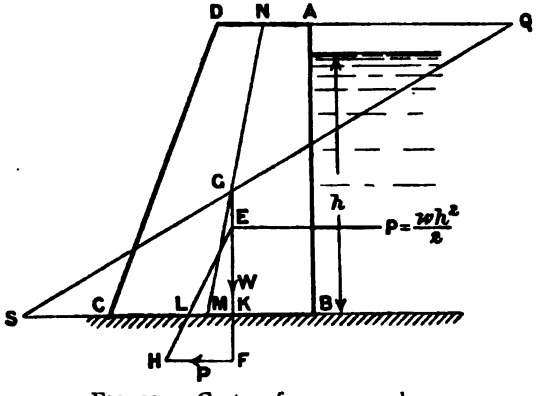



FIG. 314.—Centre of pressure on base.

*Forms of Section.*—Actual dams are rarely trapezoidal, but have the flank hollowed, and the base widened by a rake of the flank, and just so much on the face as will keep the thrust well within the middle third when the reservoir is empty.

Complete curves between dotted lines marking the middle third boundaries are shown in Fig. 315 for a typical reservoir dam section. The vector force triangles for the five horizontal sections are set out to the right-hand side of the diagram. It will be sufficient to indicate in detail the method of finding two sample points, say D and F, for the section 44′. The weight $w_4$ of the masonry 433′4′ is calculated, and its centre of gravity $g_4$ found by calculation or graphically as in Fig. 314. The weight of masonry 300′3′ ($w_1 + w_2 + w_3$) having been previously calculated and its centre of gravity $G_3$ determined, the centre of gravity $G_4$ of all the masonry above 44′ is found, as indicated for clearness just to the right of the dam section, by dividing the line $G_3g_4$ inversely as the weights $w_1 + w_2 + w_3$ to $w_4$. This is accomplished by drawing a line $G_3K$ proportional to $w_4$ and a parallel line $g_4H$ proportional to $w_1 + w_2 + w_3$, and joining HK; then the intersection of HK with $G_3g_4$

gives the centre of gravity of the masonry 432100′1′2′3′4′ at $G_4$. By dropping a perpendicular $G_4D$ on 44′ the point D in the line of pressure for the empty reservoir is obtained.

The water pressure $P_4$ on the face 01234 is taken as perpendicular to the line 04 and $\frac{2}{3}$ of its length from O, *i.e.* $\frac{140}{3}$ feet vertically below the water surface. The mean pressure is taken as that at a depth of $\frac{1}{2}$ of 70 = 35 feet below the surface, viz. 35 × 62·5 lbs. per square foot, and

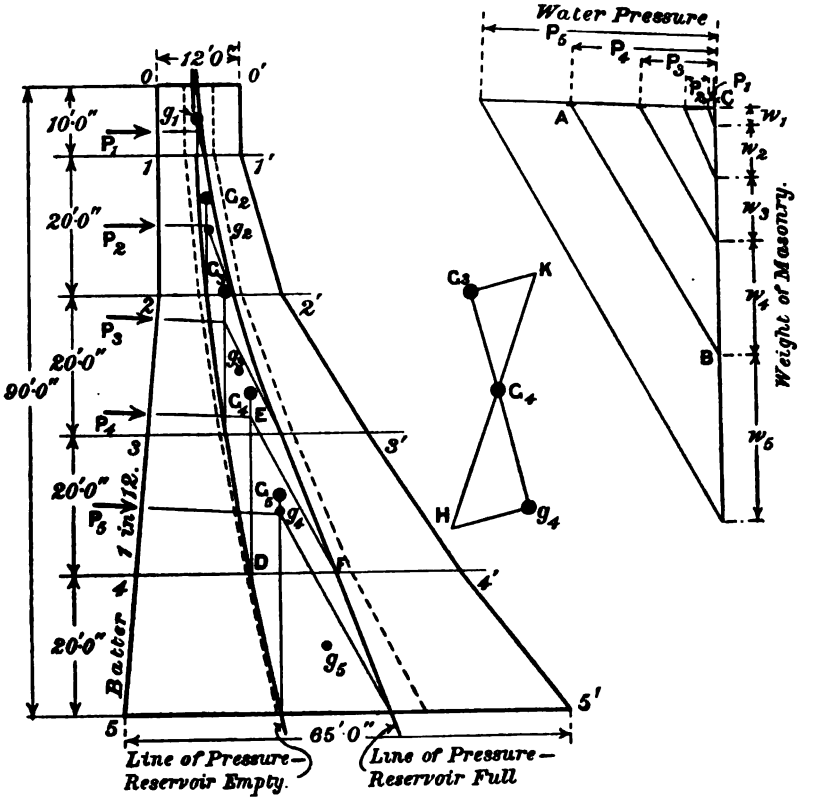



FIG. 315.—Lines of pressure.

the area per foot length of dam is equal to the number of feet in the straight line 04. The line of action of $P_4$ intersects $G_4D$ at E. The direction of the resultant of $w_1 + w_2 + w_3 + w_4$ and $P_4$ is given by the line AB in the vector diagram. Hence, drawing EF parallel to AB to meet 44′ gives the point F in the line of pressure for the reservoir full to the sill 00′. The other points are similarly obtained. The pressure vectors $P_2$, $P_3$, $P_4$, $P_5$ radiating from C are so nearly parallel that they cannot all be drawn separately. In obtaining P the effect of the weight

of the prism of water shown in profile by the triangle 024 is neglected. Its effect would be to slightly lower E and slightly increase the steepness of EF, thus bringing F very slightly closer to the middle of 44'. Alternatively $P_4$ might be found by calculating the true pressure on 01, 12, 23, 34 and obtaining their vector sum and true position of their resultant by vector and link polygons or by calculation.

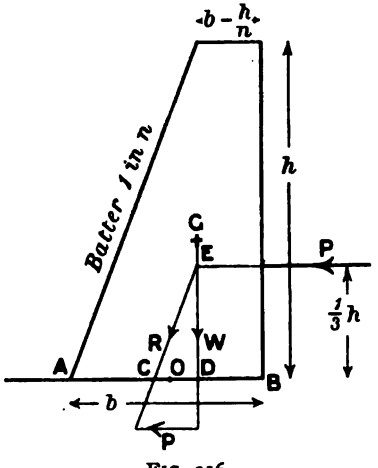


FIG. 316.

*Trapezoidal Retaining Wall or Dam.*—The limiting dimensions for a trapezoidal wall with vertical face subject to normal pressure due to level filling calculated by Rankine's rule, so that the thrust on the base just remains in the middle third, may easily be estimated by moments. Thus in Fig. 316 if the ratio of the weight of one cubic foot of masonry to $\frac{1-\sin\phi}{1+\sin\phi}$ times that of one cubic foot of filling is $s$, where $\phi$ = angle of repose (so that for water on the face $\phi = 0$, $s$ = specific gravity of the masonry), we have

$$\frac{P}{W} = \tfrac{1}{2}h^2 \div sh\left(b - \tfrac{1}{2}\frac{h}{n}\right) = \frac{h}{s\left(2b - \frac{h}{n}\right)}$$

and by moments of areas,

$$BD = \frac{3b^2 - \frac{3bh}{n} + \frac{h^2}{n^2}}{3\left(2b - \frac{h}{n}\right)}, \qquad CD = \tfrac{1}{3}h \times \frac{P}{W} = \frac{h^2}{3s\left(2b - \frac{h}{n}\right)}$$

If the intersection C falls at the limit of the middle third, equating BC + CD = $\frac{2}{3}b$, we find

$$b^2 + \frac{bh}{n} - h^2\left(\frac{1}{n^2} + \frac{1}{s}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a quadratic equation for $b$ with a given batter one in $n$ or a quadratic in $n$ for a given width of base $b$.

In the particular case of a triangular section $b = \frac{h}{n}$ this reduces to

$$b = h\sqrt{\frac{1}{s}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which also holds for a rectangular section, $n$ being infinite. If for a reservoir we put, say, $s = 2$, then $b = 0{\cdot}7h$, while for $s = 2{\cdot}5$, $b = 0{\cdot}63h$. If we take Baker's suggestion of allowing for a fluid of density 20 lbs.

per cubic foot, and take masonry at 150 lbs. per cubic foot, $s = 7{\cdot}5$, (2) gives $b = 0{\cdot}365h$, while if we put $n = 12$ in (1) we get $b = 0{\cdot}336h$.

REFERENCES TO STABILITY OF DAMS, ETC.

"On some Disregarded Points in the Stability of Masonry Dams," by L. W. Atcherly and Karl Pearson (Dulau).

"An Experimental Study of the Stresses in Masonry Dams," by Karl Pearson and A. F. C. Pollard (Dulau).

"Stresses in Masonry Dams," by Sir J. W. Ottley and A. W. Brightmore, *Proc. Inst. C.E.*, 1907–1908.

Letters and Articles in *Engineering*, vols. 79 and 80, and in the *Engineer*, 1907–1908.

**217. Masonry Arches.**—The mechanics of masonry (including brickwork) arches presents considerable difficulty, and there is no

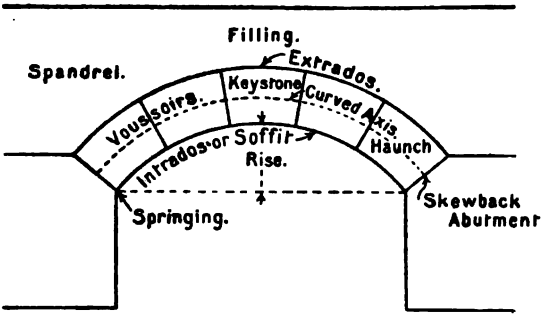



FIG. 317.

theory dealing with this point which is both simple and satisfactory. The arch ring supporting the load is made of material such as brick, or stone and mortar, which is more or less perfectly elastic. The names used in connection with this ring and the adjacent parts are shown in Fig. 317. The ring may be of uniform radial depth, or it may gradually thicken from the crown to the haunches.

Such a structure is, in any case, statically indeterminate, and the true line of thrust or linear arch for any given loading cannot be drawn with great certainty. For, in the first place, the incidence of the loading on the ring, when the force is transmitted through the spandrel, filled, it may be, with more or less loose material, is indeterminate. The pressure of granular material will not be wholly vertical, but will have a horizontal component dependent on the angle of repose. It is usual to take the loading as vertical, any error resulting being on the safe side. In some cases open spandrels are used, the load from the roadway being transmitted by vertical masonry columns connected under the roadway by short arches; in such a case the loading of the arch ring is fairly definitely vertical.

Then of all the possible reactions at the skewbacks which would satisfy the statical conditions of equilibrium, the correct values will depend upon the relative elasticities of the ring and the abutments (including heavy semi-rigid backing over the haunches and piers). It is interesting to record that in Germany attempts have been made to localise the line of thrust at three sections in the arch by the insertion of blocks of lead near the curved axis between voussoirs at those sections, thus forming quasi hinges. Masonry arches have also been constructed with actual metal pin hinges.

If we treat the arch ring as an elastic rib fixed at both ends, we may apply to it the theory of Art. 210. Experiments made by the Austrian Society of Engineers and Architects showed that masonry arches behaved very nearly as elastic arches with fixed ends.* This is, of course, equivalent to taking the rigidity of the abutments as infinite in comparison with the flexibility of the arch ring.

*Winkler's Criterion of Stability.*—Neglecting the strain from normal thrust and considering the resilience due to bending moment, (6) Art. 108, the principle of minimum resilience, Arts. 158 and 160, indicates that the average square of the bending moments will be as small as possible. And as for vertical loads the bending moment is everywhere proportional to the vertical distance of the arch axis from the line of thrust (Art. 204) the average square of the vertical distance must be as small as possible, *i.e.* the correct horizontal thrust will be such as to give minimum deviation of the line of thrust from the axis as measured by the square of the vertical deviations. If the corresponding funicular polygon or linear arch is wholly within the middle third of the arch ring, and if the maximum compressive stress is within the safe allowance for the material, the arch may be considered stable. If, therefore, any funicular polygon can be drawn for the particular system of loads such as to lie wholly within the middle third of the arch ring, the ring is stable for that system of loads. To ensure stability under all systems, the polygons for different positions of the movable load would have to be drawn, or their effect investigated. This is somewhat beyond the scope of this brief treatment of the subject, but it is usually sufficient to apply the criterion to the arch under (1) Dead load only, (2) Full load, (3) Dead load, and full movable load on half of the span only.

In an arch the dead load is a considerable proportion of the whole load, and minimises the deviation of the linear arch from the curved axis, and if it is sufficiently great will keep it within the required limits, provided the axis follows the linear arch for the dead load. In designing an arch, the ring may be made to an empirical formula, the curved axis following, say, the linear arch for the dead load, and then its stability tested by the above criterion. Or the depth of the ring, say, at the crown, and the intrados curve may be assigned, and then the lines of thrust may be drawn in by the elastic method of Art. 210 (estimating roughly the weight of ring) and then the

* For a brief account of these experiments, see Howe's "Treatise on Arches," which, with "Symmetrical Masonry Arches," by the same writer, contains much valuable information.

extrados drawn so as to form a ring of variable radial depth, the middle third of which lies entirely outside the extreme limits of the line of thrust.

The elastic method offers the most direct method of design, and gives the most probable line of thrust, according to Winkler's criterion, for in Art. 210 it was shown that (neglecting strain due to thrust) the solution given, follows from the principle of minimum resilience. If, however, an entirely empirical design is made the criterion may be applied by trial, either graphically, by drawing various trial lines of thrust, or algebraically by calculating the moments, and hence the deviation of the linear arch from the curved axis. In either case three conditions have to be assumed for any trial line of thrust. These three conditions may be three points in the linear arch (say at the crown and the abutments) or two points and one direction, or any three which make the funicular polygon determinate (see Arts. 46, 47, 48, and 51); these three assumptions correspond to the three quantities actually computed in Art. 210.

The application of the criterion graphically by trial is much facilitated by a device due to Prof. Fuller, which is illustrated in Fig. 318, which represents an arch ring, the boundaries of the middle third being shown dotted. The funicular polygon ADEKB is drawn with any pole distance, $o_1 h$ corresponding to any horizontal thrust, for the loads $W_1$, $W_2$, $W_3$, $W_4$, etc., on a horizontal base AB (Art. 51) the points A and B being the intersection of the curved axis with the skewbacks. Any length FG on the line AB is then chosen, and the highest point K of the funicular polygon is joined to F and G. Then FKG represents the funicular polygon "straightened out." The next step is to correspondingly modify the region between the middle third boundaries. This is accomplished by projecting horizontally the vertices such as D and E of the funicular polygon on to FK and KG, giving such points as D' and E'. Vertical lines through D and E intersect the upper boundary of the middle third region at $d$ and $e$; horizontal lines through $d$ and $e$ intersect vertical lines through D' and E' in $d'$ and $e'$, giving points in the modified or derived upper boundary of the middle third region; other points are similarly obtained, and both upper and lower boundaries are drawn in through the points $j$, $d'$, $e'$, $k$, etc. If now two intersecting straight lines, $mpn$, can be drawn to meet in $p$ on the vertical through K and $k$, and lie wholly within the modified middle third region, corresponding funicular polygons can be drawn within the original middle third region. Such a funicular polygon may be drawn by projecting horizontally the intersections of the lines $mp$ and $pn$, with the verticals through D' and E', etc., on to the verticals through D and E, etc., or by taking a pole distance equal to $o_1 h \times \frac{Kq}{\text{height of } p \text{ above } m}$ and starting the polygon through $p$, or through the horizontal projection of $m$, say, on the vertical through A. If several polygons are possible, a good approximation to the most probable one could be obtained by drawing in the centre line of the modified middle third region, and drawing by inspection a pair

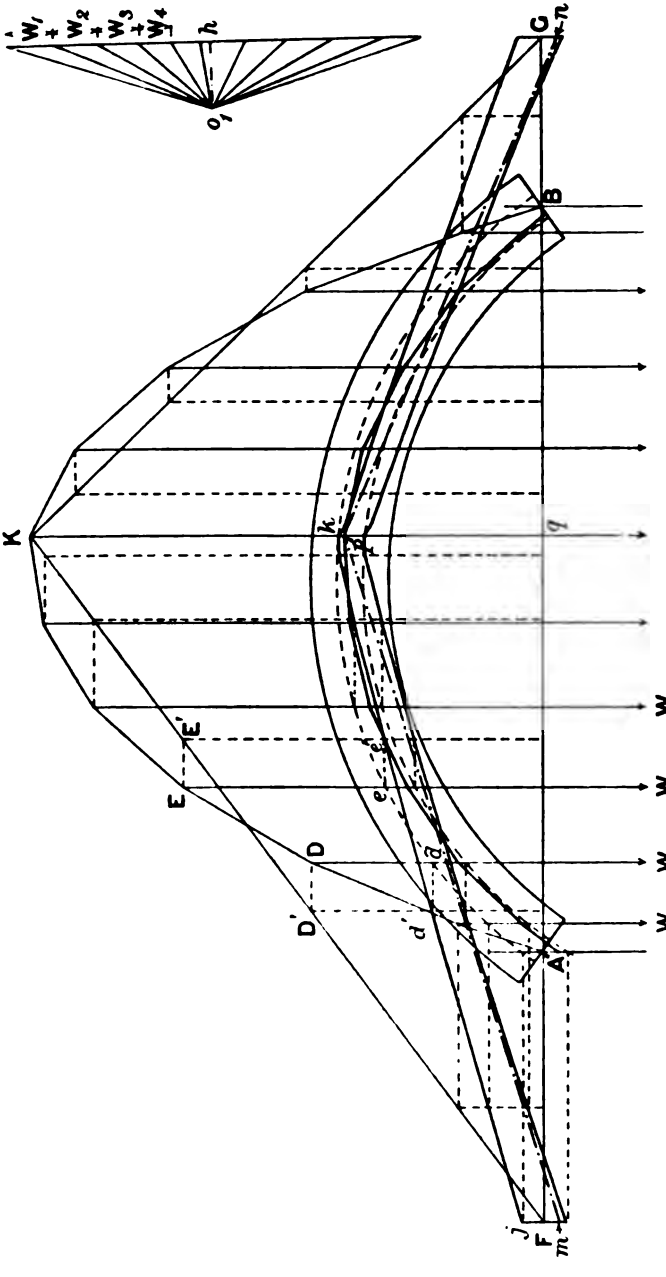



FIG. 318.

of intersecting straight lines, such as *mp* and *pn*, deviating as little as possible from the centre line, which is the axis, as modified.

*Elastic Method.*—The application of the elastic method to the determination of the line of thrust in a masonry arch may best be explained by an example. A segmental or circular arch is chosen and the approximate methods for the summations given in Art. 210 are used, although, after dividing the continuous load into several concentrated loads the exact formulæ given in Example 2, Art. 210, might have been employed. The purpose is, however, to illustrate the method, and with that object the data have been simplified as far as possible.

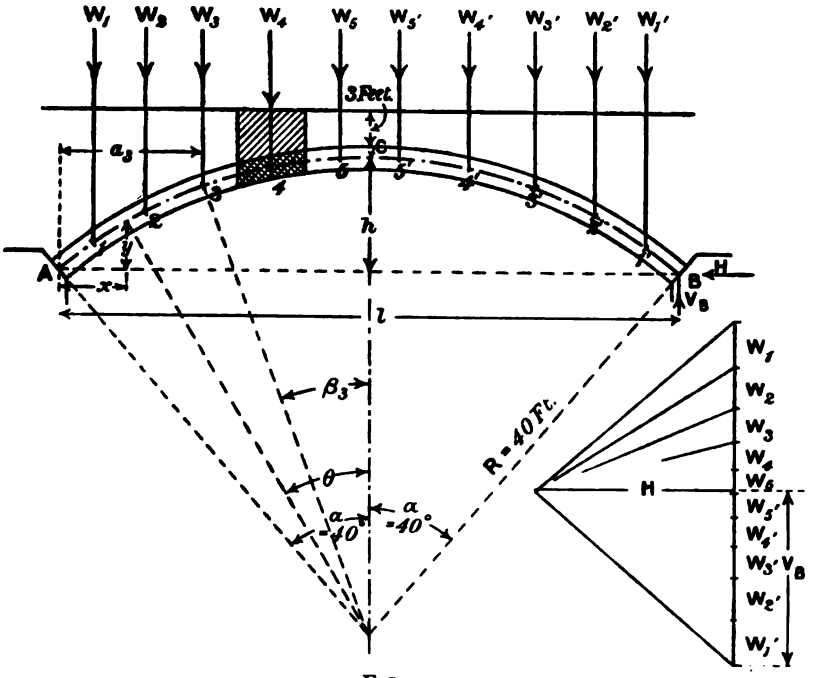



FIG. 319.

*Data.*—Radius of centre line of symmetrical arch ring 40 feet. Angle at centre of arc 80° ($\alpha = 40°$). Uniform radial depth of ring 2 feet. Filling to stand level 3 feet above the top of the ring at the crown. Average weight of filling to include roadway, 120 lbs. per cubic foot; weight of masonry in ring 160 lbs. per cubic foot. Moving load 150 lbs. per foot run per foot of width of roadway.

The arch is considered per foot width of roadway throughout. The arc is divided into 10 parts of 8° each, and both dead and live load are taken as that vertically above the 8° length of the curved axis, and acting vertically through points 1, 2, 3, 4, 5, 5′, 4′, 3′, 2′, 1′, in the centres of the 8° arcs. Thus at point 4 the dead load is that of the

spandrel filling and the arch ring shown cross hatched in Fig. 319. The estimation of the dead and moving loads assigned to these points is simple mensuration and is not shown in detail, any reasonable approximation being satisfactory. The three unknown quantities H, $V_B$ and $M_C$, as given in (14), (15), and (17), Art. 210, are first calculated for unit loads at points 2, 3, 4, and 5, the values for point 1 being zero according to the approximation. The following values are first tabulated (Table A), the notation being given in Fig. 319. The number of significant figures used may be in excess of that warranted by the assumptions, but will assist the reader in tracing the work clearly.

Half span of centre line $= 40 \sin 40^\circ = 25{\cdot}7116$ feet.
Rise of centre line $40\,(1 - \cos 40^\circ) = 9{\cdot}3582$ feet.

From Table A the summations required for the expressions (14), (15), and (17), Art. 210, are taken. Some of these are independent of the load position and magnitude. For example, the denominator for (14) is ($n$ being 5)

$$n\Sigma(y^2) - \{\Sigma(y)\}^2 = 5 \times 230{\cdot}82 - (31{\cdot}087)^2 = 187{\cdot}7$$

The denominator for (15) is

$$\Sigma\{(\tfrac{1}{2}l - x)^2\} = 1170.$$

And for (17), the coefficient

$$\left\{h - \frac{1}{n}\Sigma(y)\right\} = 9{\cdot}35822 - \frac{31{\cdot}0868}{5} = 3{\cdot}14086 \text{ feet.}$$

The values of H, $V_B$ and $M_C$ for unit load at the various points are next calculated. Sample calculations are as follows:—

For unit load at 3 or 3′, from table A, columns 4 and 8

$$\Sigma(y) \times \Sigma(a - x) = 31{\cdot}0868 \times 14{\cdot}9284 = 260{\cdot}4.$$

hence from (14), Art. 210

$$H = \tfrac{1}{2} \times \frac{260{\cdot}4}{187{\cdot}7} = 0{\cdot}6935.$$

For unit load at point 3, from (15), Art. 210, taking the value from Table A, columns 16 and 6, $V_C = V_B = \tfrac{1}{2} \times \frac{326{\cdot}8}{1170} = 0{\cdot}1398$, and for load at 3′ $V_C$ would be $-0{\cdot}1398$, and $V_B$ would be $1 - 0{\cdot}1398$.

Also for unit load at 3 or 3′, using the above values and column (8), Table A in (17), Art. 210,

$$M_C = 0{\cdot}6935 \times 3{\cdot}14086 - \tfrac{1}{10} \times 14{\cdot}9284 = +0{\cdot}6854.$$

The other quantities similarly calculated are entered in columns 2, 9, and 5 of Table C. They may be checked by the exact formulæ given in Example 2, Art. 210, and show the nearness of the method of approximate summation.

We propose to investigate one type of loading only, viz. movable load covering the left-hand half of the arch only. Then by simple mensuration we find approximately the loads in Table B.

## TABLE A.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Point. | $\theta^\circ$ | $\frac{1}{2}l - x =$ 40 sin $\theta$ ft.). | $y = 40$ (cos $\theta$ − cos 40). | $y^2$ | $(\frac{1}{2}l - x)^2$. | (By differences of column 3.) $(a - x) = 40$ (sin $\theta$ − sin $\beta$) for load at | | | | $y(a - x)$ for load at | | | | $(a - x)(\frac{1}{2}l - x)$ for load at | | | |
| | | | | | | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 |
| 1 | 36 | 23·5112 | 1·7192 | 2·9556 | 552·78 | 4·7324 | 9·8304 | 15·1948 | 20·7208 | 8·1359 | 16·9004 | 26·1229 | 35·6232 | 111·3 | 231·1 | 357·2 | 487·2 |
| 2 | 28 | 18·7788 | 4·6764 | 21·8687 | 352·64 | — | 5·0980 | 10·4624 | 15·9884 | — | 23·8403 | 48·9264 | 74·7682 | — | 95·7 | 196·2 | 300·2 |
| 3 | 20 | 13·6808 | 6·9460 | 48·2469 | 187·16 | | — | 5·3644 | 10·8904 | | — | 37·2611 | 75·6447 | | — | 73·4 | 149·0 |
| 4 | 12 | 8·3164 | 8·4844 | 71·9850 | 69·16 | | | — | 5·5260 | | | — | 46·8848 | | | — | 46·0 |
| 5 | 4 | 2·7904 | 9·2608 | 85·7624 | 7·79 | | | | — | | | | — | | | | — |
| Sums | . | | 31·0868 | 230·82 | 1169·53 | 4·7324 | 14·9284 | 31·0216 | 53·1256 | 8·1359 | 40·7407 | 112·3104 | 232·9209 | 111·3 | 326·8 | 627·1 | 982·4 |

TABLE B.

| Point. | Weight of filling (lbs.). | Weight of arch ring (lbs.). | Total dead load (lbs.). | Total live load. |
|---|---|---|---|---|
| 1 | 5640 | 1790 | 7430 | 680 |
| 2 | 4460 | 1790 | 6250 | 740 |
| 3 | 3360 | 1790 | 5150 | 790 |
| 4 | 2520 | 1790 | 4310 | 820 |
| 5 | 2070 | 1790 | 3860 | 840 |

TABLE C.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| Points. | H for unit load at either point. | H for dead load at either point. | H for live load at either point. | $M_C$ for unit load at either point. | $M_C$ for dead load at either point. | $M_C$ for live load at either point. | $V_B$ for dead load on *both* points. | $V_B = V_C$ for unit load on left point only. | $V_B = V_C$ for live load on left point only. |
| 1, 1′ | 0 | 0 | 0 | 0 | 0 | 0 | 7430 | 0 | 0 |
| 2, 2′ | 0·2835 | 1772 | 210 | +0·4172 | 2608 | 309 | 6250 | 0·0476 | 35 |
| 3, 3′ | 0·6935 | 3572 | 546 | +0·6854 | 3530 | 539 | 5150 | 0·1398 | 110 |
| 4, 4′ | 1·0730 | 4625 | 880 | +0·2680 | 1155 | 220 | 4310 | 0·2681 | 220 |
| 5, 5′ | 1·2972 | 5007 | 1085 | −1·2382 | −4780 | −1036 | 3860 | 0·4199 | 351 |
| Totals | — | 14976 | 2721 | — | 2513 | +32 | 27000 | — | 716 |

Total H for arch, $2 \times 14976 + 2721 = 32673$ lbs.
Total $M_C$ for arch, $2 \times 2513 + 32 =$ say 5060 lb.-feet.
Total $V_B$ for arch $27000 + 716 =$ say 27720 lbs.
$V_C = 716$, say 720 lbs.

The loads in Table B are used with the unit load coefficients to find the contribution of each load to the totals as shown in Table C. Thus at point 3 we have for H $0{\cdot}6935 \times 5150 = 3572$, and $0{\cdot}6935 \times 790 = 546$.

The three unknown quantities H, $M_C$, and $V_B$ or $V_C$, being now known, the funicular polygon which is the linear arch may be drawn by setting off the load line, and $V_B$ and H, as shown to the right of Fig. 319, and starting at a point $5060 \div 32673 = 0{\cdot}155$ foot vertically below C. On account of the difficulty of obtaining an accurate result in the thin ring graphically, the work may be completed arithmetically by calculating the bending moments. If V is the vertical shearing force, that is the external downward force to the left of any section (or upward to the right), $\delta V$ the increase of V on passing any load is equal to that load, hence after entering column 3, Table D, column 4 is easily obtained by addition or subtraction, since we know starting points $V_C$ or $V_B$.

TABLE D.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Point. | Angle. | Load = $\delta V$ lbs. | V to right of point (lbs.). | $\delta x$ to right of point (ft.). | $\delta y$ (ft.) to next point. | $\delta M$ to next point (lb.-ft.). | M (lb.-ft.). | P to right of points (lbs.). | Eccentricity towards intrados = $\frac{M}{P}$ (feet). | Maximum unit thrust $f_o$ lb. per sq. foot. | Maximum unit tension $f_t$ lbs. per sq. foot. |
| A | 40 | — | −30150 | 2·2004 | 1·7192 | −10170 | +15520 | 44400 | 0·350 | 45500 | 1100 |
| 1 | 36 | 8110 | −22040 | 4·7324 | 2·9572 | −7690 | +5350 | 39400 | 0·136 | | — |
| 2 | 28 | 6990 | −15050 | 5·0980 | 2·2696 | −2560 | −2340 | 35900 | −0·065 | | — |
| 3 | 20 | 5940 | −9110 | 5·3644 | 1·5384 | +1400 | −4900 | 33800 | −0·145 | | — |
| 4 | 12 | 5130 | −3980 | 5·5260 | 0·7764 | +3370 | −3500 | | | | — |
| 5 | 4 | 4700 | +720 | 2·7904 | 0·0974 | +5190 | −130 | | | | — |
| C | 0 | 0 | +720 | 2·7904 | −0·0974 | −1170 | +5060 | 32670 | +0·155 | | — |
| 5′ | −4 | 3860 | +4580 | 5·5260 | −0·7764 | −60 | +3890 | | | | — |
| 4′ | −12 | 4310 | +8890 | 5·3644 | −1·5384 | −2580 | +3830 | | | | — |
| 3′ | −20 | 5150 | +14040 | 5·0980 | −2·2696 | −2590 | +1250 | | | | — |
| 2′ | −28 | 6250 | +20290 | 4·7324 | −2·9572 | −600 | −1340 | 38300 | | | — |
| 1′ | −36 | 7430 | 27720 | 2·2004 | −1·7192 | +4820 | −1940 | 42800 | −0·051 | | — |
| B | −40 | — | — | — | — | | +2880 | — | 0·067 | | — |

Also the increase $\delta M$ in bending moment between two consecutive loads is easily found by taking moments about the second of the two to be

$$\delta M = H . \delta y + V \delta x,$$

which is also evident by differentiating (7), Art. 210, thus

$$\frac{dM}{dx} = \frac{dm}{dx} + V_B + H\frac{dy}{dx}$$

and since

$$\frac{dm}{dx} + V_B = V, \quad \delta M = V . \delta x + H \delta y.$$

Columns 4 and 5, Table D, follow by subtraction from columns 3 and 4 of Table A, and from the known values of H and V, column 7 Table D, is easily calculated. Thus from point 3 to point 4

$$\delta M = -9110 \times 5{\cdot}3644 + 32673 \times 1{\cdot}5384 = 1400.$$

Then knowing $M_C$ as a starting point, column 8, Table D, is completed by additions and subtractions from column 7. Column 9, Table D, gives the normal thrusts on the ring sections, viz.

$$P = H \cos\theta - V \sin\theta.$$

The radial eccentricity of thrust at any cross-section is $M \div P$. Line 1, column 10, shows that the eccentricity at the abutment A is 0·350 foot. The limits of the middle third of the ring have an eccentricity of $\frac{1}{6}$ of 2 ft. = 0·333 ft., so the linear arch is just outside those limits by 0·017 ft. or less than $\frac{1}{4}$ of an inch.

The values of the extreme stresses are calculated on the assumptions and by the formulæ of Art. 112, *e.g.*

$$f_c = \frac{M}{\frac{1}{6} \times 1 \times 2^2} + \frac{P}{2 \times 1} = \frac{3}{2}M + \frac{P}{2} \text{ lb. per sq. foot,}$$

and at point A

$$f_c = 1{\cdot}5 \times 15{,}520 + 0{\cdot}5 \times 44{,}400 = 45{,}500 \text{ lb. per sq. foot.}$$
$$f_t = 1{\cdot}5 \times 15{,}520 - 0{\cdot}5 \times 44{,}400 = 1100 \text{ lb. per sq. foot.}$$

It will be observed from the changes in sign of M in column 8, Table D, that the line of thrust crosses the axis of the arch ring four times.

To complete the investigation it would be necessary to find the live load positions to give maximum bending moment at every section. For example, if point A be chosen, for each unit load to the left of C, taking a section at C and moments about A, $M_A = M_C - Hh - \frac{1}{2}V_C l + 1 \times a$, where $M_C$, H and $V_C$ may be taken from columns 5, 2 and 9 respectively, of Table C. For each unit load to the right of C the corresponding values from the left are modified by the omission of the term $Wa = a$ (since $W = 1$) and reversal of the sign of $V_C$. When all the coefficients for unit loads are obtained, live load may be taken at points which give like signs for $M_A$, and $M_A$ then calculated for the extreme variations in conditions. Influence lines for M, V, H, P, etc., may also be plotted approximately by setting up ordinates at the load points 1, 2, 3, 4, 5, etc.

## EXAMPLES XIX.

1. A concrete foundation has to be provided for a wall to carry 6 tons per linear foot at 1·5 tons per square foot bearing pressure. Estimate the necessary depth of foundation according to Rankine's rule if the angle of repose of the earth is 35°, and its weight 110 lbs. per cubic foot.

2. Using British Standard beams, find suitable dimensions for a two-tier grillage foundation to carry a stanchion designed to carry 100 tons, the base being 2 feet square. The earth is to be limited to a pressure of 1·75 tons per square foot and the tensile and shear unit stresses in the joists to 7·5 and 4 tons per square inch respectively.

3. A retaining wall, trapezoidal in cross-section, 24 feet high and 8 feet wide at the base, has a vertical face and a batter of 1 in 12 at the back. Find according to Rankine's rule how far from the centre of the base the resultant thrust passes (*a*) for horizontal filling to the level of the top of the wall, (*b*) for the maximum surcharge of earth if the angle of repose is 45°, weights of earth filling 120 lbs., masonry 150 lbs. per cubic foot. Assuming uniformly varying intensity of stress in each case find the extreme values of the normal unit stress across the base of the wall.

4. For the same height, batter and constants as in Problem No. 3, find the minimum width of base to prevent the resultant passing outside the middle third of the base.

5. Assuming uniformly varying normal stress across the base, find the limit of height of a triangular masonry dam with water up to the vertical face in order that the vertical compressive stress across the base shall not exceed 6 tons per square foot if the masonry weighs 150 lbs. per cubic foot.

6. Assuming uniform variation in the intensity of vertical stress across the base of the dam in Fig. 315, find the extreme unit stresses at the upstream and downstream toes 5 and 5′, given that the widths at the levels 0, 1, 2, 3, 4, 5, are 12′, 12′, 18′, 32′, 47′, and 65′ respectively, and the masonry weighs 160 lbs. per cubic foot.

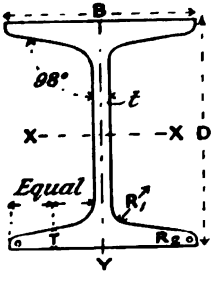
# APPENDIX.

## DIMENSIONS AND PROPERTIES OF BRITISH STANDARD SECTIONS.

THESE Tables are based on *Report No.* 6 of the Engineering Standards Committee and are published by permission of the Committee. Some of the Tables are slightly modified in form, and some contain the properties of sections of thicknesses not given in the above *Report.* All the tables have been taken by permission of Messrs. Dorman, Long & Co., Ltd., from their "Pocket Companion."

# TABLE I

## DIMENSIONS AND PROPERTIES OF




| Reference mark | Size D × B inches | Weight per foot lbs. | Diagram | | | |
|---|---|---|---|---|---|---|
| | | | Web t | Flange T | Radius R1 | Radius R2 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| B S B 30 | 24 × 7½ | 100 | 0·6 | 1·07 | 0·7 | 0·35 |
| ,, 29 | 20 × 7½ | 89 | 0·6 | 1·01 | 0·7 | 0·35 |
| ,, 28 | 18 × 7 | 75 | 0·55 | 0·928 | 0·65 | 0·325 |
| ,, 27 | 16 × 6 | 62 | 0·55 | 0·847 | 0·65 | 0·325 |
| ,, 26 | 15 × 6 | 59 | 0·5 | 0·88 | 0·6 | 0·3 |
| ,, 25 | 15 × 5 | 42 | 0·42 | 0·647 | 0·52 | 0·26 |
| ,, 24 | 14 × 6 | 57 | 0·5 | 0·873 | 0·6 | 0·3 |
| ,, 23 | 14 × 6 | 46 | 0·4 | 0·698 | 0·5 | 0·25 |
| ,, 22 | 12 × 6 | 54 | 0·5 | 0·883 | 0·6 | 0·3 |
| ,, 21 | 12 × 6 | 44 | 0·4 | 0·717 | 0·5 | 0·25 |
| ,, 20 | 12 × 5 | 32 | 0·35 | 0·55 | 0·45 | 0·225 |
| ,, 19 | 10 × 8 | 70 | 0·6 | 0·97 | 0·7 | 0·35 |
| ,, 18 | 10 × 6 | 42 | 0·4 | 0·736 | 0·5 | 0·25 |
| ,, 17 | 10 × 5 | 30 | 0·36 | 0·552 | 0·46 | 0·23 |
| ,, 16 | 9 × 7 | 58 | 0·55 | 0·924 | 0·65 | 0·325 |
| ,, 15 | 9 × 4 | 21 | 0·3 | 0·46 | 0·4 | 0·2 |
| ,, 14 | 8 × 6 | 35 | 0·44 | 0·597 | 0·54 | 0·27 |
| ,, 13 | 8 × 5 | 28 | 0·35 | 0·575 | 0·45 | 0·225 |
| ,, 12 | 8 × 4 | 18 | 0·28 | 0·402 | 0·38 | 0·19 |
| ,, 11 | 7 × 4 | 16 | 0·25 | 0·387 | 0·35 | 0·175 |
| ,, 10 | 6 × 5 | 25 | 0·41 | 0·52 | 0·51 | 0·255 |
| ,, 9 | 6 × 4½ | 20 | 0·37 | 0·431 | 0·47 | 0·235 |
| ,, 8 | 6 × 3 | 12 | 0·26 | 0·348 | 0·36 | 0·18 |
| ,, 7 | 5 × 4½ | 18 | 0·29 | 0·448 | 0·39 | 0·195 |
| ,, 6 | 5 × 3 | 11 | 0·22 | 0·376 | 0·32 | 0·16 |
| ,, 5 | 4⅜ × 1⅝ | 6·5 | 0·18 | 0·325 | 0·28 | 0·14 |
| ,, 4 | 4 × 3 | 9·5 | 0·22 | 0·336 | 0·32 | 0·16 |
| ,, 3 | 4 × 1⅝ | 5 | 0·17 | 0·24 | 0·27 | 0·135 |
| ,, 2 | 3 × 3 | 8·5 | 0·2 | 0·332 | 0·3 | 0·15 |
| ,, 1 | 3 × 1½ | 4 | 0·16 | 0·248 | 0·26 | 0·13 |

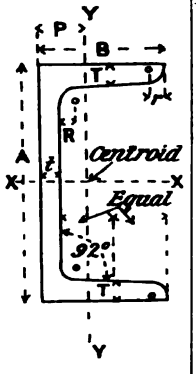
TABLE I.—*continued.*

BRITISH STANDARD I BEAMS.

| Area square inches | Moments of inertia | | Radii of gyration inches | | Section modulus | Reference mark |
|---|---|---|---|---|---|---|
| | About X - X | About Y - Y | About X - X | About Y - Y | About X - X | |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 29·4 | 2654 | 66·92 | 9·5 | 1·5 | 221·1 | B S B 30 |
| 26·17 | 1670 | 62·63 | 7·99 | 1·54 | 167·0 | ,, 29 |
| 22·06 | 1149 | 47·04 | 7·21 | 1·46 | 127·6 | ,, 28 |
| 18·23 | 725·7 | 27·08 | 6·31 | 1·21 | 90·71 | ,, 27 |
| 17·35 | 628·9 | 28·22 | 6 02 | 1·27 | 83·85 | ,, 26 |
| 12·35 | 428 | 11·81 | 5·88 | 0·978 | 57·06 | ,, 25 |
| 16·76 | 532·9 | 27·96 | 5·63 | 1·29 | 76·12 | ,, 24 |
| 13·53 | 440·5 | 21·6 | 5·7 | 1·26 | 62·92 | ,, 23 |
| 15·88 | 375·5 | 28·3 | 4·86 | 1·33 | 62·58 | ,, 22 |
| 12·94 | 315·3 | 22·27 | 4·93 | 1·31 | 52·55 | ,, 21 |
| 9·41 | 220 | 9·753 | 4·83 | 1·01 | 36·66 | ,, 20 |
| 20·6 | 344·9 | 71·67 | 4·09 | 1·86 | 68·98 | ,, 19 |
| 12·35 | 211·5 | 22·95 | 4·13 | 1·36 | 42·3 | ,, 18 |
| 8·82 | 145·6 | 9·79 | 4·06 | 1·05 | 29·12 | ,, 17 |
| 17·06 | 229·5 | 46·3 | 3·66 | 1·64 | 51·0 | ,, 16 |
| 6·176 | 81·1 | 4·2 | 3·62 | 0·824 | 18·02 | ,, 15 |
| 10·29 | 110·5 | 17·95 | 3·27 | 1·32 | 27·62 | ,, 14 |
| 8·24 | 89·32 | 10·26 | 3·29 | 1·11 | 22·33 | ,, 13 |
| 5·294 | 55·69 | 3·578 | 3·24 | 0·822 | 13·92 | ,, 12 |
| 4·706 | 39·21 | 3·414 | 2·88 | 0·851 | 11·2 | ,, 11 |
| 7·35 | 43·61 | 9·116 | 2·43 | 1·11 | 14·53 | ,, 10 |
| 5·88 | 34·62 | 5·415 | 2·42 | 0·959 | 11·54 | ,, 9 |
| 3·53 | 20·21 | 1·339 | 2·39 | 0·616 | 6·736 | ,, 8 |
| 5·29 | 22·69 | 5·664 | 2·07 | 1·03 | 9·076 | ,, 7 |
| 3·235 | 13·61 | 1·462 | 2·05 | 0·672 | 5·444 | ,, 6 |
| 1·912 | 6·73 | 0·263 | 1·87 | 0·37 | 2·833 | ,, 5 |
| 2·794 | 7·52 | 1·281 | 1·64 | 0·677 | 3·76 | ,, 4 |
| 1·47 | 3·668 | 0·186 | 1·58 | 0·355 | 1·834 | ,, 3 |
| 2·5 | 3·787 | 1·262 | 1·23 | 0·71 | 2·524 | ,, 2 |
| 1·176 | 1·659 | 0·124 | 1·18 | 0·324 | 1·106 | ,, 1 |

## TABLE II.

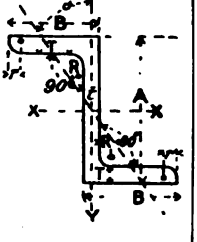
### DIMENSIONS AND PROPERTIES OF




| Reference mark | Size. A × B | Standard thicknesses | | Radii | | Weight per foot lbs. |
|---|---|---|---|---|---|---|
| | | $t$ | T | R | $r$ | |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| B S C 27 | 15×4 | 0·525 | 0·630 | 0·630 | 0·440 | 41·94 |
| ,, 26 | 12×4 | 0·525 | 0·625 | 0·625 | 0·425 | 36·47 |
| ,, 25 | 12×3½ | 0·500 | 0·600 | 0·600 | 0·425 | 32·88 |
| ,, 24 | 12×3½ | 0·375 | 0·500 | 0·500 | 0·350 | 26·10 |
| ,, 22 | 11×3½ | 0·475 | 0·575 | 0·575 | 0·400 | 29·82 |
| ,, 21 | 10×4 | 0·475 | 0·575 | 0·575 | 0·400 | 30·16 |
| ,, 20 | 10×3½ | 0·475 | 0·575 | 0·575 | 0·400 | 28·21 |
| ,, 19 | 10×3½ | 0·375 | 0·500 | 0·500 | 0·350 | 23·55 |
| ,, 17 | 9×3½ | 0·450 | 0·550 | 0·550 | 0·375 | 25·39 |
| ,, 16 | 9×3½ | 0·375 | 0·500 | 0·500 | 0·350 | 22·27 |
| ,, 15 | 9×3 | 0·375 | 0·437 | 0·437 | 0·350 | 19·37 |
| ,, 13 | 8×3½ | 0·425 | 0·525 | 0·525 | 0·375 | 22·72 |
| ,, 12 | 8×3 | 0·375 | 0·500 | 0·500 | 0·350 | 19·30 |
| ,, 10 | 7×3½ | 0·400 | 0·500 | 0·500 | 0·350 | 20·23 |
| ,, 9 | 7×3 | 0·375 | 0·475 | 0·475 | 0·325 | 17·56 |
| ,, 8 | 6×3½ | 0·375 | 0·475 | 0·475 | 0·325 | 17·9 |
| ,, 6 | 6×3 | 0·312 | 0·437 | 0·437 | 0·300 | 14·49 |

## TABLE III.

### DIMENSIONS AND PROPERTIES OF




| Reference mark | Size A × B | Standard thicknesses | | Area square inches | Weight per foot lbs. |
|---|---|---|---|---|---|
| | | $t$ | T | | |
| 1 | 2 | 3 | 4 | 5 | 6 |
| B S Z 8 | 10×3½ | 0·475 | 0·575 | 8·283 | 28·16 |
| ,, 7 | 9×3½ | 0·450 | 0·550 | 7·449 | 25·33 |
| ,, 6 | 8×3½ | 0·425 | 0·525 | 6·670 | 22·68 |
| ,, 5 | 7×3½ | 0·400 | 0·500 | 5·948 | 20·22 |
| ,, 4 | 6×3½ | 0·375 | 0·475 | 5·258 | 17·88 |
| ,, 3 | 5×3 | 0·350 | 0·450 | 4·169 | 14·17 |

## TABLE II.—*continued.*

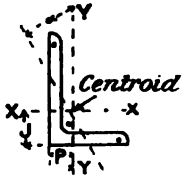
### BRITISH STANDARD CHANNELS.

| Area square inches | Dimension P | Moments of inertia | | Section moduli | | Radii of gyration inches | | Reference mark |
|---|---|---|---|---|---|---|---|---|
| | | About XX | About YY | About XX | About YY | About XX | About YY | |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 12·334 | 0·935 | 377·0 | 14·55 | 50·27 | 4·748 | 5·53 | 1·09 | B S C 27 |
| 10·727 | 1·031 | 218·2 | 13·65 | 36·36 | 4·599 | 4·51 | 1·13 | ,, 26 |
| 9·671 | 0·867 | 190·7 | 8·922 | 31·79 | 3·389 | 4·44 | 0·960 | ,, 25 |
| 7·675 | 0·860 | 158·6 | 7·572 | 26·44 | 2·868 | 4·55 | 0·993 | ,, 24 |
| 8·771 | 0·896 | 148·6 | 8·421 | 27·02 | 3·234 | 4·12 | 0·980 | ,, 22 |
| 8·871 | 1·102 | 130·7 | 12·02 | 26·14 | 4·147 | 3·84 | 1·16 | ,, 21 |
| 8·296 | 0·933 | 117·9 | 8·194 | 23·59 | 3·192 | 3·77 | 0·994 | ,, 20 |
| 6·925 | 0·933 | 102·6 | 7·187 | 20·52 | 2·800 | 3·85 | 1·02 | ,, 19 |
| 7·469 | 0·971 | 88·07 | 7·660 | 19·57 | 3·029 | 3·43 | 1·01 | ,, 17 |
| 6·550 | 0·976 | 79·90 | 6·963 | 17·76 | 2·759 | 3·49 | 1·03 | ,, 16 |
| 5·696 | 0·754 | 65·18 | 4·021 | 14·48 | 1·790 | 3·38 | 0·840 | ,, 15 |
| 6·682 | 1·011 | 63·76 | 7·067 | 15·94 | 2·839 | 3·09 | 1·03 | ,, 13 |
| 5·675 | 0·844 | 53·43 | 4·329 | 13·36 | 2·008 | 3·07 | 0·873 | ,, 12* |
| 5·950 | 1·061 | 44·55 | 6·498 | 12·73 | 2·664 | 2·74 | 1·04 | ,, 10 |
| 5·166 | 0·874 | 37·63 | 4·017 | 10·75 | 1·889 | 2·70 | 0·882 | ,, 9 |
| 5·266 | 1·119 | 29·66 | 5·907 | 9·885 | 2·481 | 2·36 | 1·06 | ,, 8 |
| 4·261 | 0·938 | 24·01 | 3·503 | 8·003 | 1·699 | 2·37 | 0·907 | ,, 6 |

## TABLE III.—*continued.*

### BRITISH STANDARD ZED BARS.

| Radii—inches | | Moments of inertia | | Section moduli | | Angle $\alpha$ degrees | Least radius of gyration inches | Reference mark |
|---|---|---|---|---|---|---|---|---|
| R | r | About XX | About YY | About XX | About YY | | | |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 0·500 | 0·350 | 117·865 | 12·876 | 23·573 | 3·947 | 14 | 0·839 | B S Z 8 |
| 0·475 | 0·350 | 87·889 | 12·418 | 19·531 | 3·792 | 16½ | 0·843 | ,, 7 |
| 0·450 | 0·325 | 63·729 | 12·024 | 15·932 | 3·657 | 19½ | 0·845 | ,, 6 |
| 0·450 | 0·300 | 44·609 | 11·618 | 12·745 | 3·521 | 23 | 0·840 | ,, 5 |
| 0·425 | 0·300 | 29·660 | 11·134 | 9·887 | 3·361 | 28½ | 0·821 | ,, 4 |
| 0·375 | 0·250 | 16·145 | 6·578 | 6·458 | 2·328 | 29½ | 0·698 | ,, 3 |




# TABLE IV.

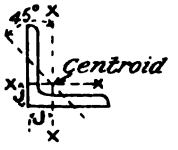
## DIMENSIONS AND PROPERTIES OF BRITISH STANDARD UNEQUAL ANGLES.

| Reference mark | Size and thickness | Area square inches | Weight per foot lbs. | Radii | | Dimensions | | Moments of inertia | | Section moduli | | Angle $\alpha$ degrees | Least radius of gyration |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Root | Toe | J | P | About XX | About YY | About XX | About YY | | |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| BSUA | | | | | | | | | | | | | |
| 25 | 7 × 3½ × ½ | 5·0 | 17·00 | 0·425 | 0·300 | 2·50 | 0·764 | 25·1 | 4·28 | 5·58 | 1·56 | 14½ | 0·74 |
| 25 | ,, ,, ⅝ | 6·172 | 20·98 | 0·425 | 0·300 | 2·55 | 0·814 | 30·55 | 5·15 | 6·86 | 1·92 | 14½ | 0·74 |
| 25 | ,, ,, ¾ | 7·313 | 24·86 | 0·425 | 0·300 | 2·60 | 0·862 | 35·68 | 5·95 | 8·11 | 2·26 | 14 | 0·73 |
| 24 | 6½ × 4½ × ½ | 5·248 | 17·84 | 0·45 | 0·325 | 2·08 | 1·09 | 22·2 | 8·75 | 5·02 | 2·57 | 25 | 0·97 |
| 24 | ,, ,, ⅝ | 6·482 | 22·04 | 0·45 | 0·325 | 2·13 | 1·14 | 27·09 | 10·60 | 6·20 | 3·15 | 25 | 0·96 |
| 24 | ,, ,, ¾ | 7·686 | 26·13 | 0·45 | 0·325 | 2·18 | 1·19 | 31·66 | 12·32 | 7·33 | 3·72 | 25 | 0·96 |
| 22 | 6½ × 3½ × ½ | 3·610 | 12·27 | 0·425 | 0·300 | 2·22 | 0·741 | 15·7 | 3·27 | 3·67 | 1·18 | 16½ | 0·75 |
| 22 | ,, ,, ⅝ | 4·750 | 16·15 | 0·425 | 0·300 | 2·28 | 0·792 | 20·4 | 4·20 | 4·83 | 1·55 | 16½ | 0·75 |
| 22 | ,, ,, ¾ | 5·860 | 19·92 | 0·425 | 0·300 | 2·33 | 0·841 | 24·83 | 5.06 | 5·95 | 1·90 | 16 | 0·74 |
| 21 | 6 × 4 × ½ | 3·610 | 12·27 | 0·425 | 0·300 | 1·91 | 0·923 | 13·2 | 4·73 | 3·23 | 1·54 | 23½ | 0·87 |
| 21 | ,, ,, ⅝ | 4·750 | 16·15 | 0·425 | 0·300 | 1·96 | 0·974 | 17·1 | 6·10 | 4·23 | 2·02 | 23½ | 0·86 |
| 21 | ,, ,, ¾ | 5·860 | 19·92 | 0·425 | 0·300 | 2·02 | 1·02 | 20·8 | 7·36 | 5·23 | 2·47 | 23½ | 0·86 |
| 20 | 6 × 3½ × ½ | 3·424 | 11·64 | 0·40 | 0·275 | 2·01 | 0·773 | 12·6 | 3·22 | 3·16 | 1·18 | 19 | 0·76 |
| 20 | ,, ,, ⅝ | 4·502 | 15·31 | 0·40 | 0·275 | 2·06 | 0·823 | 16·4 | 4·14 | 4·16 | 1·55 | 19 | 0·75 |
| 20 | ,, ,, ¾ | 5·549 | 18·87 | 0·40 | 0·275 | 2·11 | 0·872 | 19·88 | 4·97 | 5·11 | 1·89 | 18½ | 0·75 |
| 19 | 5½ × 3½ × ½ | 3·236 | 11·00 | 0·40 | 0·275 | 1·80 | 0·807 | 9·93 | 3·15 | 2·68 | 1·17 | 22 | 0·76 |
| 19 | ,, ,, ⅝ | 4·252 | 14·46 | 0·40 | 0·275 | 1·85 | 0·857 | 12·80 | 4·05 | 3·51 | 1·53 | 22 | 0·75 |
| 19 | ,, ,, ¾ | 5·236 | 17·80 | 0·40 | 0·275 | 1·90 | 0·905 | 15·6 | 4·86 | 4·33 | 1·87 | 21½ | 0·75 |
| 18 | 5½ × 3 × ½ | 3·050 | 10·37 | 0·375 | 0·250 | 1·90 | 0·662 | 9·45 | 2·02 | 2·62 | 0·86 | 17 | 0·64 |
| 18 | ,, ,, ⅝ | 4·003 | 13·61 | 0·375 | 0·250 | 1·95 | 0·711 | 12·2 | 2·58 | 3·44 | 1·13 | 16½ | 0·64 |
| 18 | ,, ,, ¾ | 4·925 | 16·74 | 0·375 | 0·250 | 2·00 | 0·759 | 14·7 | 3·08 | 4·20 | 1·37 | 16½ | 0·63 |
| 17 | 5 × 4 × ½ | 3·236 | 11·00 | 0·40 | 0·275 | 1·51 | 1·01 | 7·96 | 4·53 | 2·28 | 1·52 | 32 | 0·85 |
| 17 | ,, ,, ⅝ | 4·252 | 14·46 | 0·40 | 0·275 | 1·56 | 1·06 | 10·3 | 5·82 | 2·99 | 1·98 | 32 | 0·84 |
| 17 | ,, ,, ¾ | 5·236 | 17·80 | 0·40 | 0·275 | 1·60 | 1·11 | 12·4 | 7·01 | 3·66 | 2·43 | 32 | 0·83 |
| 16 | 5 × 3½ × ½ | 3·050 | 10·37 | 0·375 | 0·250 | 1·59 | 0·848 | 7·64 | 3·09 | 2·24 | 1·17 | 25½ | 0·75 |
| 16 | ,, ,, ⅝ | 4·003 | 13·61 | 0·375 | 0·250 | 1·64 | 0·897 | 9·86 | 3·96 | 2·93 | 1·52 | 25½ | 0·75 |
| 16 | ,, ,, ¾ | 4·925 | 16·74 | 0·375 | 0·250 | 1·69 | 0·944 | 11·9 | 4·75 | 3·60 | 1·86 | 25 | 0·74 |

## TABLE IV.—*continued.*

## DIMENSIONS AND PROPERTIES OF BRITISH STANDARD UNEQUAL ANGLES.

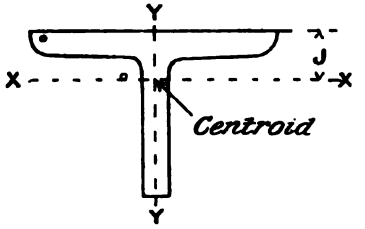
| Reference mark | Size and thickness | Area square inches | Weight per foot lbs. | Radii | | Dimensions | | Moments of inertia | | Section moduli | | Angle $\alpha$ degrees | Least radius of gyration |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Root | Toe | J | P | About XX | About YY | About XX | About YY | | |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| BSUA | | | | | | | | | | | | | |
| 15 | 5 × 3 × 5/16 | 2·402 | 8·17 | 0·350 | 0·250 | 1·66 | 0·667 | 6·14 | 1·68 | 1·84 | 0·72 | 20 | 0·65 |
| 15 | ,, ,, 5/16 | 2·859 | 9·72 | 0·350 | 0·250 | 1·68 | 0·693 | 7·24 | 1·97 | 2·18 | 0·85 | 19½ | 0·65 |
| 15 | ,, ,, ½ | 3·749 | 12·75 | 0·350 | 0·250 | 1·73 | 0·742 | 9·33 | 2·51 | 2·85 | 1·11 | 19½ | 0·64 |
| 15 | ,, ,, 5/8 | 4·609 | 15·67 | 0·350 | 0·250 | 1·78 | 0·789 | 11·25 | 3·00 | 3·49 | 1·36 | 19 | 0·64 |
| 14 | 4½ × 3½ × 5/16 | 2·402 | 8·17 | 0·350 | 0·250 | 1·36 | 0·866 | 4·82 | 2·55 | 1·54 | 0·97 | 30½ | 0·74 |
| 14 | ,, ,, 5/16 | 2·859 | 9·72 | 0·350 | 0·250 | 1·39 | 0·891 | 5·69 | 3·00 | 1·83 | 1·15 | 30½ | 0·74 |
| 14 | ,, ,, ½ | 3·749 | 12·75 | 0·350 | 0·250 | 1·44 | 0·940 | 7·31 | 3·84 | 2·39 | 1·5 | 30 | 0·74 |
| 14 | ,, ,, 5/8 | 4·609 | 15·67 | 0·350 | 0·250 | 1·48 | 0·987 | 8·81 | 4·61 | 2·92 | 1·83 | 30 | 0·74 |
| 12 | 4 × 3½ × 5/16 | 2·246 | 7·64 | 0·350 | 0·250 | 1·16 | 0·915 | 3·46 | 2·47 | 1·22 | 0·96 | 37 | 0·72 |
| 12 | ,, ,, 5/16 | 2·671 | 9·08 | 0·350 | 0·250 | 1·19 | 0·941 | 4·08 | 2·90 | 1·45 | 1·13 | 37 | 0·72 |
| 12 | ,, ,, ½ | 3·499 | 11·90 | 0·350 | 0·250 | 1·24 | 0·990 | 5·23 | 3·71 | 1·89 | 1·48 | 37 | 0·71 |
| 12 | ,, ,, 5/8 | 4·296 | 14·61 | 0·350 | 0·250 | 1·28 | 1·04 | 6·28 | 4·44 | 2·31 | 1·80 | 36½ | 0·71 |
| 11 | 4 × 3 × 5/16 | 2·091 | 7·11 | 0·325 | 0·225 | 1·24 | 0·746 | 3·31 | 1·59 | 1·20 | 0·71 | 28½ | 0·64 |
| 11 | ,, ,, 5/16 | 2·485 | 8·45 | 0·325 | 0·225 | 1·27 | 0·771 | 3·89 | 1·87 | 1·42 | 0·84 | 28½ | 0·64 |
| 11 | ,, ,, ½ | 3·251 | 11·05 | 0·325 | 0·225 | 1·31 | 0·819 | 4·98 | 2·37 | 1·85 | 1·09 | 28½ | 0·63 |
| 11 | ,, ,, 5/8 | 3·985 | 13·55 | 0·325 | 0·225 | 1·36 | 0·865 | 5·96 | 2·83 | 2·26 | 1·33 | 28 | 0·63 |
| 9 | 3½ × 3 × 5/16 | 1·934 | 6·58 | 0·325 | 0·225 | 1·04 | 0·792 | 2·27 | 1·53 | 0·92 | 0·69 | 35½ | 0·62 |
| 9 | ,, ,, 5/16 | 2·298 | 7·81 | 0·325 | 0·225 | 1·07 | 0·819 | 2·67 | 1·80 | 1·10 | 0·83 | 35½ | 0·62 |
| 9 | ,, ,, ½ | 3·001 | 10·20 | 0·325 | 0·225 | 1·11 | 0·867 | 3·40 | 2·28 | 1·42 | 1·07 | 35½ | 0·61 |
| 9 | ,, ,, 5/8 | 3·673 | 12·49 | 0·325 | 0·225 | 1·16 | 0·912 | 4·05 | 2·71 | 1·73 | 1·30 | 35 | 0·61 |
| 8 | 3½ × 2½ × 5/16 | 1·779 | 6·05 | 0·30 | 0·20 | 1·12 | 0·627 | 2·15 | 0·910 | 0·90 | 0·49 | 26½ | 0·54 |
| 8 | ,, ,, 5/16 | 2·111 | 7·18 | 0·30 | 0·20 | 1·15 | 0·652 | 2·52 | 1·06 | 1·07 | 0·57 | 26 | 0·53 |
| 8 | ,, ,, ½ | 2·752 | 9·36 | 0·30 | 0·20 | 1·20 | 0·699 | 3·20 | 1·34 | 1·39 | 0·74 | 26 | 0·53 |
| 7 | 3 × 2½ × ¼ | 1·312 | 4·46 | 0·275 | 0·20 | 0·895 | 0·648 | 1·14 | 0·716 | 0·54 | 0·39 | 34 | 0·52 |
| 7 | ,, ,, 5/16 | 1·921 | 6·53 | 0·275 | 0·20 | 0·945 | 0·697 | 1·62 | 1·02 | 0·79 | 0·57 | 34 | 0·52 |
| 7 | ,, ,, ½ | 2·499 | 8·50 | 0·275 | 0·20 | 0·992 | 0·744 | 2·05 | 1·28 | 1·02 | 0·73 | 33½ | 0·52 |
| 6 | 3 × 2 × ¼ | 1·187 | 4·04 | 0·275 | 0·20 | 0·976 | 0·482 | 1·06 | 0·373 | 0·52 | 0·25 | 23½ | 0·43 |
| 6 | ,, ,, 5/16 | 1·733 | 5·89 | 0·275 | 0·20 | 1·03 | 0·532 | 1·50 | 0·525 | 0·76 | 0·36 | 23 | 0·42 |
| 6 | ,, ,, ½ | 2·249 | 7·65 | 0·275 | 0·20 | 1·07 | 0·578 | 1·89 | 0·656 | 0·98 | 0·46 | 22½ | 0·42 |
| 5 | 2½ × 2 × ¼ | 1·063 | 3·61 | 0·250 | 0·175 | 0·774 | 0·527 | 0·636 | 0·359 | 0·37 | 0·24 | 32 | 0·42 |
| 5 | ,, ,, 5/16 | 1·309 | 4·45 | 0·250 | 0·175 | 0·799 | 0·552 | 0·770 | 0·433 | 0·45 | 0·30 | 31½ | 0·42 |
| 5 | ,, ,, ⅜ | 1·547 | 5·26 | 0·250 | 0·175 | 0·823 | 0·575 | 0·895 | 0·502 | 0·53 | 0·35 | 31½ | 0·42 |
| 4 | 2 × 1½ × 5/16 | 0·622 | 2·11 | 0·225 | 0·150 | 0·627 | 0·381 | 0·240 | 0·115 | 0·17 | 0·10 | 28½ | 0·32 |
| 4 | ,, ,, ¼ | 0·814 | 2·77 | 0·225 | 0·150 | 0·653 | 0·407 | 0·308 | 0·146 | 0·23 | 0·13 | 28 | 0·31 |
| 4 | ,, ,, 5/16 | 0·997 | 3·39 | 0·225 | 0·150 | 0·678 | 0·431 | 0·369 | 0·174 | 0·28 | 0·16 | 28 | 0·31 |




# TABLE V.

## DIMENSIONS AND PROPERTIES OF BRITISH STANDARD EQUAL ANGLES.

| Reference mark | Size and thickness | Area square inches | Weight per foot lbs. | Radii | | Dimension J | Moment of inertia XX | Section modulus XX | Least radius of gyrt'n. |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Root | Toe | | | | |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| BSEA | | | | | | | | | |
| 16 | 8 × 8 × ½ | 7·75 | 26·35 | 0·600 | 0·425 | 2·15 | 47·4 | 8·10 | 1·58 |
| 16 | ,, ,, ⅝ | 9·609 | 32·67 | 0·600 | 0·425 | 2·20 | 58·2 | 10·03 | 1·57 |
| 16 | ,, ,, ¾ | 11·437 | 38·89 | 0·600 | 0·425 | 2·25 | 68·5 | 11·91 | 1·56 |
| 14 | 6 × 6 × 7/16 | 5·062 | 17·21 | 0·475 | 0·325 | 1·64 | 17·3 | 3·97 | 1·18 |
| 14 | ,, ,, ½ | 7·112 | 24·18 | 0·475 | 0·325 | 1·71 | 23·8 | 5·55 | 1·18 |
| 14 | ,, ,, ⅝ | 8·441 | 28·70 | 0·475 | 0·325 | 1·76 | 27·8 | 6·56 | 1·17 |
| 13 | 5 × 5 × ⅜ | 3·610 | 12·27 | 0·425 | 0·300 | 1·37 | 8·51 | 2·34 | 0·98 |
| 13 | ,, ,, ½ | 4·750 | 16·15 | 0·425 | 0·300 | 1·42 | 11·0 | 3·07 | 0·98 |
| 13 | ,, ,, ⅝ | 5·860 | 19·92 | 0·425 | 0·300 | 1 47 | 13·4 | 3·80 | 0·98 |
| 12 | 4½ × 4½ × ⅜ | 3·236 | 11·00 | 0·400 | 0·275 | 1·22 | 6·14 | 1·87 | 0·88 |
| 12 | ,, ,, ½ | 4·252 | 14·46 | 0·400 | 0·275 | 1·29 | 7·92 | 2·47 | 0·87 |
| 12 | ,, ,, ⅝ | 5·236 | 17·80 | 0·400 | 0·275 | 1·34 | 9·56 | 3·03 | 0·87 |
| 11 | 4 × 4 × ⅜ | 2·859 | 9·72 | 0·350 | 0·250 | 1·12 | 4·26 | 1·48 | 0·78 |
| 11 | ,, ,, ½ | 3·749 | 12·75 | 0·350 | 0·250 | 1·17 | 5·46 | 1·93 | 0·77 |
| 11 | ,, ,, ⅝ | 4·609 | 15·67 | 0·350 | 0·250 | 1·22 | 6·56 | 2·36 | 0·77 |
| 10 | 3½ × 3½ × 7/16 | 2·091 | 7·11 | 0·325 | 0·225 | 0·975 | 2·39 | 0·95 | 0·68 |
| 10 | ,, ,, ½ | 2·485 | 8·45 | 0·325 | 0·225 | 1·00 | 2·80 | 1·12 | 0·68 |
| 10 | ,, ,, ⅝ | 3·251 | 11·05 | 0·325 | 0·225 | 1·05 | 3·57 | 1·46 | 0·68 |
| 10 | ,, ,, ¾ | 3·985 | 13·55 | 0·325 | 0·225 | 1·09 | 4·27 | 1·77 | 0·68 |
| 9 | 3 × 3 × ⅜ | 1·44 | 4·90 | 0·300 | 0·200 | 0·827 | 1·21 | 0·56 | 0·59 |
| 9 | ,, ,, 7/16 | 2·111 | 7·18 | 0·300 | 0·200 | 0·877 | 1·72 | 0·81 | 0·58 |
| 9 | ,, ,, ½ | 2·752 | 9·36 | 0·300 | 0·200 | 0·924 | 2·19 | 1·05 | 0·58 |
| 9 | ,, ,, ⅝ | 3·362 | 11·43 | 0·300 | 0·200 | 0·970 | 2·59 | 1·28 | 0·58 |
| 7 | 2½ × 2½ × ⅜ | 1·187 | 4·04 | 0·275 | 0·200 | 0·703 | 0·677 | 0·38 | 0·48 |
| 7 | ,, ,, 7/16 | 1·464 | 4·98 | 0·275 | 0·200 | 0·728 | 0·822 | 0·46 | 0·48 |
| 7 | ,, ,, ½ | 1·733 | 5·89 | 0·275 | 0·200 | 0·752 | 0·962 | 0·55 | 0·48 |
| 7 | ,, ,, ⅝ | 2·249 | 7·65 | 0·275 | 0·200 | 0·799 | 1·21 | 0·71 | 0·48 |
| 6 | 2½ × 2½ × 5/16 | 0·809 | 2·75 | 0·250 | 0·175 | 0·616 | 0·378 | 0·23 | 0·44 |
| 6 | ,, ,, ⅜ | 1·063 | 3·61 | 0·250 | 0·175 | 0·643 | 0·489 | 0·30 | 0·44 |
| 6 | ,, ,, 7/16 | 1·309 | 4·45 | 0·250 | 0·175 | 0·668 | 0·592 | 0·37 | 0·43 |
| 6 | ,, ,, ½ | 1·547 | 5·26 | 0·250 | 0·175 | 0·692 | 0·686 | 0·44 | 0·43 |
| 5 | 2 × 2 × 5/16 | 0·715 | 2·43 | 0·250 | 0·175 | 0·554 | 0·260 | 0·18 | 0·39 |
| 5 | ,, ,, ⅜ | 0·938 | 3·19 | 0·250 | 0·175 | 0·581 | 0·336 | 0·24 | 0·39 |
| 5 | ,, ,, 7/16 | 1·153 | 3·92 | 0·250 | 0·175 | 0·605 | 0·401 | 0·29 | 0·38 |
| 5 | ,, ,, ½ | 1·36 | 4·62 | 0·250 | 0·175 | 0·629 | 0·467 | 0·34 | 0·38 |
| 4 | 1¾ × 1¾ × 5/16 | 0·622 | 2·11 | 0·225 | 0·150 | 0·495 | 0·172 | 0·14 | 0·34 |
| 4 | ,, ,, ¼ | 0·814 | 2·77 | 0·225 | 0·150 | 0·520 | 0·220 | 0·18 | 0·34 |
| 4 | ,, ,, 7/16 | 0·997 | 3·39 | 0·225 | 0·150 | 0·544 | 0·264 | 0·22 | 0·34 |
| 3 | 1½ × 1½ × 5/16 | 0·526 | 1·79 | 0·200 | 0·150 | 0·434 | 0·105 | 0·10 | 0·29 |
| 3 | ,, ,, ¼ | 0·686 | 2·33 | 0·200 | 0·150 | 0·458 | 0·134 | 0·13 | 0·29 |
| 3 | ,, ,, 7/16 | 0·839 | 2·85 | 0·200 | 0·150 | 0·482 | 0·159 | 0·16 | 0·29 |
| 2 | 1¼ × 1¼ × 5/16 | 0·433 | 1·47 | 0·200 | 0·150 | 0·371 | 0·058 | 0·07 | 0·24 |
| 2 | ,, ,, ¼ | 0·561 | 1·91 | 0·200 | 0·150 | 0·396 | 0·073 | 0·09 | 0·23 |




# TABLE VI.

## DIMENSIONS AND PROPERTIES OF BRITISH STANDARD TEES.

| Reference mark | Size and thickness | Area square inches | Weight per foot lbs. | Radii | | Dimension | Moments of inertia | | Section moduli | | Radii of gyration | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Table root | Table toe | J | About XX | About YY | About XX | About YY | About XX | About YY |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| BST | | | | | | | | | | | | |
| 21 | 6 × 4 × ⅜ | 3·634 | 12·36 | 0·425 | 0·300 | 0·915 | 4·700 | 6·344 | 1·52 | 2·11 | 1·137 | 1·321 |
| 21 | ,, ,, ½ | 4·771 | 16·22 | 0·425 | 0·300 | 0·968 | 6·070 | 8·621 | 2·00 | 2·87 | 1·128 | 1·344 |
| 21 | ,, ,, ⅝ | 5·878 | 19·99 | 0·425 | 0·300 | 1·02 | 7·350 | 10·912 | 2·47 | 3·64 | 1·118 | 1·362 |
| 20 | 6 × 3 × ⅜ | 3·260 | 11·08 | 0·400 | 0·275 | 0·633 | 2·062 | 6·389 | 0·87 | 2·13 | 0·795 | 1·400 |
| 20 | ,, ,, ½ | 4·272 | 14·53 | 0·400 | 0·275 | 0·684 | 2·635 | 8·649 | 1·14 | 2·88 | 0·785 | 1·423 |
| 20 | ,, ,, ⅝ | 5·256 | 17·87 | 0·400 | 0·275 | 0·732 | 3·144 | 10·938 | 1·39 | 3·65 | 0·773 | 1·443 |
| 19 | 5 × 4 × ⅜ | 3·257 | 11·07 | 0·400 | 0·275 | 0·998 | 4·471 | 3·691 | 1·49 | 1·48 | 1·172 | 1·065 |
| 19 | ,, ,, ½ | 4·268 | 14·51 | 0·400 | 0·275 | 1·05 | 5·772 | 5·017 | 1·96 | 2·01 | 1·163 | 1·084 |
| 17 | 5 × 3 × ⅜ | 2·875 | 9·78 | 0·350 | 0·250 | 0·691 | 1·973 | 3·716 | 0·85 | 1·49 | 0·828 | 1·137 |
| 17 | ,, ,, ½ | 3·762 | 12·79 | 0·350 | 0·250 | 0·741 | 2·516 | 5·031 | 1·11 | 2·01 | 0·818 | 1·156 |
| 15 | 4 × 4 × ⅜ | 2·872 | 9·77 | 0·350 | 0·250 | 1·11 | 4·189 | 1·901 | 1·45 | 0·95 | 1·208 | 0·814 |
| 15 | ,, ,, ½ | 3·758 | 12·78 | 0·350 | 0·250 | 1·16 | 5·402 | 2·590 | 1·90 | 1·29 | 1·199 | 0·830 |
| 14 | 4 × 3 × ⅜ | 2·498 | 8·49 | 0·325 | 0·225 | 0·767 | 1·860 | 1·914 | 0·83 | 0·96 | 0·863 | 0·875 |
| 14 | ,, ,, ½ | 3·262 | 11·08 | 0·325 | 0·225 | 0·816 | 2·365 | 2·599 | 1·08 | 1·30 | 0·851 | 0·893 |
| 13 | 3½ × 3½ × ⅜ | 2·496 | 8·49 | 0·325 | 0·225 | 0·988 | 2·768 | 1·284 | 1·10 | 0·73 | 1·053 | 0·717 |
| 13 | ,, ,, ½ | 3·259 | 11·08 | 0·325 | 0·225 | 1·04 | 3·543 | 1·752 | 1·44 | 1·00 | 1·043 | 0·733 |
| 11 | 3 × 3 × ⅜ | 2·121 | 7·21 | 0·300 | 0·200 | 0·868 | 1·708 | 0·816 | 0·80 | 0·54 | 0·897 | 0·620 |
| 11 | ,, ,, ½ | 2·76 | 9·38 | 0·300 | 0·200 | 0·918 | 2·165 | 1·115 | 1·04 | 0·74 | 0·886 | 0·636 |
| 10 | 3 × 2½ × ⅜ | 1·929 | 6·56 | 0·275 | 0·200 | 0·695 | 1·015 | 0·814 | 0·56 | 0·54 | 0·725 | 0·650 |
| 10 | ,, ,, ½ | 2·506 | 8·52 | 0·275 | 0·200 | 0·742 | 1·275 | 1·109 | 0·73 | 0·74 | 0·713 | 0·665 |
| 8 | 2½ × 2½ × ⅜ | 1·197 | 4·07 | 0·275 | 0·200 | 0·697 | 0·677 | 0·302 | 0·38 | 0·24 | 0·752 | 0·502 |
| 8 | ,, ,, ⅝ | 1·474 | 5·01 | 0·275 | 0·200 | 0·724 | 0·823 | 0·387 | 0·46 | 0·31 | 0·747 | 0·512 |
| 8 | ,, ,, ¾ | 1·742 | 5·92 | 0·275 | 0·200 | 0·750 | 0·959 | 0·473 | 0·55 | 0·38 | 0·742 | 0·521 |
| 7 | 2½ × 2½ × ⅜ | 1·071 | 3·64 | 0·250 | 0·175 | 0·638 | 0·488 | 0·224 | 0·30 | 0·20 | 0·675 | 0·457 |
| 7 | ,, ,, ½ | 1·554 | 5·28 | 0·250 | 0·175 | 0·689 | 0·685 | 0·349 | 0·44 | 0·31 | 0·664 | 0·474 |
| 6 | 2 × 2 × ⅜ | 0·947 | 3·22 | 0·250 | 0·175 | 0·579 | 0·337 | 0·157 | 0·24 | 0·16 | 0·597 | 0·407 |
| 6 | ,, ,, ½ | 1·367 | 4·64 | 0·250 | 0·175 | 0·628 | 0·469 | 0·246 | 0·34 | 0·25 | 0·586 | 0·424 |
| 5 | 1½ × 2 × ⅜ | 0·820 | 2·79 | 0·225 | 0·150 | 0·648 | 0·307 | 0·068 | 0·23 | 0·09 | 0·612 | 0·288 |
| 5 | ,, ,, ⅝ | 1·003 | 3·41 | 0·225 | 0·150 | 0·674 | 0·369 | 0·088 | 0·28 | 0·12 | 0·607 | 0·296 |
| 4 | 1¾ × 1¾ × ⅜ | 0·820 | 2·79 | 0·225 | 0·150 | 0·519 | 0·221 | 0·107 | 0·18 | 0·12 | 0·520 | 0·361 |
| 4 | ,, ,, ⅝ | 0·999 | 3·40 | 0·225 | 0·150 | 0·544 | 0·265 | 0·137 | 0·22 | 0·16 | 0·515 | 0·370 |
| 3 | 1½ × 1½ × ⅜ | 0·531 | 1·81 | 0·200 | 0·150 | 0·435 | 0·106 | 0·048 | 0·10 | 0·06 | 0·447 | 0·301 |
| 3 | ,, ,, ½ | 0·692 | 2·35 | 0·200 | 0·150 | 0·460 | 0·135 | 0·067 | 0·13 | 0·09 | 0·442 | 0·312 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 2 3 4 | 5 | 6 7 8 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4 9 13 17<br>4 8 12 16 | 21<br>20 | 25 30 34 38<br>24 28 32 37 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4 8 12 15 | 19 | 23 27 31 35 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 4 7 11 15<br>3 7 11 14<br>3 7 10 14 | 19<br>18<br>17 | 22 26 30 33<br>21 25 28 32<br>20 24 27 31 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3 7 10 13 | 16 | 20 23 26 30 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3 7 10 12<br>3 6 9 12<br>3 6 9 12 | 16<br>15<br>15 | 19 22 25 29<br>18 21 24 28<br>17 20 23 26 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3 6 9 11<br>3 6 8 11 | 14<br>14 | 17 20 23 26<br>16 19 22 25 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3 5 8 11 | 14 | 16 19 22 24 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 3 5 8 10<br>3 5 8 10<br>2 5 7 10 | 13<br>13<br>12 | 15 18 21 23<br>15 18 20 23<br>15 17 19 22 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2 5 7 9 | 12 | 14 16 19 21 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2 5 7 9<br>2 4 7 9<br>2 4 6 8 | 11<br>11<br>11 | 14 16 18 21<br>13 16 18 20<br>13 15 17 19 |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 4 6 8 | 11 | 13 15 17 19 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 4 6 8 | 10 | 12 14 16 18 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 4 6 8 | 10 | 12 14 15 17 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 4 6 7 | 9 | 11 13 15 17 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 4 5 7 | 9 | 11 12 14 16 |
| 25 | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 3 5 7 | 9 | 10 12 14 15 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 3 5 7 | 8 | 10 11 13 15 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 3 5 6 | 8 | 9 11 13 14 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 3 5 6 | 8 | 9 11 12 14 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 3 4 6 | 7 | 9 10 12 13 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 3 4 6 | 7 | 9 10 11 13 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1 3 4 6 | 7 | 8 10 11 12 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 3 4 5 | 7 | 8 9 11 12 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 3 4 5 | 6 | 8 9 10 12 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 3 4 5 | 6 | 8 9 10 11 |
| 35 | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 2 4 5 | 6 | 7 9 10 11 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 2 4 5 | 6 | 7 8 10 11 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 2 3 5 | 6 | 7 8 9 10 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 2 3 5 | 6 | 7 8 9 10 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 2 3 4 | 5 | 7 8 9 10 |
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 2 3 4 | 5 | 6 8 9 10 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 2 3 4 | 5 | 6 7 8 9 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 2 3 4 | 5 | 6 7 8 9 |
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| ·95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| ·96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| ·97 | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| ·98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| ·99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |

| Angle. Degrees. | Angle. Radians. | Chord. | Sine. | Tangent. | Co-tangent. | Cosine. | . | | |
|---|---|---|---|---|---|---|---|---|---|
| 0° | 0 | 000 | 0 | 0 | ∞ | 1 | 1·414 | 1·5708 | 90° |
| 1 | ·0175 | ·017 | ·0175 | ·0175 | 57·2900 | ·9998 | 1·402 | 1·5533 | 89 |
| 2 | ·0349 | ·035 | ·0349 | ·0349 | 28·6363 | ·9994 | 1·389 | 1·5359 | 88 |
| 3 | ·0524 | ·052 | ·0523 | ·0524 | 19·0811 | ·9986 | 1·377 | 1·5184 | 87 |
| 4 | ·0698 | ·070 | ·0698 | ·0699 | 14·3007 | ·9976 | 1·364 | 1·5010 | 86 |
| 5 | ·0873 | ·087 | ·0872 | ·0875 | 11·4301 | ·9962 | 1·351 | 1·4835 | 85 |
| 6 | ·1047 | ·105 | ·1045 | ·1051 | 9·5144 | ·9945 | 1·338 | 1·4661 | 84 |
| 7 | ·1222 | ·122 | ·1219 | ·1228 | 8·1443 | ·9925 | 1·325 | 1·4486 | 83 |
| 8 | ·1396 | ·140 | ·1392 | ·1405 | 7·1154 | ·9903 | 1·312 | 1·4312 | 82 |
| 9 | ·1571 | ·157 | ·1564 | ·1584 | 6·3138 | ·9877 | 1·299 | 1·4137 | 81 |
| 10 | ·1745 | ·174 | ·1736 | ·1763 | 5·6713 | ·9848 | 1·286 | 1·3963 | 80 |
| 11 | ·1920 | ·192 | ·1908 | ·1944 | 5·1446 | ·9816 | 1·272 | 1·3788 | 79 |
| 12 | ·2094 | ·209 | ·2079 | ·2126 | 4·7046 | ·9781 | 1·259 | 1·3614 | 78 |
| 13 | ·2269 | ·226 | ·2250 | ·2309 | 4·3315 | ·9744 | 1·245 | 1·3439 | 77 |
| 14 | ·2443 | ·244 | ·2419 | ·2493 | 4·0108 | ·9703 | 1·231 | 1·3265 | 76 |
| 15 | ·2618 | ·261 | ·2588 | ·2679 | 3·7321 | ·9659 | 1·218 | 1·3090 | 75 |
| 16 | ·2793 | ·278 | ·2756 | ·2867 | 3·4874 | ·9613 | 1·204 | 1·2915 | 74 |
| 17 | ·2967 | ·296 | ·2924 | ·3057 | 3·2709 | ·9568 | 1·190 | 1·2741 | 73 |
| 18 | ·3142 | ·313 | ·3090 | ·3249 | 3·0777 | ·9511 | 1·176 | 1·2566 | 72 |
| 19 | ·3316 | ·330 | ·3256 | ·3443 | 2·9042 | ·9455 | 1·161 | 1·2392 | 71 |
| 20 | ·3491 | ·347 | ·3420 | ·3640 | 2·7475 | ·9397 | 1·147 | 1·2217 | 70 |
| 21 | ·3665 | ·364 | ·3584 | ·3839 | 2·6051 | ·9336 | 1·133 | 1·2043 | 69 |
| 22 | ·3840 | ·382 | ·3746 | ·4040 | 2·4751 | ·9272 | 1·118 | 1·1868 | 68 |
| 23 | ·4014 | ·399 | ·3907 | ·4245 | 2·3559 | ·9205 | 1·104 | 1·1694 | 67 |
| 24 | ·4189 | ·416 | ·4067 | ·4452 | 2·2460 | ·9135 | 1·089 | 1·1519 | 66 |
| 25 | ·4363 | ·433 | ·4226 | ·4663 | 2·1445 | ·9063 | 1·075 | 1·1345 | 65 |
| 26 | ·4538 | ·450 | ·4384 | ·4877 | 2·0503 | ·8988 | 1·060 | 1·1170 | 64 |
| 27 | ·4712 | ·467 | ·4540 | ·5095 | 1·9626 | ·8910 | 1·045 | 1·0996 | 63 |
| 28 | ·4887 | ·484 | ·4695 | ·5317 | 1·8807 | ·8829 | 1·030 | 1·0821 | 62 |
| 29 | ·5061 | ·501 | ·4848 | ·5543 | 1·8040 | ·8746 | 1·015 | 1·0647 | 61 |
| 30 | ·5236 | ·518 | ·5000 | ·5774 | 1·7321 | ·8660 | 1·000 | 1·0472 | 60 |
| 31 | ·5411 | ·534 | ·5150 | ·6009 | 1·6643 | ·8572 | ·985 | 1·0297 | 59 |
| 32 | ·5585 | ·551 | ·5299 | ·6249 | 1·6003 | ·8480 | ·970 | 1·0123 | 58 |
| 33 | ·5760 | ·568 | ·5446 | ·6494 | 1·5399 | ·8387 | ·954 | ·9948 | 57 |
| 34 | ·5934 | ·585 | ·5592 | ·6745 | 1·4826 | ·8290 | ·939 | ·9774 | 56 |
| 35 | ·6109 | ·601 | ·5736 | ·7002 | 1·4281 | ·8192 | ·923 | ·9599 | 55 |
| 36 | ·6283 | ·618 | ·5878 | ·7265 | 1·3764 | ·8090 | ·908 | ·9425 | 54 |
| 37 | ·6458 | ·635 | ·6018 | ·7536 | 1·3270 | ·7986 | ·892 | ·9250 | 53 |
| 38 | ·6632 | ·651 | ·6157 | ·7813 | 1·2799 | ·7880 | ·877 | ·9076 | 52 |
| 39 | ·6807 | ·668 | ·6293 | ·8098 | 1·2349 | ·7771 | ·861 | ·8901 | 51 |
| 40 | ·6981 | ·684 | ·6428 | ·8391 | 1·1918 | ·7660 | ·845 | ·8727 | 50 |
| 41 | ·7156 | ·700 | ·6561 | ·8693 | 1·1504 | ·7547 | ·829 | ·8552 | 49 |
| 42 | ·7330 | ·717 | ·6691 | ·9004 | 1·1106 | ·7431 | ·813 | ·8378 | 48 |
| 43 | ·7505 | ·733 | ·6820 | ·9325 | 1·0724 | ·7314 | ·797 | ·8203 | 47 |
| 44 | ·7679 | ·749 | ·6947 | ·9657 | 1·0355 | ·7193 | ·781 | ·8029 | 46 |
| 45° | ·7854 | ·765 | ·7071 | 1·0000 | 1·0000 | ·7071 | ·765 | ·7854 | 45° |
| | | | Cosine. | Co-tangent. | Tangent. | Sine. | Chord. | Radians. | Degrees. |
| | | | | | | | | Angle. | |

# ANSWERS TO EXAMPLES

## EXAMPLES I.

(1) 3·96 tons per square inch; 13,700 tons per square inch; 1·98 tons per square inch.
(2) 20° 54½′; 2·62 tons per square inch; 2·80 tons per square inch.
(3) 3·27 tons per square inch; 3·60 tons per square inch.
(4) 0·0318 inch.
(5) 23,200,000 lbs. per square inch; 3·385.
(6) 3·5 tons per square inch; 0·866 ton per square inch; 3·60 tons per square inch inclined 76° 5′ to the plane.
(7) 32·5° and 3·54 tons per square inch, or 72° and 2·27 tons per square inch.
(8) 4·58 tons per square inch 40·9° to plane; 4 tons per square inch.
(9) 8·12 tons per square inch; normal of plane inclined 38° to axis of 5-ton stress.
(10) 6·65 tons per square inch; normal of plane inclined 22½° to axis of 5-ton stress.
(11) 4·828 tons per square inch tensile on plane inclined 22½° to cross-section. 0·828 ton per square inch compressive on plane inclined 67½° to cross-section.
(12) 4·16 and 3·16 tons per square inch.
(13) 4·375 tons per square inch.
(14) $\frac{m^2 - 1}{m^2}$.
(15) 19,556 lbs. per square inch (steel); 10,222 lbs. per square inch (brass); 48·89 per cent.

## EXAMPLES II.

(1) 32·4 and 21·6 tons per square inch; 23·5 per cent.; 13,120 tons per square inch.
(2) (*a*) 15·77 tons; (*b*) 69·1 tons.
(3) 7·03 inch-tons.
(4) 620 inch-pounds.
(5) 2760 and 16·26 inch-pounds.
(6) 8 tons per square inch; 0·0738 inch; 4·06 tons.
(7) (*a*) 55 tons; 4·07 square inches; (*b*) 25 tons: 1·85 square inches.
(8) 5·46 tons per square inch.
(9) 3·50 inches.
(10) 4·17 tons per square inch (Launhardt); 3·33 tons per square inch (Dynamic.)
(11) 1·56 square inch (Launhardt); 1·71 square inch (Dynamic).

## EXAMPLES III.

(1) 11·46 lbs., 28·6 inches, 197°.
(2) 17·7 tons right, 11·3 tons left.
(3) Left 10 tons, right 3 tons.
(4) 21·6 lbs., 134° measured clockwise.
(6) 3·55 inches.
(7) 2·52 inches from outside of flange.
(8) 312 (inches)$^4$.
(9) 74·1 (inches)$^4$, 2·47 inches.

## EXAMPLES IV.

(1) 158 tons-feet ; 20 tons ; 50 tons-feet ; 14 tons.
(2) 2650 tons-feet.
(3) 8 tons-feet ; 6 feet from left end ; 9·75 tons-feet.
(4) 10·958 feet from left support ; 88·1 tons-feet ; 87 tons-feet.
(5) $\frac{1}{\sqrt{3}}l$ feet ; $\frac{wl^2}{9\sqrt{3}}$ tons-feet ; 10·4 feet ; 41·5 tons-feet.
(6) 11·76 feet from A.
(7) 13·1 feet from A.
(8) 32 and 40 tons-feet ; 3·05 feet from supports.
(9) $0{\cdot}207l$ and $0{\cdot}293l$ from ends.
(10) 4·6 tons-feet ; 0·5 ton-foot ; 4·9 feet from left support ; 4·74 feet from right support.
(11) 13 tons-feet ; 2·89 feet from left support ; 1·46 feet from right support.
(12) 27·5 tons-feet ; 52 tons-feet ; 16 tons-feet ; 4·15 feet (left) and 1·41 feet (right).

## EXAMPLES V.

(1) 4·8 tons per square inch.
(2) 217·5 tons-inches.
(3) 15·625 tons ; 7·812 tons.
(4) 937·5 feet ; 253·2 tons-inches.
(5) 1470 lbs. per square inch ; 609·5 feet.
(6) $3\frac{1}{8}$ inches.
(7) 13·1 inches.
(8) 1·414.
(9) 12 feet.
(10) 3·27 to 1.
(11) 7 tons per square inch.
(12) 21,750 lb.-inches.
(13) 5·96 (inches)$^4$.
(14) 4·57 inches ; 930 (inches)$^4$ ; 1·36 ton ; 1·95 ton per square inch.
(15) 30·7 feet.
(16) 7·15 tons per square inch.
(17) 16 inches.
(18) $\frac{7}{8}$ inch.
(19) 1437 lbs. ; 6930 lbs. per square inch.

(20) 0·63 square inch ; 386 lbs.
(21) 4·65 square inches.
(22) 0·565 square inch ; 14,580 lbs. per square inch.
(23) 3 square inches ; 18,000 lbs. per square inch.
(24) 9580 lbs. per square inch ; 1,040,000 lb.-inches.
(25) 351,900 lb.-inches ; 18,000 lbs. per square inch.
(26) 1·867.
(27) 5·80 tons per square inch ; 3·93.
(28) 4·68 tons per square inch tension inclined 53° 44′ to section ; 2·60 tons per square inch inclined 36° 46′ to section.

## EXAMPLES VI.

(1) 1·875 tons; 7·5 tons; 337·5 and 450 tons-feet.
(2) Positive, 0·33, 0·67, and 1 ton; negative, 1·67, 1·33 and 1 ton ; 8·33 tons-feet ; 13·33 tons-feet ; 15 tons-feet.
(3) 1·125, 3·75 and 5·25 tons ; 162·5, 306 and 318·75 tons-feet ; 0·255 ton per foot.
(4) 243 tons-feet ; 2·5 feet from centre ; 240 tons-feet.
(5) 100 tons-feet at centre ; 27·24 feet.
(6) 31·2 feet from an abutment ; 779 tons-feet.
(7) 3,238,500 lb.-feet ; 615,000 lb.-feet.
(8) 137,700 lbs.
(9) 5,500 lbs. per foot ; 4,400,000 lb.-feet ; 4,320,000 lb.-feet.
(10) 12·52 feet.
(11) 612 tons-feet ; 7·5 tons ; 13·5 tons.

## EXAMPLES VII.

(1) 4·96 tons ; 4·74 tons per square inch ; 7·94 tons ; 3·79 tons per square inch.

(2) $\frac{1}{384}\frac{Wl^3}{EI}$ ; $\frac{\frac{W}{2}}{1+\frac{48EI}{el^3}}$

(3) 3 inches (nearly) from centre of span ; 0·262 inch.

(4) $\frac{5}{32}W$ ; $\frac{7}{960}\frac{Wl^3}{EI}$.

(5) $\frac{5}{16}W$ ; $\frac{5}{32}Wl$ ; $\frac{3}{16}Wl$ ; $\frac{1}{\sqrt{5}}l$ from free end ; $\frac{1}{48\sqrt{5}}\frac{Wl^3}{EI}$ ; 0·2038 W.

(6) $\frac{19}{32}$.

(7) $\frac{1}{30}\frac{wl^4}{EI}$.

(8) 0·134 inch ; 0·148 inch ; 9·25 inches from centre ; 0·148 inch.
(9) 9·18 tons ; 3·3 tons.
(10) 8·8 inches from centre ; 0·342 inch.
(11) 12·083 tons (centre) ; 3·958 tons (ends).
(12) 0·414 ; 0·68.
(13) 0·29 ; 0·337 ; 0·644.
(14) $\frac{1}{8}$ ; $\frac{13}{16}$.
(15) 0·0186 inch ; 0·224 inch ; 0·0181 inch (upward) ; 9·87 feet.

(16) 0·0988 ; 0·073 inch (upward) ; 0·409 inch ; 4·63 feet to left of D.

(17) $0{\cdot}544\,\frac{Wl^3}{EI_0}$.

(18) 2·98 inches.

(19) $0{\cdot}0241\,\frac{Wl^3}{EI_0}$.

(20) $0{\cdot}0153\,\frac{Wl^3}{EI_0}$.

## EXAMPLES VIII.

(1) 6·55 tons per square inch ; 0·152 inch.

(2) $\frac{1}{30}wl^2$ ; $\frac{1}{20}wl^2$ ; $\frac{2}{30}wl$ ; $\frac{3}{20}wl$ ; 0·025$l$ from centre.

(3) $\frac{2}{3}Wl$ ; $\frac{1}{3}Wl$ ; $\frac{5}{648}\frac{Wl^3}{EI}$ ; $\frac{1}{162}\frac{Wl^3}{EI}$ ; $\frac{2}{3}l$ from ends.

(4) $\frac{2}{27}Wl$ ; $\frac{4}{27}Wl$ ; $\frac{7}{27}W$ ; $\frac{20}{27}W$ ; $\frac{5}{12960}\frac{Wl^3}{EI}$ ; $\frac{8}{2187}\frac{Wl^3}{EI}$ ; $\frac{4}{7}l$ from light end ; $\frac{16}{3969}\frac{Wl^3}{EI}$ ; $\frac{3}{7}l$ and $\frac{4}{7}l$ from light end.

(5) 22·025 tons-feet (left) ; 19·475 tons-feet (right).

(6) $\frac{11}{192}wl^2$ and $\frac{5}{192}wl^2$ ; 0·182$l$ and $\frac{13}{18}l$ from heavy end ; 0·443 from heavy end ; $0{\cdot}00134\,\frac{wl^4}{EI}$.

(7) 0·1108W$l$ ; 0·1392W$l$ ; $0{\cdot}007\,\frac{Wl^3}{EI}$.

(8) 0·0759W$l$ ; 0·0491W$l$ ; $0{\cdot}0037\,\frac{Wl^3}{EI}$.

(9) 0, $\frac{1}{10}wl^2$, $\frac{1}{10}wl^2$, 0 ; $\frac{4}{10}wl$, $\frac{11}{10}wl$, $\frac{11}{10}wl$, $\frac{4}{10}wl$.

(10) 0, 175 tons-feet, 125 tons-feet, 0 ; 24·16 tons, 57·083 tons, 55 tons, 23·75 tons.

(11) 7·429 tons-feet at B, 4·913 tons-feet at C ; in order A, B, C, D, 3·45, 7·34, 6·39, 3·82 tons.

(12) (*a*) From fixed end, $\frac{9}{104}wl^2$, $\frac{1}{13}wl^2$, $\frac{11}{104}wl^2$, 0 ; $\frac{53}{104}wl$, $\frac{25}{26}wl$, $\frac{52}{52}wl$, $\frac{41}{104}wl$. (*b*) $\frac{1}{13}wl^2$ at each ; $\frac{wl}{2}$ at ends, $wl$ at inner supports.

(13) In order A, B, C, D, 6·193, 5·661, 5·486, 0 tons-feet ; 4·441, 6·03, 6·843, 3·703 tons.

(14) 2·94 and 8·65 tons-feet ; 4·01, 5·60, 8·32, 3·07 tons.

(15) 3·2 to 1, 1 to 3.

(16) 7·4 per cent.

## EXAMPLES IX.

(1) 1·936 and 0·844 tons per square inch.

(2) 5·6 and 2·4 tons per square inch.

(3) 7·417 and 6·583 tons per square inch.

(4) 14·85 feet.

(5) 72·8 tons.

(6) 4 feet 6·6 inches.

(7) 989 tons.

(8) 354 tons.
(9) 324 tons.
(10) 36·6 tons.
(11) 121·3 tons.
(12) 0·48 inch.
(13) 9·5 inches.
(14) 3·43 inches.
(15) 2·441 and 0·339 tons per square inch.
(16) 0·309 inch.
(17) 46·3 inches; 0·34 ton per square inch.
(18) 770 tons.
(19) 19·06 tons; 5·42 tons per square inch.
(20) 2·275 inches.
(21) 13·2 tons; 4·06 tons per square inch.
(22) 4571 and 521 pounds per square inch compression.
(23) 0·0308 inch; 3173 pounds per square inch.

## EXAMPLES X.

(1) At bearings 392 lbs., at apex and struts 784 lbs., at bearings and apex 940 lbs., at strut 1880 lbs.
(2) 7390 lbs.
(3) At shoe and apex 1155 lbs.; at intermediate joints 2310 lbs.

## EXAMPLES XI.

(1) *Dead loads.* Main rafters and short strut 2630, 2280 and 700 lbs. thrust. Main ties and inclined ties 2350, 1568 and 786 lbs. tension. *Wind loads.* Main rafters and short strut 3290, 3290 and 1880 lbs. thrust. Main ties and inclined tie 3680, 1575 and 2100 lbs. tension.

(2) Main rafters 7700, 6060, 7700 lbs. thrust, short struts 3980 lbs. thrust, main ties 8610 and 3100 lbs. tension, inclined tie 5510 lbs. tension.

(3) Main rafters 14,680, 13,830, 13,000, 12,300 lbs. thrust; truss struts 1680, 3360, 1680 lbs. thrust; main ties 13,130, 11,250, 7500 lbs., second truss ties 3750 and 5625 lbs., sub-truss tie 1875 lbs.

(4) Add to No. 3 answers in order, 11,140, 11,140, 11,140, 11,140 lbs., 2475, 4950, 2475 lbs., 12,440, 9680, 4150 lbs., 5530, 8300 lbs., 3760 lbs.

(5) From left end, + for tension, − for thrust. Diagonals − 16·96, + 16·96, − 11·19, + 11·19, − 5·089, + 5·089, + 0·36, − 0·36, + 6·135, − 6·135, + 9·02, − 9·02, + 9·02, − 9·02, + 9·02, − 9·02 tons. Top chord thrusts 16·96, 28·16, 33·26, 32·90, 26·78, 17·78, 8·78 tons. Lower chord tensions 8·48, 22·56, 30·71, 33·08, 29·84, 22·28, 13·28, 4·28 tons.

(6) Coefficients of W from left end. Diagonals (tension) $\frac{25}{28}$, $\frac{15}{7}$, $\frac{25}{28}$, (0), $\frac{5}{14}$, $\frac{45}{28}$, $\frac{135}{56}$, $\frac{135}{56}$. Verticals (thrust) $\frac{12}{7}$, $\frac{12}{7}$, $\frac{5}{7}$, 0, $\frac{2}{7}$, $\frac{9}{7}$, $\frac{25}{14}$, $\frac{25}{14}$. Top chord (thrust) $\frac{45}{28}$, $\frac{93}{28}$, $\frac{27}{7}$, $\frac{27}{7}$, $\frac{41}{14}$, $\frac{76}{28}$, $\frac{75}{56}$. Lower chord (tension) 0, $\frac{87}{28}$, $\frac{93}{28}$, $\frac{41}{14}$, $\frac{75}{28}$, $\frac{75}{56}$, 0.

(7) $\frac{l^2}{h}\frac{(n-1)(n+1)}{6n} \cdot W.$

(8) $\frac{21}{16} \cdot \frac{l}{h} \cdot W.$

(9) Top chord thrusts firm support to centre 1294, 1230, 1295, 1423, 1402, 1378 tons. Lower chord tensions, 915, 915, 931, 1290, 1290, 1444, 1444, 1422 tons.

## EXAMPLES XII.

(1) From support to centre (in tons). Lower chord maximum tensions, 44·1, 44·1, 75·6, 94·5; minimum tensions, 12·6, 12·6, 21·6, 27·0; Top chord maximum thrusts, 75·6, 94·5, 100·8.; minimum thrusts, 21·6, 27·0, 28·8.

(2) + tension, − thrust (in tons). End posts, − 73·5, − 21. Diagonals, support to centre first, + 54·4, + 13·1, second +37·1, + 3·4, third + 21·8, −8·3. More exactly diagonals: first +53·7, +13·9, second +35·8, +4·8, third + 20·1, −6·6.

(3) From support to centre (in tons); lower chord tensions (max.) 23·3, 60·3, 78·7, (min.) 6·03, 15·3, 19·9; upper cord thrusts (max.) 46·1, 73·9, 83·1, (min.) 11·5, 18·5, 20·8; extreme stresses in diagonals, end to centre (tension +) max. −46·8, +45·65, −29·65, +28·50, −15·35, +14·20, min. −12·1, +10·95, −6·11, +4·96, +2·66, −3·81.

(4) 2·8 tons per foot.

From end to centre (in tons); lower chord maximum tensions 0, 30, 42·86; top chord maximum thrusts 30·9, 43·3, 48·2; diagonals maximum tensions 42·1, 23·3, 12·6; verticals maximum thrusts 37·5, 17·8, 9·7, 0.

## EXAMPLES XIII.

(1) $-1250w$, $+3750w$ and $+2500w$ tons-feet.
(2) Maximum tension 115·8 tons; maximum thrust 26·1 tons-feet.
(3) Bay QE; 67·0 tons; 12·1 tons.
(4) MF and FG.
(5) 111·4 and 13·5 tons (tension).
(6) 104 and 12·6 tons.
(7) 49·3 tons thrust; 216·4 tons tension.
(8) 62·8 and 21·2 tons tension.
(9) 185,600 and 66,900 lbs. tension.
(10) Thrusts 14 tons; tension 17·4 tons.
(11) Thrusts 14 tons, tension 9·9 tons.
(12) Stresses in lbs.; tension +; tie, +1000; jib −1732; shear legs (*a*)+650 each, (*b*)+1060 and +170, (*c*)+1154 and 0; post (*a*)+370, (*b*) +310, (*c*)+260; strict RS, (*a*)−485, (*b*)−785, (*c*)−870; strut RT (*a*)− 485, (*b*)− 125, (*c*) 0.
(13) AB 1450 lbs., AD 1280 lbs., AC 800 lbs.

## EXAMPLES XIV.

(1) 0·0252 inch, 0·00762 inch.
(2) 0·2124 inch.
(3) 0·2395 inch.
(4) AC, 2·40 tons; BC 5·49 tons, DC 5·25 tons.
(5) Sides 207 lbs.; vertical diagonal 707 lbs. tension; horizontal diagonal 293 lbs. thrust.
(6) 0·387W and 0·467W.
(7) 1540 lbs. tension, 2180 lbs. thrust.
(8) 25·98 tons, (*a*) 40 tons, (*b*) 38 tons.

## EXAMPLES XV.

(1) 242 lbs.; 6·03 tons per square inch; 1·33 ton per square inch.
(2) 59,130 lb-feet.
(3) 56,318 lb.-feet.
(4) 2·625 tons-feet; 1·125 tons-feet.
(5) 0·726 inch.
(6) 2·5 tons-feet; 1·25 tons-feet.
(7) 0·655 inch.
(8) 12·455; 10·446 and 21·797 tons-inches.
(9) 0·1148 inch.

## EXAMPLES XVI.

(1) to (5) Indefinite; refer to Plate II.
(6) 4·91 tons.
(7) 2·749 tons.
(8) 2·943 tons.

## EXAMPLES XVII.

(1) 6 feet 8 inches and 12 feet 10 inches.
(2) 16 feet 1 inch.
(3) 1·25 inch; 23 feet 6 inches; 32 feet.
(4) $\frac{7}{8}$ inch rivets, pitch 5·8 inches theoretical, 4 inches actual; or 1 inch rivets, 6-inch pitch changing to $\frac{7}{8}$-inch rivets at first stiffener.
(5) 24 inches or 21 inches according to 4-inch or 6-inch pitch.
(6) (*a*) 6 tons 12 cwt. 1 qr. 13 lbs.; (*b*) 17 cwt. 2 qrs. 22 lbs.; (*c*) 8 cwt. 3 qrs. 17 lbs.

## EXAMPLES XVIII.

(1) 3710 feet.
(2) 53·85 tons; 10·77 sq. inches; 82·13 feet, 57·7 tons; 48·85 tons; 3·36 tons; 46·9 tons.
(3) (by calculation) left end 30·79, right end 31·10 tons.
(4) 47·3 tons; 48 tons-feet; left − 15 tons-feet, right + 10 tons-feet.
(5) + 18·83 tons-feet; − 18·83 tons-feet at 23·4 feet from left, 60·5 feet from right and 39·5 feet from left loaded.
(6) + 1·5625 and − 3·4375 tons; + 2·5 and − 2·5 tons; + 2·8125 and − 2·1875 tons.
(7) + and − 0·9450, 0·625, 1·055 tons.
(8) 40·39 tons + and − 62·5 tons-feet; − 15 tons-feet, + 10 tons-feet.
(9) + and − 31·25 tons-feet; for + value, 25 feet from each end; for − value 50 feet central.
(10) + and − 2·5 tons for all sections.
(11) + and − 1·25 tons for all sections.
(12) 1·523 tons per sq. inch.
(13) 31·25 tons; 8·4 tons-feet.
(14) 20·83 tons; 6·51 tons-feet; 25·1 tons-feet; 25·1 tons; 0·57 ton.
(15) 3·125 tons; − 23·4375 tons-feet; + 7·8125 tons-feet; 6·25 tons; − 15·625 tons-feet.

(16) + and − 0·563 W and 0·9945 W.
(17) Thrust 1·032 W ; tension 1·1628 W ; (W = load per 50 feet panel).
(18) 30·5 tons.
(19) 9·76 tons ; 21·9 tons-feet.
(20) 6·96 tons ; 5·66 tons-feet ; 5·8 tons ; 6·4 tons-feet.
(21) 0·421 ton.
(22) 0·43 ton per sq. inch.
(23) 11·72 tons ; − 18·75 tons-feet.
(24) Ends − 0·0553 $Wl$ ; $-\frac{W}{2}$ ; 0·459 W ; Crown − 0·0757 $Wl$ ; 0·459 W ; zero.
(25) 0·3103 ton.
(26) 1990 lbs. per sq. inch ; 50·5°.
(27) 1·62 tons per sq. inch.

## EXAMPLES XIX.

(1) 2·24 feet.
(2) Three 12″ × 5″ beams on eight 8″ × 4″ beams all 7 feet long in concrete 8 ft. by 8 ft..
(3) (*a*) 1·405 ft. ; 6468 lbs. thrust and 168 lbs. tension per sq. ft. ; (*b*) 1·3446 ft. ; 10,665 lbs. thrust and 45 lbs. tension per sq. ft.
(4) (*a*) 8·17 ft. ; (*b*) 8·04 feet.
(5) 89·6 feet.
(6) Upstream toe 6·22 tons per sq. ft. ; downstream toe 4·98 tons per sq. foot.

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